Revealed preference tests of stability in aggregate matching with single-peaked preferences *

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Abstract

This paper examines the rationalizability of two-sided aggregate matchings with non-transferable utility under single-peaked preferences. We build upon Echenique et al. (2013), which considers unrestricted preference domains. In contrast, we restrict preferences to be single-peaked, motivated by the fact that many observable characteristics of agents, such as age and income, are naturally ordered. A matching is rationalizable by single-peaked preferences if there exists a single-peaked preference profile under which the observed matching is stable. We find a characterization for such matchings and extend the result to a multidimensional type setting.

Keywords: Revealed Preference, Aggregate Matching, Single-peaked Preference *JEL Classifications:* C78, D11

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In the matching literature, pairwise stability, which requires that no pair of agents prefer each other to their current partners, is the prevailing solution concept. Whether a matching is stable or not depends on the preferences of the agents, but true preferences are often not observed, even when market participants are asked to disclose them. Echenique et al. (2013) study this problem from a revealed preference standpoint, where a matching is observed but not the agents' preferences. In particular, they focus on observations in the form of aggregate matchings, where people are grouped into types, and the number of matches between each pair of types is recorded. In order to derive testable implications, they make three assumptions: (i) all individuals of the same type are viewed as identical, (ii) all individuals with the same type have identical preferences, and (iii) preferences over types are strict. An aggregate matching is said to be rationalizable if there exist preferences such that the observed matching is stable. Their paper derives characterizations of rationalizable aggregate matching for both the transferable utility model and the non-transferable utility (NTU) model.

This paper focuses on the revealed preference test of aggregate matching with single-peaked preferences in the NTU setting. Echenique et al. (2013) map the aggregate matching data into a graph and obtain a characterization that an aggregate matching is rationalizable if, and only if, its associated graph does not contain two connected, distinct minimal cycles. This condition is invariant to the way in which the types of agents are indexed or ordered. However, if the types are grouped by observable characteristics such as age, education, or height, they follow some natural orderings, and there might be cases where a given matching can only be rationalized by preferences that seem unreasonable for such natural orderings. For example, suppose men are indexed by height. It could be that any preference profile that rationalizes a matching requires a type of woman to rank 5'9 men above 5'7 men above 5'8 men, which seems unnatural: if 5'9 is preferred to 5'7, then why should 5'8 be less preferred than 5'7?

Here, we discipline the preferences that rationalize a matching to be single-peaked. We say that a matching is rationalizable by single-peaked preferences if there exists a single-peaked preference profile under which the matching is stable. We obtain a characterization of this restricted test: a matching is rationalizable by single-peaked preference if and only if the associated graph does not contain two connected, distinct minimal cycles; two crossing edges; two externally connected, distinct split nodes; or a minimal cycle that is externally connected to a split node.

Our paper is most closely related to works that also follow the framework from Echenique et al. (2013) on the rationalizability of aggregate matchings. Echenique et al. (2021) provides a sufficient condition under which an aggregate matching is rationalizable as the median stable matching, which assigns each agent the median outcome among all stable matching outcomes. Demuynck and Salman (2022) develop equivalent characterizations of rationaliz-

able matchings, and study the minimum number of matches that need to be removed from an unrationalizable matching to make it rationalizable.[1] Our paper complements this body of literature by showing how the test can be modified to a refined test for rationalizability by restricted, namely single-peaked, preferences.

The condition of single-peakedness has a long tradition in economics, especially in the social choice literature, following the seminal work by Black (1948) on the Median Voter Theorem. In the marriage literature, the pioneering work of Becker (1973) on assortative matching assumes monotone preferences in the spouse's productivity in marital output, which is a special case of single-peaked preferences. In the literature that models matching as between masses or continua of agents, Clark (2004, 2007, 2020) explores the condition of "like attracts like", which is also a special case of single-peakedness, and Flanders (2013) studies general single-peaked preferences. In matching theory with finite markets, Bade (2019) studies single-peaked preferences in matching with one-sided preferences with a focus on housing markets. In the field of revealed preference theory, Arlegi and Teschl (2022) study Pareto rationalizability by single-peaked preferences. Our paper lies in the intersection of two-sided matching in finite markets, revealed preference, and single-peaked preferences.

The paper proceeds as follows: Section 1 outlines the framework and the result in Echenique et al. (2013). Sections 2 and 3 present the additional conditions needed for rationalizability by single-peaked preferences. Section 4 extends the results to settings with multidimensional types, and Section 5 concludes.

1 Framework and previous results

We follow the notation and definitions of Echenique et al. (2013). Let the set of types of men be $M = \{m_1, m_2, \ldots, m_{|M|}\}$ and the set of types of women be $W = \{w_1, w_2, \ldots, w_{|W|}\}$. A matching is an $|M| \times |W|$ matrix $X = (x_{m,w})$, where $x_{m,w}$ represents the number of men of type m matched to women of type w. Assume that there are equal numbers of men and women, and that no one is unmatched. Regarding preferences, assume that all individuals of the same type are viewed as identical, that all individuals of the same type have identical preferences, and that all preferences are strict.

[1] There are also other revealed preference approaches to matching models, although none consider single-peaked preferences. For example, Echenique (2008) studies rationalizability of individual matchings, which asks whether there exists a preference profile such that multiple observed individual matchings are all stable. Cherchye et al. (2017) develop a revealed preference characterization for whether a household consumption data set is rationalizable by a stable matching, followed by an empirical application. Haeringer and Iehlé (2019), Hu et al. (2020), and Tai (2022) focus on matching markets with one-sided preferences.

Given a preference profile P, which consists of preferences (denoted P_m and P_w) for every type of man and woman, a pair of types (m, w) is *a blocking pair* for X if there exist m' and w' such that $x_{m,w'} > 0$, $x_{m',w} > 0$, $mP_w m'$, and $wP_m w'$. A matching X is *stable* if there is no blocking pair. A matching X is *rationalizable* if there exists a preference profile under which X is a stable matching.

The associated graph of matching X is $G = (V, L)$, where that $V = \{(m, w) \in M \times W : x_{m,w} > 0\}$ is the set of vertices and $L = \{((m, w), (m', w')) \in V^2 : m = m' \text{ or } w = w'\}$ is the set of edges. Figure 1 shows an example from Echenique et al. (2013). The number 11, for example, indicates that there are 11 couples formed between type m_1 and type w_1. In the associated graph, each positive entry of the matching is a vertex, and edges are drawn between any pair of vertices that are either in the same row or in the same column.

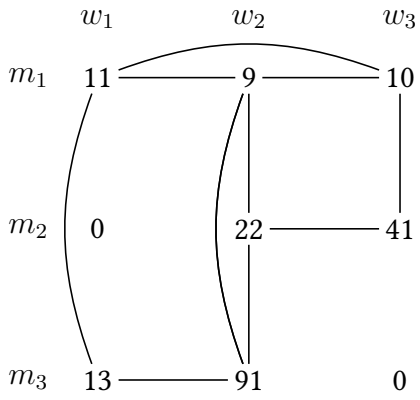

Figure 1: Example from Echenique et al. (2013)

To state the result from Echenique et al. (2013), we need the following definitions pertaining to the structure of a graph:

Definition. *Given graph $G = (V, L)$:*

1. *A* path *in G is a sequence $\langle v_0, \ldots, v_N \rangle$ such that $(v_n, v_{n+1}) \in L$ for all $n \in \{0, \ldots N-1\}$. The path* connects *the vertices v_0 and v_N.*

2. *A path is* minimal *if it contains no proper subsequence that also connects v_0 and v_N.*

3. *A* cycle *is a path $\langle v_0, \ldots, v_N \rangle$ with $v_0 = v_N$.*

4. *A cycle is* minimal *if for any two of its vertices, say v_n abd $v_{n'}$, the paths (in the cycle) from v_n to $v_{n'}$ and from $v_{n'}$ to v_n are minimal and distinct (i.e., they contain different sets of edges).*

5. *Two cycles are* connected *if there is a path from a vertex in one to a vertex in the other.*

Theorem (Echenique et al. (2013))**.** *A matching is rationalizable if, and only if, its associated graph does not contain two connected, distinct minimal cycles.*

In the example above, there are three minimal cycles. For easy reference, we describe them using the numbers instead of the formal description. Cycles

$$\mathfrak{c}_1 = \langle 11 - 9 - 91 - 13 - 11 \rangle,$$

$$\mathfrak{c}_2 = \langle 11 - 10 - 41 - 22 - 91 - 13 - 11 \rangle,$$

and

$$\mathfrak{c}_3 = \langle 9 - 10 - 41 - 22 - 9 \rangle$$

are minimal. Cycles

$$\langle 11 - 9 - 22 - 91 - 13 - 11 \rangle$$

and

$$\langle 11 - 9 - 10 - 41 - 22 - 91 - 13 - 11 \rangle$$

are not minimal because $\langle 9 - 22 - 91 \rangle$ and $\langle 9 - 10 - 41 - 22 \rangle$ are not minimal paths. The minimal cycles $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are connected via $\langle 11 - 10 \rangle$; $\mathfrak{c}_1$ and $\mathfrak{c}_3$ are connected via $\langle 9 - 10 \rangle$; and $\mathfrak{c}_2$ and $\mathfrak{c}_3$ are connected via $\langle 11 - 10 \rangle$. Since any two of these minimal cycles are connected, this matching is not rationalizable.

The intuition for the necessity direction facilitates the reasoning behind our results. Whenever two edges form a right angle, there must be a flow in the preference orientation; and whenever there are two connected minimal cycles, those flows must clash. We demonstrate this intuition with the example in Figure 2.

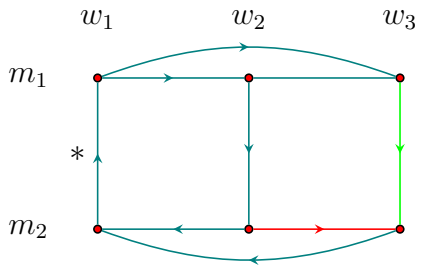

Figure 2: Intuition for the necessity of the condition

Let us start at w_1. Since preferences are strict, w_1 must prefer either m_1 or m_2. Suppose w_1 prefers m_1, i.e. $m_1 P_{w_1} m_2$, which is indicated by the arrow labeled with a star. Now, if m_1 also

prefers w_1 to either w_2 or w_3, (m_1, w_1) would be a blocking pair, which makes the matching unstable. Therefore, if the matching is stable, it must be that $w_2 P_{m_1} w_1$ and $w_3 P_{m_1} w_1$, as indicated by the arrows on the horizontal edges in the upper row. By similar arguments, $w_2 P_{m_1} w_1$ implies $m_2 P_{w_2} m_1$, and $w_3 P_{m_1} w_1$ implies $m_2 P_{w_3} m_1$, which is indicated by the green arrow. Furthermore, $m_2 P_{w_2} m_1$ implies $w_3 P_{m_2} w_2$, indicated by the red arrow. Notice that the red arrow and the green arrow point toward each other, which indicates that (m_2, w_3) is a blocking pair. A similar conclusion is obtained if we start with $m_2 P_{w_1} m_1$ instead of $m_1 P_{w_1} m_2$. Hence, the matching is not rationalizable.

2 Testable implications under single-peaked preferences

With the basic framework from Echenique et al. (2013) in place, we can now consider restricting the test to single-peaked preferences.

Fixing the ordering of indices of types of men and women, we say that a man of type m *has single-peaked preferences* if for all i, j, k with $i < j < k$ or $i > j > k$, $w_i P_m w_k \Rightarrow w_j P_m w_k$, and similarly for a woman of type w. A preference profile is single-peaked if all types of men and all types of women have single-peaked preferences. A matching is rationalizable by single-peaked preferences if there exists a single-peaked preference profile for which the matching is stable.

Notice that rationalizability by single-peaked preferences is a stronger condition than rationalizability. If a matching is not rationalizable, it is not rationalizable by single-peaked preferences; while if a matching is rationalizable, it may still not be rationalizable by single-peaked preferences. Therefore, in addition to not containing two connected, distinct minimal cycles, other conditions must hold for a matching to be rationalizable by single-peaked preferences.

Definition 1. *The* edges $((m_j, w_p), (m_j, w_r))$ and $((m_i, w_q), (m_k, w_q))$ cross *if* $p < q < r$ *and* $i < j < k$.

Figure 3 is an illustration of two edges that cross. The first necessary condition that we identify for a matching to be rationalized with single-peaked preferences is that there can be no crossing edges.

Proposition 1. *If a matching's associated graph contains crossing edges, it is not rationalizable by single-peaked preferences.*

Proof. Suppose two edges $((m_j, w_p), (m_j, w_r))$ and $((m_i, w_q), (m_k, w_q))$ cross, with $p < q < r$ and $i < j < k$. Since P_{m_j} is strict, either $w_p P_{m_j} w_r$ or $w_r P_{m_j} w_p$ (but not both). Then, by

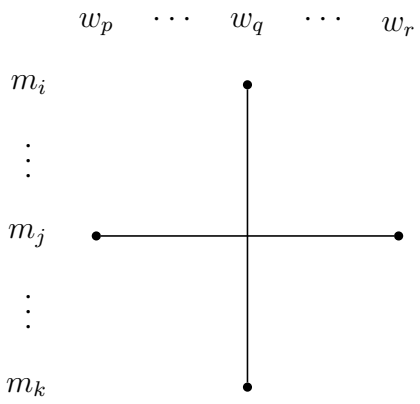

Figure 3: Crossing edges

single-peakedness, $w_q P_{m_j} w_p$ or $w_q P_{m_j} w_r$. Similarly for P_{w_q}, $m_i P_{w_q} m_k$ or $m_k P_{w_q} m_i$ (but not both), and $m_j P_{w_q} m_i$ or $m_j P_{w_q} m_k$. Since $x_{m_j,w_p} > 0$, $x_{m_j,w_r} > 0$, $x_{m_i,w_q} > 0$, and $x_{m_k,w_q} > 0$, (m_j, w_q) is a blocking pair. □

Consider a 3×3 matching whose associated graph only contains four vertices, say (m_2, w_1), (m_2, w_3), (m_1, w_2), and (m_3, w_2). The associated graph can be obtained from Figure 3 by letting $m_i = m_1, m_j = m_2, m_k = m_3, w_p = w_1, w_q = w_2$, and $w_r = w_3$. There is no cycle in the graph while there are two crossing edges, making this an example of a matching that is rationalizable but not by single-peaked preferences.

The other type of structure that interacts with single-peaked preferences is a T-shaped structure that we will call a split. We define *split nodes*, *splits*, and *split ends* as follows:

Definition 2. Vertex $(m, w) \in V$ is a horizontal split node *if there exist* m', m'', w', *with* $m' < m < m''$ *and* $w' \neq w$, *such that* $(m', w), (m'', w), (m, w') \in V$. *Given such a split node* (m, w), *we call* $((m, w), (m, w')) \in L$ *(in teal in Figure 4) a* split *for* (m, w) *and* (m, w') *(filled in blue in the same figure) a* split end *for* (m, w).

In the same vein, vertex $(m, w) \in V$ is a vertical split node *if the symmetric condition holds.*[2] *(See Figure 5.)*

When there is a split node, single-peakedness, together with stability, puts restrictions on the potential preferences that rationalize the matching. Take the above vertical split as an

[2] That is, if there exist w', w'', m', with $w' < w < w''$ and $m' \neq m$ (none needs to be unique) such that $(m, w'), (m, w''), (m', w) \in V$. An edge $((m, w), (m', w)) \in L$ is a *split* (the teal edge in Figure 5) for the split node (m, w) (filled in red), that (m', w) is a *split end* (filled in blue) for (m, w), and that (m, w) has a vertical split.

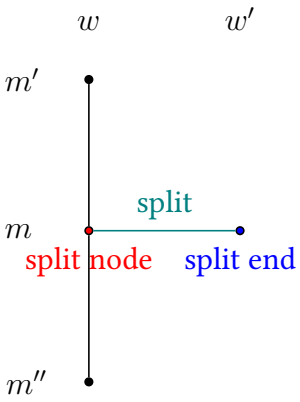

Figure 4: Horizontal split node

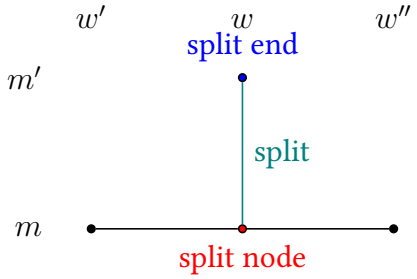

Figure 5: Vertical split node

example. If w prefers m to m', then stability requires that m prefer w' to w and w'' to w, violating single-peakedness. Therefore, any single-peaked preference profile that rationalizes the matching must require w to prefer m' over m. In general, single-peakedness and stability require the preference orientation arrow to point away from a split node to each of its split ends. Now, it is conceivable that when a graph contains multiple split nodes, incompatibilities may arise. This observation brings us to the next necessary condition.

Definition 3. *Path* $\langle(\hat{m}, \hat{w})_n : n = 0, \ldots, N\rangle$ connects the split nodes $(\bar{m}, \bar{w})$ and $(\tilde{m}, \tilde{w})$ externally *if*

1. $(\hat{m}, \hat{w})_0 = (\bar{m}, \bar{w})$,
2. $(\hat{m}, \hat{w})_1$ *is a split end for* $(\bar{m}, \bar{w})$,
3. $(\hat{m}, \hat{w})_{N-1}$ *is a split end for* $(\tilde{m}, \tilde{w})$, *and*
4. $(\hat{m}, \hat{w})_N = (\tilde{m}, \tilde{w})$,

with $(\bar{m}, \bar{w})$ *and* $(\tilde{m}, \tilde{w})$ *both appearing exactly once. In this case, we say that* $(\bar{m}, \bar{w})$ *and* $(\tilde{m}, \tilde{w})$ *are* externally connected.

Proposition 2. *If a matching's associated graph contains two externally connected split nodes, it is not rationalizable by single-peaked preference.*

The proof is in the appendix, while Figure 6 illustrates the intuition. By stability and single-peakedness of all the preferences, the split node (m_2, w_1) requires $w_2 P_{m_2} w_1$. For stability, this further induces $m_5 P_{w_2} m_2$ and hence $w_3 P_{m_5} w_2$. Now we have reached the other split node (m_5, w_3) and stability requires $m_4 P_{w_3} m_5$ and $m_6 P_{w_3} m_5$, violating single-peakedness. Notice

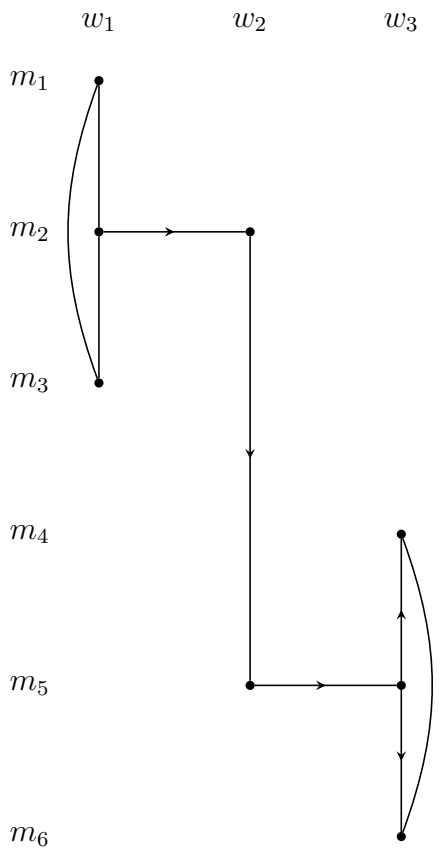

Figure 6: Two externally connected split nodes

that this is also an example where the matching is rationalizable but not rationalizable by single-peaked preferences, as the graph does not contain any cycles.

Definition 4. Path $\langle(\hat{m}, \hat{w})_n : n = 0, \ldots, N\rangle$ connects cycle $\mathfrak{c}$ and split node $(\bar{m}, \bar{w}) \notin \mathfrak{c}$ externally *if*

1. $(\hat{m}, \hat{w})_0 = (\bar{m}, \bar{w})$,
2. $(\hat{m}, \hat{w})_1$ *is a split end for* $(\bar{m}, \bar{w})$*, and*
3. $(\hat{m}, \hat{w})_N \in \mathfrak{c}$.

In this case, the cycle $\mathfrak{c}$ *and the split node* $(\bar{m}, \bar{w})$ *are* externally connected.

At this point, we hope to have developed the intuition that preference flow must go out from any split end. Recall that the insight from Echenique et al. (2013) is that, as preference flows must go out from any minimal cycle, when two minimal cycles are connected, the flows must

clash. So, when a minimal cycle is externally connected to a split node, the flows must clash, too. Strictly speaking, Proposition 1 already rules out this possibility. It is convenient for us, however, to state it as the last necessary condition.

Corollary 1. *If in the associated graph of a matching, a minimal cycle is externally connected to a split node, the matching is not rationalizable by single-peaked preference.*

Proof. This result follows from Proposition 2, since the cycle considered in that proposition would contain a second split node. □

The example in Figure 7 again gives the intuition. By stability and single-peaked preferences, the split node requires $w_2 P_{m_2} w_1$ and $w_3 P_{m_2} w_1$. Following $w_2 P_{m_2} w_1$, we have $m_4 P_{w_2} m_2$, $w_3 P_{m_4} w_2$, $m_2 P_{w_3} m_4$, and eventually $w_1 P_{m_2} w_3$, contradicting $w_3 P_{m_2} w_1$.

Note that this is also an example where the matching is rationalizable but not rationalizable by single-peaked preferences, as the graph does not contain two connected minimal cycles.

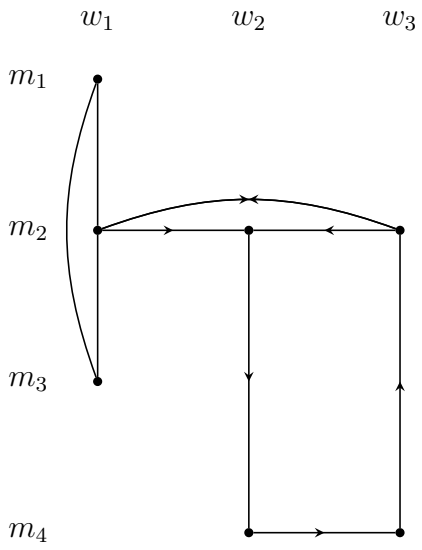

Figure 7: A minimal cycle that is externally connected to a split node

3 Sufficiency of the conditions

The three necessary conditions presented above, together with the requirement that there be no two connected, distinct minimal cycles, turn out to exhaust the testable implications of rationalizability under single-peaked preferences.

Theorem 1. *A matching is rationalizable by single-peaked preferences if the associated graph does not contain two connected, distinct minimal cycles, crossing edges, two externally connected split nodes, or a minimal cycle that is externally connected to a split node.*

For the argument, we construct a single-peaked preference profile such that the matching is stable. The detailed construction is in the appendix, but the essence is as follows: For each split node, we assign the preference orientation to point towards each of its split ends, which then assigns preference orientations to all the subsequent flows. For each distinct minimal cycle that does not contain a split node, we arbitrarily assign preference orientations along the cycle, which then assign preference orientations to all the subsequent flows. Next, we assign other preference orientations in a way that is consistent with single-peakedness and the existing flows. Finally, we show that the constructed preference profile is well defined, single-peaked, and rationalizes the given matching. Figure 8 is an example of how the preferences are assigned.

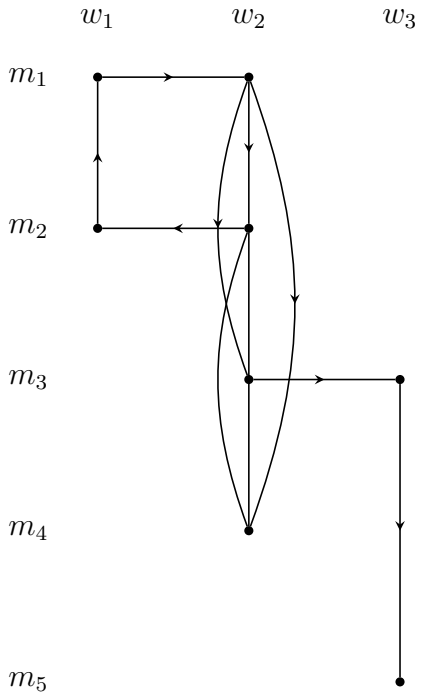

Figure 8: Illustration of the construction of preference

First, we assign the preference orientation on the splits, namely $((m_3, w_2), (m_3, w_3))$ and $((m_2, w_2), (m_2, w_1))$, to be $w_3 P_{m_3} w_2$ and $w_1 P_{m_2} w_2$. Recall that stability requires that whenever two edges form a right angle, there must be a flow in the preference orientation. Therefore, $w_3 P_{m_3} w_2$ implies $m_5 P_{w_3} m_3$. Similarly, $w_1 P_{m_2} w_2$ implies a clockwise preference orientation on the cycle, i.e. $w_2 P_{m_1} w_1, m_2 P_{w_2} m_1, w_1 P_{m_2} w_2$, and $m_1 P_{w_1} m_2$, followed by the flow that

this orientation creates, which are $m_3 P_{w_2} m_1$ and $m_4 P_{w_2} m_1$. Notice that the subsequent flows that $m_3 P_{w_2} m_1$ creates are $w_3 P_{m_3} w_2$ and $m_5 P_{w_3} m_3$, which are consistent with the preferences assigned due to the other split. Finally, we fill in other preference relations in a consistent and single-peaked manner: so, for the men,

$$\begin{aligned} P_{m_1} &= w_2, w_1, w_3, \\ P_{m_2} &= w_1, w_2, w_3, \\ P_{m_3} &= w_3, w_2, w_1, \\ P_{m_4} &= w_2, w_1, w_3, \\ P_{m_5} &= w_3, w_2, w_1, \end{aligned}$$

while for the women

$$\begin{aligned} P_{w_1} &= m_1, m_2, m_3, m_4, m_5, \\ P_{w_2} &= m_4, m_3, m_2, m_1, m_5, \\ P_{w_3} &= m_5, m_4, m_3, m_2, m_1 \end{aligned}$$

These preferences are single-peaked and rationalize a matching whose associated graph is as in Figure 8.

4 Multidimensional types

We have extended the analysis of Echenique et al. (2013) to rationalizablility by single-peaked preferences in the context that the indices of types capture a one-dimensional observed characteristic. Next, we will generalize our result to the case of multidimensional observed characteristics.

Let $\Gamma_1, \ldots, \Gamma_{\ell^*}, \ldots, \Gamma_L$, where $\Gamma_\ell = \{\gamma_{\ell,1}, \gamma_{\ell,2}, \ldots, \gamma_{\ell,A_\ell}\}$, be the dimensions of characteristics of men. For example, $\Gamma_1 = \{$twenties, thirties, forties$\}$ can be the age dimension, $\Gamma_2 = \{$green, blue, brown, black$\}$ can be the eye color dimension, and so on. Then the set of types of men can be written as $M = \{m_\gamma : \gamma \in \times_\ell \Gamma_\ell\}$.

Similarly, let $\Delta_1, \ldots, \Delta_{n^*}, \ldots, \Delta_N$ be the dimensions of characteristics of women, where $\Delta_n = \{\delta_{n,1}, \delta_{n,2}, \ldots, \delta_{n,B_n}\}$. Then the set of types of women can be written as $W = \{w_\delta : \delta \in \times_n \Delta_n\}$.

Let $\gamma_{-\ell} = \{\gamma_1, \ldots, \gamma_{\ell-1}, \gamma_{\ell+1}, \ldots, \gamma_L\}$ and $\delta_{-n} = \{\delta_1, \ldots, \delta_{n-1}, \delta_{n+1}, \ldots, \delta_N\}$. We say that m-type men have single-peaked preferences in Δ_n if for all δ_{-n}, for all i, j, k such that $i < j < k$ or $i > j > k$,

$$w_{(\delta_{n,i}, \delta_{-n})} P_m w_{(\delta_{n,k}, \delta_{-n})} \Rightarrow w_{(\delta_{n,j}, \delta_{-n})} P_m w_{(\delta_{n,k}, \delta_{-n})}.$$

Similarly we define type w women having single-peaked preference in Γ_ℓ.

Let $\Gamma_1, \ldots, \Gamma_{\ell^*}$ and $\Delta_1, \ldots, \Delta_{n^*}$ be the dimensions of characteristics where single-peaked preferences are relevant. We say that m-type men have single-peaked preference if they have single-peaked preference in $\Delta_1, \ldots, \Delta_{n^*}$. Similarly, we say that w-type women have single-peaked preference if they have single-peaked preference in $\Gamma_1, \ldots, \Gamma_{\ell^*}$.

We say that there are *crossing edges* if men have single-peaked preference in Δ_n, women have single-peaked preference in Γ_ℓ, and there exist $\gamma_{-\ell}$, δ_{-n}, $p < q < r$, and $i < j < k$, such that $(m_{(j,-\ell)}, w_{(p,-n)}), (m_{(j,-\ell)}, w_{(r,-n)}), (m_{(i,-\ell)}, w_{(q,-n)}), (m_{(k,-\ell)}, w_{(q,-n)}) \in V$.

Split nodes, splits, and split ends are redefined as follows:

1. If women are to have single-peaked preferences in Γ_ℓ, $(m_{(j,-\ell)}, w) \in V$ is a ***split node*** if there exist $\gamma_{-\ell}$, $i < j < k$, and $w' \neq w$, such that

   $$(m_{(i,-\ell)}, w), (m_{(k,-\ell)}, w), (m_{(j,-\ell)}, w') \in V.$$

   $((m_{(j,-\ell)}, w), (m_{(j,-\ell)}, w')) \in L$ is a ***split*** for the split node $(m_{(j,-\ell)}, w)$, $(m_{(j,-\ell)}, w')$ is a ***split end*** for $(m_{(j,-\ell)}, w)$, and we say that $(m_{(j,-\ell)}, w)$ has a ***horizontal*** split.

2. If men are to have single-peaked preferences in Δ_n, $(m, w_{(q,-n)}) \in V$ is a ***split node*** if there exist δ_{-n}, $p < q < r$, and $m' \neq m$, such that

   $$(m, w_{(p,-n)}), (m, w_{(r,-n)}), (m', w_{(q,-n)}) \in V.$$

   $((m, w_{(q,-n)}), (m', w_{(q,-n)})) \in L$ is a ***split*** for the split node $(m, w_{(q,-n)})$, $(m', w_{(q,-n)})$ is a ***split end*** for $(m, w_{(q,-n)})$, and we say that $(m, w_{(q,-n)})$ has a ***vertical*** split.

The definitions for two split nodes to be externally connected and for a split node to be externally connected to a cycle are the same as in the previous section. The analogous results hold:

Theorem 2. *In the multidimensional type setting, a matching is rationalizable by single-peaked preferences if, and only if, the associated graph does not contain two connected, distinct minimal cycles, crossing edges, two externally connected split nodes, or a minimal cycle that is externally connected to a split node.*

The proofs are in the appendix.

5 Conclusion

In this paper, we restrict the revealed preference test of aggregate matching by Echenique et al. (2013) to single-peaked preferences. As rationalizability by single-peaked preferences is a

stronger requirement than rationalizability by any preferences, there must be extra conditions that a matching needs to satisfy, on top of the characterization for rationalizability without restrictions. These conditions are: no crossing edges, no externally connected distinct split nodes, and no minimal cycle that is externally connected to a split node. Along with the no two connected distinct minimal cycles condition from Echenique et al. (2013), they are also sufficient for rationalizability by single-peaked preferences. Moreover, the analogous results hold when the model is extended to multidimensional type spaces.

The revealed preference test of aggregate matching with a restriction to single-peaked preferences becomes relevant when the types of agents depend on observed characteristics that have natural orderings, such as age, income, or height. A matching can potentially be rationalizable but not rationalizable by single-peaked preferences, in which case the stability implications differ. In the case of multidimensional type setting, the test result may also change depending on which dimensions are considered relevant for single-peaked preferences. Depending on the context, the restricted test, in conjunction with the unrestricted test, may provide richer insight into an empirical exercise than the unrestricted test alone.

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A Proofs

We borrow the following notation from Echenique et al. (2013). Let P be a preference profile and d the orientation derived from P. Then, write $d((m,w),(m,w'))=1$ if wP_mw', and $d((m,w),(m,w'))=0$ otherwise. Similarly, write $d((m,w),(m',w))=1$ if mP_wm', and $d((m,w),(m',w))=0$ otherwise. A path $\langle (m,w)_n : n=0,\dots,N\rangle$ is a *flow* for d if either

$$d((m,w)_n,(m,w)_{n+1}))=1 \text{ for all } n\in\{0,\dots,N-1\}$$

or

$$d((m,w)_n,(m,w)_{n+1}))=0 \text{ for all } n\in\{0,\dots,N-1\}.$$

In the second case, the path is a *forward flow*.

We also use some intermediate results from Echenique et al. (2013). For their statements, fix an orientation d derived from a preference profile that rationalizes X.

Lemma (Lemma 1 in Echenique et al., 2013)**.** *Let $\mathfrak{p}=\langle (m,w)_n : n=0,\dots,N\rangle$ be a minimal path. If $d((m,w)_0,(m,w)_1))=0$ or $d((m,w)_{N-1},(m,w)_N))=1$, then $\mathfrak{p}$ is a flow for d.*

Lemma (Lemma 2 in Echenique et al., 2013)**.** *If $\mathfrak{c}=\langle (m,w)_n : n=0,\dots,N\rangle$ is a minimal cycle, then $\mathfrak{c}$ is a flow for d.*

Lemma (Lemma 3 in Echenique et al., 2013)**.** *Let $\mathfrak{c}=\langle (m,w)_n : n=0,\dots,N\rangle$ be a minimal cycle and $\mathfrak{p}=\langle (\bar{m},\bar{w})_n : n=0,\dots,\bar{N}\rangle$ be a minimal path connecting $\mathfrak{c}$ and $(\bar{m},\bar{w})\notin\mathfrak{c}$. Then $\langle (\bar{m},\bar{w})_n : n=1,\dots,\bar{N}\rangle$ is a forward flow for d.*

A.1 Necessity

For the purposes of this subsection, fix a matching X, suppose that P is a single-peaked preference profile that rationalizes X, and let d be the orientation derived from P.

Lemma 1. *If (m,w) is a split node and $(\check{m},\check{w})$ is one of its split ends, $d((m,w),(\check{m},\check{w}))=0$.*[3]

Proof. By way of contradiction, suppose $d((m,w),(\check{m},\check{w}))=1$. We consider two separate cases:

1. *Supppose (m,w) is a split node with a horizontal split, so $\check{m}=m$.* Then, there exist m',m'', with $m'<m<m''$ and $\check{w}\neq w$, such that $(m',w),(m'',w),(m,\check{w})\in V$. Since $\mathfrak{p}_1=\langle(\check{m},\check{w}),(m,w),(m',w)\rangle$ and $\mathfrak{p}_2=\langle(\check{m},\check{w}),(m,w),(m'',w)\rangle$ are minimal paths, by Lemma 1 in Echenique et al. (2013), they are forward flows. Hence,

$$d((m,w),(m',w))=0 \text{ and } d((m,w),(m'',w))=0,$$

   contradicting single-peakedness of P_w.

2. *Alternatively, (m,w) is a split node with a vertical split, so $\check{w}=w$.* Here, there exist w',w'', with $w'<w<w''$ and $(\check{m}\neq m$, such that $(m,w'),(m,w''),((\check{m},w)\in V$. Hence , $\mathfrak{p}_1=\langle(\check{m},\check{w}),(m,w),(m,w')\rangle$ and $\mathfrak{p}_2=\langle(\check{m},\check{w}),(m,w),(m,w'')\rangle$ are minimal paths, as before they are forward flows. Again,

$$d((m,w),(m,w'))=0 \text{ and } d((m,w),(m,w''))=0,$$

   which contradicts single-peakedness $of P_m$. □

[3] And, equivalently, $d((\check{m},\check{w}),(m,w))=1$.

Corollary 2. *Let (m, w) be a split node and $(\check{m}, \check{w})$ be one of its split ends. Let $\mathbb{p} = \langle(\hat{m}, \hat{w})_n : n = 0, \ldots, N\rangle$ be a minimal path connecting (m, w), $(\check{m}, \check{w})$, and $(\dot{m}, \dot{w}) \notin \{(m, w), (\check{m}, \check{w})\}$. Then $\mathbb{p}$ is a forward flow.*

Proof. This follows from Lemma 1 above and Lemma 1 in Echenique et al. (2013) . □

The following claim is immediate, but we state it as a lemma for ease of reference.

Lemma 2. *Two distinct split nodes are externally connected if, and only if, they are externally connected through a minimal path.*

These results allow us to prove Proposition 2, which claims that if a matching's associated graph contains two externally connected distinct split nodes, then it is not rationalizable by single-peaked preference. Recall that at this point we are assuming that P is a single-peaked preference profile that rationalizes matching X, and that d is the orientation derived from P.

Proof of Proposition 2: Suppose, by way of contradiction, that the associated graph of matching X contains two distinct externally connected split nodes. Let $(\bar{m}, \bar{w})$ and $(\tilde{m}, \tilde{w})$ be these externally connected split nodes with $\bar{m} \neq \tilde{m}$ or $\bar{w} \neq \tilde{w}$. Then by Lemma 2, there exist a minimal path $\mathbb{p} = \langle(m, w)_n : n = 0, \ldots, N\rangle$ where $(m, w)_0 = (\bar{m}, \bar{w})$, $(m, w)_1$ is a split end for $(\bar{m}, \bar{w})$, $(m, w)_{N-1}$ is a split end for $(\tilde{m}, \tilde{w})$, and $(m, w)_N = (\tilde{m}, \tilde{w})$, with $(\bar{m}, \bar{w})$ and $(\tilde{m}, \tilde{w})$ both appearing exactly once. We have $N \geq 1$, because $N > 1$ if $\bar{m} \neq \tilde{m}$ and $\bar{w} \neq \tilde{w}$, whereas $N = 1$ if $\bar{m} = \tilde{m}$ or $\bar{w} = \tilde{w}$.

By Lemma 1, $d((m, w)_0, (m, w)_1) = 0$ and $d((m, w)_{N-1}, (m, w)_N) = 1$, which implies, by definition, that $\mathbb{p}$ is not a flow. By Lemma 1 in Echenique et al. (2013), however, $\mathbb{p}$ is a flow for d, since it is a minimal path. □

A.2 Sufficiency

We now argue the sufficiecy part of Theorem 1, namely that if the associated graph of a matching X does not contain two connected, distinct minimal cycles, crossing edges, two externally connected distinct split nodes, or a minimal cycle externally connected to a split node, then X is rationalizable by single-peaked preference. The argument is partially constructive, so we first define the preference profile and then argue that the desired properties hold.

A.2.1 Constructing individual preferences

Suppose the associated graph of matching X does not contain two connected, distinct minimal cycles, crossing edges, two externally connected distinct split nodes, or a minimal cycle externally connected to a split node. Construct a preference profile as follows:

1. For each minimal path $\mathbb{p}$ that externally connects a split node $(\bar{m}, \bar{w})$ to another vertex, let $\mathbb{p}$ be a forward flow.

2. For each minimal cycle $\mathbb{c}$ that does not contain a split node, let $\mathbb{c}$ be a forward flow.

3. For each minimal path $\mathbb{p} = \langle(\hat{m}, \hat{w})_n : n = 0, \ldots, N\rangle$ that connects $\mathbb{c}$ to a vertex not in $\mathbb{c}$ and not externally connected to a split node, let $\langle(\hat{m}, \hat{w})_n : n = 1, \ldots, N\rangle$ be a forward flow.[4]

[4] Although no minimal cycle is externally connected to a split node, there can be vertices that are connected to a cycle and externally connected to a split node. These are already "reached" in the first step.

4. For each connected component that consists of a split node but not a cycle, among the vertices that are not connected to the split node externally, find a vertex that has the longest almost minimal path (a path that does not contain a repeating vertex) from the split node. We call it a border vertex. If there are multiple, pick an arbitrary one. Let each minimal path that starts from this border vertex be a forward flow.

5. For each connected component of more than one vertex that does not contain any minimal cycle or split node, find the vertex with the smallest lexicographic index of (m, w). Let each minimal path that starts from this vertex be a forward flow.

6. For any type m that is matched to only one type of the other side, say w_j, let

$$P_m = w_j, w_{j-1}, w_{j-2}, \ldots, w_1, w_{j+1}, w_{j+2}, \ldots, w_{|W|}.$$

   Similarly for any type w.

7. Complete the preferences as follows. For any type m whose preference has been partially defined, there exist a $\underline{w}_m$ such that all the assigned orientations are $d((m, \underline{w}_m), (m, w)) = 0$ for all $w \neq \underline{w}_m$ and $\underline{w}_m$ has the smallest or the largest index among $W_m = \{w : (m, w) \in V\}$.

   (a) If $\underline{w}$ has the smallest index among W_m, let λ denote the index of the w with the largest index among W_m, and let

$$P_m = w_\lambda, w_{\lambda-1}, w_{\lambda-2}, \ldots, \underline{w}, \ldots, w_1, w_{\lambda+1}, w_{\lambda+2}, \ldots, w_{|W|};$$

   (b) Else, if $\underline{w}$ has the largest index among W_m, let λ denote the index of the w with the smallest index among W_m, and let

$$P_m = w_\lambda, w_{\lambda+1}, w_{\lambda+2}, \ldots, \underline{w}, \ldots, w_{|W|}, w_{\lambda-1}, w_{\lambda-2}, \ldots, w_1.$$

   Do similarly for each type w, *mutatis mutandis*.

A.2.2 Properties of the preferences

We now argue that the preferences constructed above are well defined and are single-peaked.

Lemma 3. *Let P be the profile of binary relations constructed in Appendix A.2.1. For each agent a, the corresponding P_a is a complete order over the types of the other gender.*

Proof. We need to argue that the orientations of the first five steps of the construction are consistent with the orders defined in steps 6 and 7.

For any type of man or woman that is matched to only one type of the other gender, the result is immediate by step 6 in the construction, so we only consider types that are matched to more than one type of the other gender.

Suppose that man type m is matched to more than one type of woman. P_m is then constructed according to step 7, so:

$$\begin{aligned} &\exists i \in \{\min\{j : (m, w_j) \in V\}, \max\{k : (m, w_k) \in V\}\} : \\ &\quad \underline{w} = w_i \Rightarrow \forall w \neq \underline{w} : (m, w) \in V, d((m, \underline{w}), (m, w)) = 0. \end{aligned} \tag{$*$}$$

Let $W_m = \{w : (m, w) \in V\}$, $p = \min\{j : w_j \in W_m\}$, and $r = \max\{k : w_k \in W_m\}$. Since $|W_m| \geq 2$, we have $p < r$. As before, we consider an exhaustive list of cases:

1. *If one of* $\{(m, w) : w \in W_m\}$ *is a split node:*

   (a) (m, w_p) and (m, w_r) cannot both be split nodes because there are no externally connected split nodes.

   (b) For any $w_q \in W_m$ with $p < q < r$, (m, w_q) cannot be a split node because there is no crossing.

   (c) If one of (m, w_p) and (w, w_r) is a split node, then the split node is $(m, \underline{w})$ and step 1 assigns orientation that is consistent with $(*)$. Since no split node is externally connected to any minimal cycle, steps 2 and 3 do not reassign orientations. Step 4 extends the orientation given by step 1, while steps 5 and 6 do not apply.

2. *If none of* $\{(m, w) : w \in W_m\}$ *is a split node and one of* $\{(m, w) : w \in W_m\}$ *is externally connected to a split node:* In this case, all of $\{(m, w) : w \in W_m\}$ are externally connected to a split node. Since there are no externally connected split nodes, they are externally connected to only one split node, say $(\bar{m}, \bar{w})$.

   We argue that $(m, \underline{w})$ is on all the minimal paths from the split node to each vertex in $\{(m, w) : w \in W_m\}$. To see that this is true, assume for contradiction that $(m, \underline{w})$ is *not* on the of the minimal path from the split node to a vertex in $\{(m, w) : w \in W_m\}$. Then, there must exist (m, w_i) and $(m, w_j)$with $w_i, w_j \in W_m$ such that there exist minimal paths that externally connect them to the split node and do not share any common vertex on row m. For this to happen, there must be a minimal cycle that is externally connected to the split node, which is a contradiction. (Moreover, $\underline{w}$ must be either w_p or w_r because otherwise $(m, \underline{w})$ would be a split node.)

   Again, step 1 assigns orientation that satisfies $(*)$. Since no split node is externally connected to any minimal cycle, steps 2 and 3 do not reassign orientation, step 4 extends the orientation given by step 1, and steps 5 and 6 do not apply.

3. *If none of* $\{(m, w) : w \in W_m\}$ *is a split node, none of them is externally connected to a split node, and one of them is connected to a minimal cycle that does not contain a split node:*

   (a) If one of $\{(m, w) : w \in W_m\}$ is a vertex in the minimal cycle, it can only be either w_p or w_r, because otherwise it would be a split node. Moreover, exactly w_p and w_r are vertices of the minimal cycle.

   (b) If none of $\{(m, w) : w \in W_m\}$ is a vertex in the minimal cycle, $(m, \underline{w})$ is on all the minimal paths from the split node to each vertex in $\{(m, w) : w \in W_m\}$, for otherwise there would be two connected minimal cycles. Again, $\underline{w}$ can only be either w_p or w_r, because otherwise it would be a split node.

   Under these premises, steps 2 and 3 assign orientations that satisfy $(*)$, while steps 1, 4, 5, and 6 do not apply.

4. *If none of* $\{(m, w) : w \in W_m\}$ *is a split node, none of them is externally connected to a split node, none of them is connected to a minimal cycle, but one of them is internally connected to a split node:* In this case, steps 1, 2, 3, 5, and 6 do not apply. Step 4 identifies a border vertex, and $(m, \underline{w})$ is on all the minimal paths from that border vertex to each vertex in $\{(m, w) : w \in W_m\}$, because otherwise there would be a cycle. This implies that, step 4 assigns an orientation that is consistent with $(*)$.

5. *If none of* $\{(m, w) : w \in W_m\}$ *is a split node, none of them is connected to a split node, and none of them is connected to a minimal cycle:* Then, steps 1, 2, 3, 4, and 6 do not apply. Step 5 identifies a vertex with the smallest lexicographic index, and $(m, \underline{w})$ is on all the minimal paths from this smallest lexicographic index to each vertex in $\{(m, w) : w \in W_m\}$, because otherwise there would be a cycle. Again, step 5 assigns an orientation that is consistent with $(*)$. □

Lemma 4. *Let P be the profile of complete orders constructed in Appendix A.2.1. For each agent a, the corresponding P_a is single-peaked.*

Proof. This is true by the last two steps of the construction. □

A.3 Stability

We complete the proof of Theorem 1 by proving that the matching X used in the construction of Appendix A.2.1 is rationalized by the preference profile constructed in that Appendix.

If the matching is unstable, there exist two vertices $(m_j, w_p), (m_i, w_q) \in V$, with $j \neq i$ and $p \neq q$, such that

$$d((m_j, w_q), (m_j, w_p)) = 1 \text{ and } d((m_j, w_q), (m_i, w_q)) = 1. \tag{1}$$

As before, the argument is by contradiction, considering an exhaustive (and exhausting) list of cases.

A.3.1 If $(m_j, w_q) \in V$:

1. If (m_j, w_q) is a split node, then exactly one of (m_j, w_p) and (m_i, w_q) must be a split end that is externally connected to the split node. By step 1, either $d((m_j, w_q), (m_i, w_q)) = 0$ or $d((m_j, w_q), (m_j, w_p)) = 0$, contradicting (1).

2. If (m_j, w_q) is not a split node, we argue that there exists (m, w) such that

   $$\hat{\pi} = \langle (m_j, w), (m_j, w_q), (m_i, w_q) \rangle \text{ or } \tilde{\pi} = \langle (m, w_q), (m_j, w_q), (m_j, w_p) \rangle$$

   is a subsequence of one of the minimal paths defined in steps 1 through 5.

   (a) If one of (m_j, w_p), (m_i, w_q) and (m_j, w_q) is externally connected to a split node $(\bar{m}, \bar{w})$, then all of them are externally connected to the split node (because (m_j, w_q) is not the split node).

   Let $\hat{\mathbb{p}} = \langle (\hat{m}, \hat{w})_n : n = 0, \ldots, \hat{N} \rangle$ be a minimal path that externally connects the split node and (m_j, w_q). Let $\bar{\mathbb{p}} = \langle (\bar{m}, \bar{w})_n : n = 0, \ldots, \bar{N} \rangle$ be a minimal path that externally connects the split node and (m_j, w_p). Let $\tilde{\mathbb{p}} = \langle (\tilde{m}, \tilde{w})_n : n = 0, \ldots, \tilde{N} \rangle$ be a minimal path that externally connects the split node and (m_i, w_q).

   By step 1, $\hat{\mathbb{p}}$, $\bar{\mathbb{p}}$, and $\tilde{\mathbb{p}}$ are forward flows, and:

   i. If $(\bar{m}, \bar{w})_{\bar{N}-1} = (m_j, w_q)$, since $(\bar{m}, \bar{w})_{\bar{N}} = (m_j, w_p)$, we have a contradiction of (1), as $d((m_j, w_q), (m_j, w_p)) = 0$.

   ii. If $(\tilde{m}, \tilde{w})_{\tilde{N}-1} = (m_j, w_q)$, since $(\tilde{m}, \tilde{w})_{\tilde{N}} = (m_j, w_p)$, we have again have a contradiction of (1), as $d((m_j, w_q), (m_i, w_q)) = 0$.

   iii. If $(\bar{m}, \bar{w})_{\bar{N}-1} \neq (m_j, w_q)$ and $(\tilde{m}, \tilde{w})_{\tilde{N}-1} \neq (m_j, w_q)$, then (m_j, w_q) is neither in $\bar{\mathbb{p}}$ nor in $\tilde{\mathbb{p}}$, because $\bar{\mathbb{p}}$ and $\tilde{\mathbb{p}}$ are minimal. Let I be the set of vertices that are in both $\bar{\mathbb{p}}$ and $\tilde{\mathbb{p}}$. $I \neq \varnothing$, as it contains the split node $(\bar{m}, \bar{w})$.
   Let

   $$\bar{n} = \max\{n : (\bar{m}, \bar{w})_n \in \bar{\mathbb{p}} \text{ and } (\bar{m}, \bar{w})_n \in \tilde{\mathbb{p}}\}$$

   and

   $$\tilde{n} = \max\{n : (\tilde{m}, \tilde{w})_n \in \tilde{\mathbb{p}} \text{ and } (\tilde{m}, \tilde{w})_n \in \bar{\mathbb{p}}\}.$$

   The "latest" intersection is $(m^*, w^*) = (\bar{m}, \bar{w})_{\bar{n}} = (\tilde{m}, \tilde{w})_{\tilde{n}}$, so there is a minimal cycle

   $$\langle (m_j, w_q), (m_j, w_p), ..., (m^*, w^*), ..., (m_i, w_q), (m_j, w_q) \rangle$$

   that is externally connected to the split node $(\bar{m}, \bar{w})$, which is impossible.

(b) If one of $(m_j, w_p), (m_i, w_q)$ and (m_j, w_q) is connected to a minimal cycle that does contain a split node, then one of them is externally connected to that split node, which is case (a) and leads to a contradiction.

(c) If one of $(m_j, w_p), (m_i, w_q)$ and (m_j, w_q) is connected to a minimal cycle $\mathfrak{c}$ that does not contain a split node, and none of them is externally connected to a split node:

i. If either $\langle (m_j, w_p), (m_j, w_q), (m_i, w_q) \rangle$ or $\langle (m_i, w_q), (m_j, w_q), (m_j, w_p) \rangle$ is a subsequence of $\mathfrak{c}$, then by (2), there is a contradiction:

$$d((m_j, w_q), (m_i, w_q)) = 0 \text{ or } d((m_j, w_q), (m_j, w_p)) = 0.$$

ii. Else, if either $\langle (m_j, w_p), (m_j, w_q) \rangle$, or $\langle (m_i, w_q), (m_j, w_q) \rangle$, or $\langle (m_j, w_q), (m_i, w_q) \rangle$ or $\langle (m_j, w_q), (m_j, w_p) \rangle$ is a subsequence of $\mathfrak{c}$, then $(m_j, w_q) \in \mathfrak{c}$ and it must be a split node. Contradiction.

iii. Else, if $(m_j, w_q) \in \mathfrak{c}$, it must be a split node. Contradiction.

iv. Else, if $(m_j, w_p) \in \mathfrak{c}$ or $(m_i, w_q) \in \mathfrak{c}$, then either $\langle (m_j, w_p), (m_j, w_q), (m_i, w_q) \rangle$ or $\langle (m_i, w_q), (m_j, w_q), (m_j, w_p) \rangle$ is a minimal path that connects $\mathfrak{c}$ to a vertex not in $\mathfrak{c}$. By (3), $\langle (m_j, w_q), (m_i, w_q) \rangle$ or $\langle (m_j, w_q), (m_j, w_p) \rangle$ is a forward flow. Therefore, $d((m_j, w_q), (m_i, w_q)) = 0$ or $d((m_j, w_q), (m_j, w_p)) = 0$. Contradiction.

v. Else, none of $(m_j, w_p), (m_i, w_q)$ and (m_j, w_q) is in $\mathfrak{c}$. Let $\mathbb{p}^{jq} = \langle (m, w)_n^{jq} : n = 0, \ldots, N \rangle$ be a minimal path that connects $(\bar{m}, \bar{w}) \in \mathfrak{c}$ and (m_j, w_q). Let $\mathbb{p}^{jp} = \langle (m, w)_s^{jp} : n = 0, \ldots, S \rangle$ be a minimal path that connects $(\bar{m}, \bar{w}) \in \mathfrak{c}$ and (m_j, w_p). Let $\mathbb{p}^{iq} = \langle (m, w)_t^{iq} : t = 0, \ldots, T \rangle$ be a minimal path that connects $(\bar{m}, \bar{w}) \in \mathfrak{c}$ and (m_i, w_q). By (3), $\langle (m, w)_r^{jq} : r = 1, \ldots, R \rangle$, $\langle (m, w)_s^{jp} : s = 1, \ldots, S \rangle$, and $\langle (m, w)_t^{iq} : t = 1, \ldots, T \rangle$ are forward flows.

v.i. If $(m, w)_{S-1}^{jp} = (m_j, w_q)$: since $(m_j, w_q) \notin \mathfrak{c}$, we have $S \geq 2$; since $(m, w)_S^{jp} = (m_j, w_p)$, we have $d((m_j, w_q), (m_j, w_p)) = 0$. Contradiction.

v.ii. If $(m, w)_{T-1}^{iq} = (m_j, w_q)$: since $(m_j, w_q) \notin \mathfrak{c}$, we have $T \geq 2$; since $(m, w)_T^{iq} = (m_j, w_p)$, we have $d((m_j, w_q), (m_i, w_q)) = 0$. Contradiction.

v.iii. If $(m, w)_{S-1}^{jp} \neq (m_j, w_q)$ and $(m, w)_{T-1}^{iq} \neq (m_j, w_q)$, then $(m_j, w_q) \notin \mathbb{p}^{jp}$ and $(m_j, w_q) \notin \mathbb{p}^{iq}$, because $\mathbb{p}^{jp}$ and $\mathbb{p}^{iq}$ are minimal. Let $I = \{(m, w) : (m, w) \in \mathbb{p}^{jp} \text{ and } (m, w) \in \mathbb{p}^{iq}\}$ be the set of vertices that are in both $\mathbb{p}^{jp}$ and $\mathbb{p}^{iq}$. This set is non-empty as it contains $(\bar{m}, \bar{w}) \in \mathfrak{c}$. Let the "latest" intersection be $(\tilde{m}, \tilde{w}) = (m, w)_g^{jp} = (m, w)_h^{iq}$ with

$$g = \max \left\{ s : (m, w)_s^{jp} \in \mathbb{p}^{jp} \text{ and } (m, w)_s^{jp} \in \mathbb{p}^{iq} \right\}$$

and

$$h = \max \left\{ t : (m, w)_t^{iq} \in \mathbb{p}^{iq} \text{ and } (m, w)_t^{iq} \in \mathbb{p}^{jp} \right\}.$$

Then,

$$\langle (m_j, w_q), (m_j, w_p), (m, w)_{S-1}^{jp}, ..., (\tilde{m}, \tilde{w}), ..., (m, w)_{T-1}^{iq}, (m_i, w_q), (m_j, w_q) \rangle$$

is a minimal cycle that is connected to $\mathfrak{c}$, which is distinct from $\mathfrak{c}$ as none of $(m_j, w_p), (m_i, w_q)$ and (m_j, w_q) is in $\mathfrak{c}$. Contradiction.

(d) If none of $(m_j, w_p), (m_i, w_q)$ and (m_j, w_q) is connected to a minimal cycle or externally connected to a split node, but one of them is internally connected to a split node, then all of them are internally connected to a split node. Let $(\bar{m}, \bar{w})$ be the vertex that (4) identifies as the border vertex. $(\bar{m}, \bar{w})$ exists because $(m_j, w_p), (m_i, w_q)$ and (m_j, w_q) are internally connected to a split node and not connected to a minimal cycle, i.e. they are candidates for $(\bar{m}, \bar{w})$.

i. If $(\bar{m}, \bar{w}) = (m_j, w_p)$ or $(\bar{m}, \bar{w}) = (m_i, w_q)$, by (4) there is a contradiction, as before, since $d((m_j, w_q), (m_i, w_q)) = 0$ or $d((m_j, w_q), (m_j, w_p)) = 0$.

ii. If $(\bar{m}, \bar{w}) = (m_j, w_q)$, then by (4), we have that $d((m_j, w_q), (m_i, w_q)) = 0$ and $d((m_j, w_q), (m_j, w_p)) = 0$. Contradiction.

iii. If none of $(m_j, w_p), (m_i, w_q)$ and (m_j, w_q) is $(\bar{m}, \bar{w})$. Let $\mathbb{p}^{jq} = \langle (m, w)_n^{jq} : n = 0, \ldots, N \rangle$ be a minimal path that connects $(\bar{m}, \bar{w}) \in \mathfrak{c}$ and (m_j, w_q). Let $\mathbb{p}^{jp} = \langle (m, w)_s^{jp} : n = 0, \ldots, S \rangle$ be a minimal path that connects $(\bar{m}, \bar{w}) \in \mathfrak{c}$ and (m_j, w_p). Let $\mathbb{p}^{iq} = \langle (m, w)_t^{iq} : t = 0, \ldots, T \rangle$ be a minimal path that connects $(\bar{m}, \bar{w}) \in \mathfrak{c}$ and (m_i, w_q). By (4), $\mathbb{p}^{jq}$, $\mathbb{p}^{jp}$, and $\mathbb{p}^{iq}$ are forward flow.

iii.i. If $(m, w)_{S-1}^{jp} = (m_j, w_q)$: since $(m_j, w_q) \notin \mathfrak{c}$, we have $S \geq 2$; since $(m, w)_S^{jp} = (m_j, w_p)$, we have $d((m_j, w_q), (m_j, w_p)) = 0$. Contradiction.

iii.ii. If $(m, w)_{T-1}^{iq} = (m_j, w_q)$: since $(m_j, w_q) \notin \mathfrak{c}$, $T \geq 2$; since $(m, w)_T^{iq} = (m_j, w_p)$, $d((m_j, w_q), (m_i, w_q)) = 0$. Contradiction.

iii.iii. If $(m, w)_{S-1}^{jp} \neq (m_j, w_q)$ and $(m, w)_{T-1}^{iq} \neq (m_j, w_q)$, then $(m_j, w_q) \notin \mathbb{p}^{jp}$ and $(m_j, w_q) \notin \mathbb{p}^{iq}$, because $\mathbb{p}^{jp}$ and $\mathbb{p}^{iq}$ are minimal. Let $I = \{(m, w) : (m, w) \in \mathbb{p}^{jp} \text{ and } (m, w) \in \mathbb{p}^{iq}\}$ be the set of vertices that are in both $\mathbb{p}^{jp}$ and $\mathbb{p}^{iq}$. This set is non-empty as it contains $(\bar{m}, \bar{w})$. Let the "latest" intersection be $(\tilde{m}, \tilde{w}) = (m, w)_g^{jp} = (m, w)_h^{iq}$ with

$$g = \max\{s : (m, w)_s^{jp} \in \mathbb{p}^{jp} \text{ and } (m, w)_s^{jp} \in \mathbb{p}^{iq}\}$$

and

$$h = \max\{t : (m, w)_t^{iq} \in \mathbb{p}^{iq} \text{ and } (m, w)_t^{iq} \in \mathbb{p}^{jp}\}.$$

Then,

$$\langle (m_j, w_q), (m_j, w_p), (m, w)_{S-1}^{jp}, ..., (\tilde{m}, \tilde{w}), ..., (m, w)_{T-1}^{iq}, (m_i, w_q), (m_j, w_q) \rangle.$$

is a minimal cycle. Contradiction.

(e) Finally, if none of $(m_j, w_p), (m_i, w_q)$ and (m_j, w_q) is connected to a minimal cycle or to a split node, then they are in a connected component that does not contain any minimal cycle or split node. Let $(\bar{m}, \bar{w})$ be the vertex that (5) identifies as the vertex with the smallest lexicographic index of (m, w). $(\bar{m}, \bar{w})$ exists because $(m_j, w_p), (m_i, w_q)$ and (m_j, w_q) are candidates.

i. If $(\bar{m}, \bar{w}) = (m_j, w_p)$ or $(\bar{m}, \bar{w}) = (m_i, w_q)$, by (4), $d((m_j, w_q), (m_i, w_q)) = 0$ or $d((m_j, w_q), (m_j, w_p)) = 0$. Contradiction.

ii. If $(\bar{m}, \bar{w}) = (m_j, w_q)$, then by (4), we have that $d((m_j, w_q), (m_i, w_q)) = 0$ and $d((m_j, w_q), (m_j, w_p)) = 0$. Contradiction.

iii. If none of $(m_j, w_p), (m_i, w_q)$ and (m_j, w_q) is $(\bar{m}, \bar{w})$. Let $\mathbb{p}^{jq} = \langle (m, w)_n^{jq} : n = 0, \ldots, N \rangle$ be a minimal path that connects $(\bar{m}, \bar{w}) \in \mathfrak{c}$ and (m_j, w_q). Let $\mathbb{p}^{jp} = \langle (m, w)_s^{jp} : n = 0, \ldots, S \rangle$ be a minimal path that connects $(\bar{m}, \bar{w}) \in \mathfrak{c}$ and (m_j, w_p). Let $\mathbb{p}^{iq} = \langle (m, w)_t^{iq} : t = 0, \ldots, T \rangle$ be a minimal path that connects $(\bar{m}, \bar{w}) \in \mathfrak{c}$ and (m_i, w_q). By (4), $\mathbb{p}^{jq}$, $\mathbb{p}^{jp}$, and $\mathbb{p}^{iq}$ are forward flow.

iii.i. If $(m, w)_{S-1}^{jp} = (m_j, w_q)$: since $(m_j, w_q) \notin \mathfrak{c}$, we have $S \geq 2$; since $(m, w)_S^{jp} = (m_j, w_p)$, we have $d((m_j, w_q), (m_j, w_p)) = 0$. Contradiction.

iii.ii. If $(m, w)_{T-1}^{iq} = (m_j, w_q)$: since $(m_j, w_q) \notin \mathfrak{c}$, $T \geq 2$; since $(m, w)_T^{iq} = (m_j, w_p)$, $d((m_j, w_q), (m_i, w_q)) = 0$. Contradiction.

iii.iii. If $(m,w)^{jp}_{S-1} \neq (m_j, w_q)$ and $(m,w)^{iq}_{T-1} \neq (m_j, w_q)$, then $(m_j, w_q) \notin \mathbb{p}^{jp}$ and $(m_j, w_q) \notin \mathbb{p}^{iq}$, because $\mathbb{p}^{jp}$ and $\mathbb{p}^{iq}$ are minimal. Let $I = \{(m,w) : (m,w) \in \mathbb{p}^{jp} \text{ and } (m,w) \in \mathbb{p}^{iq}\}$ be the set of vertices that are in both $\mathbb{p}^{jp}$ and $\mathbb{p}^{iq}$. This set is non-empty as it contains $(\bar{m}, \bar{w})$. Let the "latest" intersection be $(\tilde{m}, \tilde{w}) = (m,w)^{jp}_g = (m,w)^{iq}_h$ with

$$g = \max\{s : (m,w)^{jp}_s \in \mathbb{p}^{jp} \text{ and } (m,w)^{jp}_s \in \mathbb{p}^{iq}\}$$

and

$$h = \max\{t : (m,w)^{iq}_t \in \mathbb{p}^{iq} \text{ and } (m,w)^{iq}_t \in \mathbb{p}^{jp}\}.$$

Then

$$\langle (m_j, w_q), (m_j, w_p), (m,w)^{jp}_{S-1}, ..., (\tilde{m}, \tilde{w}), ..., (m,w)^{iq}_{T-1}, (m_i, w_q), (m_j, w_q)\rangle.$$

is a minimal cycle. Contradiction.

A.4 Necessity for multidimensional types

Proposition 3. *If a matching's associated graph contains crossing edges, it is not rationalizable by single-peaked preferences.*

Proof. Suppose there are crossing edges. Then, there exist Δ_n that men have single-peaked preference in and Γ_ℓ that women have single-peaked preference in. Moreover, there exist $\gamma_{-\ell}$, δ_{-n}, $p < q < r$, and $i < j < k$, such that

$$(m_{(j,-\ell)}, w_{(p,-n)}), (m_{(j,-\ell)}, w_{(r,-n)}), (m_{(i,-\ell)}, w_{(q,-n)}), (m_{(k,-\ell)}, w_{(q,-n)}) \in V.$$

By strict preferences of $m_{(j,-\ell)}$-type men, either $w_{(p,-n)} P_{m_{(j,-\ell)}} w_{(r,-n)}$ or $w_{(r,-n)} P_{m_{(j,-\ell)}} w_{(p,-n)}$, but not both. Since $m_{(j,-\ell)}$-type men have single-peaked preference in Δ_n, either $w_{(q,-n)} P_{m_{(j,-\ell)}} w_{(p,-n)}$ or $w_{(q,-n)} P_{m_{(j,-\ell)}} w_{(r,-n)}$. Similarly for $w_{(q,-n)}$: $m_{(i,-\ell)} P_{w_{(q,-n)}} m_{(k,-\ell)}$ or $m_{(k,-\ell)} P_{w_{(q,-n)}} m_{(i,-\ell)}$, but not both. Since $w_{(q,-n)}$ women have single-peaked preference in Γ_ℓ, $m_{(j,-\ell)} P_{w_{(q,-n)}} m_{(i,-\ell)}$ or $m_{(j,-\ell)} P_{w_{(q,-n)}} m_{(k,-\ell)}$. Since

$$x_{m_{(j,-\ell)}, w_{(p,-n)}} > 0,\ x_{m_{(j,-\ell)}, w_{(r,-n)}} > 0,\ x_{m_{(i,-\ell)}, w_{(q,-n)}} > 0, \text{ and } x_{m_{(k,-\ell)}, w_{(q,-n)}} > 0,$$

$(m_{(j,-\ell)}, w_{(q,-n)})$ is a blocking pair. □

Lemma 5. *Let (m,w) be a split node and $(\check{m}, \check{w})$ be one of its split ends. Then $d((m,w),(\check{m},\check{w})) = 0$ (equivalently, $d((\check{m},\check{w}),(m,w)) = 1$).*

Proof. Suppose $d((m,w),(\check{m},\check{w})) = 1$.

1. Suppose (m,w) is a split node with a horizontal split, so $\check{m} = m$. Then, there exists Γ_ℓ that women have single-peaked preference in. Furthermore, there exist $\gamma_{-\ell}$, $i < j < k$, and $w' \neq w$, such that $(m_{(i,-\ell)}, w), (m_{(k,-\ell)}, w), (m_{(j,-\ell)}, w') \in V$, where $m_{(j,-\ell)} = m$.

   Since $\langle (\check{m},\check{w}), (m_{(j,-\ell)}, w), (m_{(i,-\ell)}, w)\rangle$ and $\langle (\check{m},\check{w}), (m_{(j,-\ell)}, w), (m_{(k,-\ell)}, w)\rangle$ are minimal paths, by Echenique et al. (2013), Lemma 1, they are forward flows. This implies that

   $$d((m_{(j,-\ell)}, w), (m_{(i,-\ell)}, w)) = 0 \text{ and } d((m_{(j,-\ell)}, w), (m_{(k,-\ell)}, w)) = 0,$$

   contradicting that w-type women have single-peaked preference in Γ_ℓ.

2. Else, (m, w) is a split node with a vertical split, so $\check{w} = w$. In this case there exists Δ_n that men have single-peaked preference in. Furthermore, there exist δ_{-n}, $p < q < r$, and $m' \neq m$, such that $(m, w_{(p,-n)}), (m, w_{(r,-n)}), (m', w_{(q,-n)}) \in V$, where $w_{(q,-n)} = w$.

   Since $\langle(\check{m}, \check{w}), (m, w_{(q,-n)}), (m, w_{(p,-n)})\rangle$ and $\langle(\check{m}, \check{w}), (m, w_{(q,-n)}), (m, w_{(r,-n)})\rangle$ are minimal paths, by Echenique et al. (2013), Lemma 1, they are forward flows. Hence

$$d((m, w_{(q,-n)}), (m, w_{(p,-n)})) = 0 \text{ and } d((m, w_{(q,-n)}), (m, w_{(r,-n)})) = 0,$$

   contradicting that m men has single-peaked preference in Δ_n. □

Lemma 6. *Let (m, w) be a split node and $(\check{m}, \check{w})$ be one of its split ends. Let $\mathbb{p} = \langle(\hat{m}, \hat{w})_n : n = 0, \ldots, N\rangle$ be a minimal path connecting (m, w), $(\check{m}, \check{w})$, and $(\dot{m}, \dot{w}) \notin \{(m, w), (\check{m}, \check{w})\}$. Then $\mathbb{p}$ is a forward flow.*

Proof. Follows from Lemma 6 and Echenique et al. (2013) Lemma 1. □

Proposition 4. *If a matching's associated graph contains two externally connected distinct split nodes, it is not rationalizable by single-peaked preference.*

Proof. The argument is identical to the proof of Proposition 2, using Lemma 6 instead of Lemma 1. □

Proposition 5. *If in the associated graph of a matching, a minimal cycle is externally connected to a split node, the matching is not rationalizable by single-peaked preference.*

Proof. The argument is identical to the proof of Proposition 3, using Lemma 6 instead of Lemma 1. □

A.5 Sufficiency for multidimensional types

A.5.1 Construction of preferences

Suppose the associated graph of a matching X does not contain two connected, distinct minimal cycles, crossing edges, two externally connected distinct split nodes, or a minimal cycle externally connected to a split node.

The first four steps are the same as in the one-dimensional case. After that

5. For each connected component of more than one vertex that does not contain any minimal cycle or split node, find the vertex with the smallest lexicographic index of $(\gamma_1, \ldots, \gamma_L, \delta_1, \ldots, \delta_N)$. Let each minimal path that starts from this vertex be a forward flow.

6. For any type (say m) that is matched to only one type of the other side (say $w_{(\delta_{1,i}, \delta_{2,j}, \ldots, \delta_{N,k})}$), let P_m be such that

$$w_{(\delta_{1,i'}, \delta_{2,j'}, \ldots, \delta_{N,k'})} P_m w_{(\delta_{1,i''}, \delta_{2,j''}, \ldots, \delta_{N,k''})}$$

   if

$$|i' - i| - \tfrac{1}{2}\mathbb{1}_{i'<i}(i') < |i'' - i| - \tfrac{1}{2}\mathbb{1}_{i''<i}(i''),$$

   or $i' = i''$ and

$$|j' - j| - \tfrac{1}{2}\mathbb{1}_{j'<i}(j') < |j'' - j| - \tfrac{1}{2}\mathbb{1}_{j''<j}(j''),$$

   or ..., or if $i' = i'', j' = j'', \ldots$, and

$$|k' - k| - \tfrac{1}{2}\mathbb{1}_{k'<k}(k') < |k'' - k| - \tfrac{1}{2}\mathbb{1}_{k''<k}(k'').$$

   Similarly for any w.

7. Completion of the preferences:

   (a) For any type, say m, whose preference had been partially assigned, there exist a $\underline{w}$ such that all the assigned orientation must be $d((m,\underline{w}),(m,w)) = 0$ for all $w \neq \underline{w})\}$. Moreover, $\underline{w}$ has the smallest or the largest index in one of the dimension (let it be Δ_n) among $\{w : (m,w) \in V\} =: W_m$.

   (b) If $\underline{w}$ has the smallest index in Δ_n among W_m, let $(\delta_{1,i}, \delta_{2,j}, \ldots, \delta_{N,k})$ denote the index of the w with the largest index in Δ_n among W_m. Let P_m be such that

   $$w_{(\delta_{1,i'},\delta_{2,j'},\ldots,\delta_{N,k'})} P_m w_{(\delta_{1,i''},\delta_{2,j''},\ldots,\delta_{N,k''})}$$

   if

   $$|x' - x| - \tfrac{1}{2}\mathbb{1}_{x'<x}(x') < |x'' - x| - \tfrac{1}{2}\mathbb{1}_{x''<x}(x''),$$

   or if $x' = x''$ and

   $$|i' - i| - \tfrac{1}{2}\mathbb{1}_{i'<i}(i') < |i'' - i| - \tfrac{1}{2}\mathbb{1}_{i''<i}(i''),$$

   or if $x' = x''$, $i' = i''$ and

   $$|j' - j| - \tfrac{1}{2}\mathbb{1}_{j'<i}(j') < |j'' - j| - \tfrac{1}{2}\mathbb{1}_{j''<j}(j''),$$

   or ..., or if $x' = x''$, $i' = i''$, $j' = j''$, ..., and

   $$|k' - k| - \tfrac{1}{2}\mathbb{1}_{k'<k}(k') < |k'' - k| - \tfrac{1}{2}\mathbb{1}_{k''<k}(k'').$$

   (c) else if $\underline{w}$ has the largest index in Δ_n among W_m, let $(\delta_{1,i}, \delta_{2,j}, \ldots, \delta_{b,x}, \ldots, \delta_{S,k})$ denote the index of the w with the small index in Δ_n among W_m. Let P_m be as above.

Do similarly for each w.

A.5.2 This preference profile rationalizes X and is single peaked

With slight modifications on proofs for Lemma 3-5, we can show that if the associated graph of a matching X does not contain two connected, distinct minimal cycles, crossing edges, two externally connected distinct split nodes, or a minimal cycle externally connected to a split node, then the constructed preferences are well defined and single peaked. Moreover, it does not produce blocking pairs. Therefore, we have:

Proposition 6. *If the associated graph of a matching X does not contain two connected distinct minimal cycles, crossing edges, two externally connected distinct split nodes, or a minimal cycle externally connected to a split node, then X is rationalizable by single-peaked preference.*