# University of California, Davis
# ECN200, General Equilibrium Theory
# LN0: The History of General Equilibrium

The first person to talk about the concept of general equilibrium was Leon Walras (France, 1834-1910). Son of an economist, Walras was one of the most prominent marginalists, alongside W. Jevons (England, 1831-1882) and C. Menger (Austria, 1840-1921). On top of the methodological importance of his ideas, which started the process of mathematization of economics, the first contributions of Walras gave foundation to a great deal of modern economic thought. On one hand, it was Walras who first considered the systematic determination of supply and demand in multiple markets.[1] He also was the first scholar to explicitly derive demand and supply as solutions to optimization problems, and to define equilibrium by combining this determination of demand and supply with market clearing.

In spite of the normative nature of his research interests, Walras decided that the first questions that had to be answered were of a positive character. The first problem he tackled was existence of equilibrium. His answer was simplistic: he observed that the equilibrium concept displayed consistency in terms of the number of equations required for market clearing and the number of variables available to obtain it. Similarly, he introduced the idea of a fictitious auctioneer who adjusts prices according to excess demand/supply until equilibrium is attained. Unfortunately, he oversimplified things and took it for granted that the mentioned coincidence of free variables and equations guaranteed existence of equilibrium and thought it obvious that the auctioneer would guarantee its stability. Under the impression that these positives issues were dealt with, Walras moved on to normative questions such as how aggregate wealth should be distributed and increased.

Walras, by then a Professor at Lausanne, failed in his attempts to popularize his ideas in the profession and, in fact, only his definitions and his flawed positive results are nowadays part of economic thought. Finding himself in poor health,[2] he decided to find the appropriate person to take over his position at Lausanne and continue his research agenda. An Italian friend of his recommended a young engineer and economist, with strong math background: Wilfredo Pareto (Italian noble, born during the exile of his father in France, 1884-1923). Despite great personal and ideological differences,[3] Walras left Pareto the position at Lausanne and his intellectual project... or so he thought.

Pareto's contributions were numerous, but so were also the differences with Walras's analysis.[4] The first difference was Pareto's departure from utilitarianism, which had been implicit in Walras's work and which at the time was, in fact, mainstream economics. Pareto

[1] Having learned about F. Quesnay's *Tableau*, which is the basis of modern input-output analysis.

[2] Mental, some say.

[3] Walras being a bourgeois, shy, social idealist; Pareto not only a noble, but in fact a rather arrogant, pragmatic, *laissez-faire* liberal.

[4] Pareto himself said that, had he wanted, he could have written his theory in total independence from Walras's work, which is true.

thought that one can actually do away with the concept of utility function, using instead the more general concept of preference relation.

Besides, Pareto completely departed from the equilibrium concept introduced by Walras. For him, equilibrium in a social system occurred at any situation in which the tension between what individuals want and what is socially possible is maximal: with the available resources, improving someone's condition would make someone else worse off—the exact opposite of utilitarianism, as interpersonal comparisons are ruled out. Additionally, Pareto opened the debate about the implementation of desired outcomes through policies, when he proposed the idea that, being Walrasian equilibrium just the solution to a system of equations, a government could simply solve this system and impose the allocation without the need for functioning markets—an idea that ignores a crucial informational problem.[5]

In spite of his math background, Pareto's work was far from formal. It is nowadays clear, though, that his work was possible thanks in part to the contemporaneous contribution of Francis Ysidro Edgeworth (Irish/English/half-Catalonian, 1845-1926). After his parents and six siblings died,[6] Edgeworth had inherited very handsomely and could dedicate his life to pure academic work in spite of great difficulties getting a position in any prestigious institution. A self-educated mathematician and lawyer, his first works in economics were under the marginalist-utilitarian tradition,[7] and led him to the definition of social indifference curve. On top of this enormous methodological contribution, Edgeworth had other influential ideas: he studied the set of exchange outcomes to which no individual, or group of individuals could independently oppose by isolating themselves from the exchange. He conjectured that as the number of agents increased such set would reduce to the set of equilibrium outcomes, as defined by Walras.

Edgeworth dropped his work in economics for a while and started to work in probability theory, where he made important contributions, until he was offered an economics professorship at Oxford and the editorship of the prestigious Economic Journal. From then on, he continued to contribute to Walrasian equilibrium theory, obtaining in particular some paradoxical results (that were then dismissed, but which we now know are correct) and working on imperfect competition models.

Edgeworth's work was not massively read on his days.[8] Besides Pareto, a notable exception was Irving Fisher (USA, 1867-1947). An economist at Yale, Fisher was important not only because of his formalization of Walras's ideas, pretty much to how we ex-

[5] Pareto retired from Lausanne, somewhat depressed given Italy's political situation. The Fascist party tried in many ways to recruit him, but he never accepted. After his death, the party used his name rather liberally, though.

[6] WTF!

[7] He really believed in utilitarianism: he at some point said that, since men are more capable of experiencing pleasure than women, it is optimal for the society that men consume more than women. No wonder he died single!

[8] Being very elegant from a mathematical point of view, many of the economists of the time found his papers unreadable. On top of that, it didn't help that he used to shy away from controversy, even when the attacks to his ideas were incorrect.

press them today, but also because, independently of Edgeworth, he defined the individual indifference curve, and, indirectly, gave a correct existence argument: he designed and built a hydraulic machine that correctly computed equilibrium prices for exchange economies.[9]

To a large extent, between 1910 and 1940 the important contributions to equilibrium theory stopped. This situation changed when, by coincidence, Kenneth Arrow (USA, 1921-2017) and Gerard Debreu (France, 1921-2004) arrived at the Cowles Commission in Chicago in 1946.

Arrow was a mathematical statistician, working his Ph.D. dissertation at Columbia under H. Hotelling (USA, 1895-1973). Right since the dissertation, Arrow began his career with seminal contributions.[10] First, he substantially weakened the theoretical foundations of utilitarianism, when he showed that under certain axioms it is impossible to aggregate individual preferences into a social welfare function. Secondly, while in Cowles, Arrow showed that the differences between Walras's and Pareto's ideas were not as deep as previously thought: first, equilibria in the sense of Walras were also equilibria in the sense of Pareto, and, second, equilibria in the sense of Pareto can be implemented via equilibria in the sense of Walras, upon redistribution of individual resources.

These two last results, the crux of the Paretian agenda, are currently known as the fundamental theorems of welfare economics. It was a coincidence that, independently but at the same time, and also in Cowles, Debreu was successfully working on the same problems.

A mathematician by formation, Debreu also arrived in Cowles upon suggestion of his Ph.D. advisor, M. Allais (France, 1911-2010). It was obvious for him that Walras's existence argument was flawed and worked on a general, mathematically correct alternative argument. T. Koopmans (Holland, 1910-1985), then director of Cowles, read early drafts of what both Arrow and Debreu were writing and realized that, despite differences in approach and formalization, they were obtaining essentially the same results. Koopmans convinced them of the benefits of joining forces, and in doing so formed the team that obtained what some people consider the most important result ever in economic theory: using novel mathematical tools, they showed weak conditions under which competitive equilibrium is guaranteed to exist.[11] It is now commonly thought that a good understand-

[9] Fisher also designed a tent for the treatment of tuberculosis, a device that computed the caloric content of diets, and the Rolodex—the patent of which he sold to Remington for $660.000. Unfortunately for him, just before the big crash of the NYSE on 1929, he, publicly, convinced some investors that shares were not overpriced and had, instead, found a correct, higher equilibrium price (to quote him: "Stocks have reached what looks like a permanently high plateau").

[10] And not only in economics: during WWII, Arrow studied the way in which wind currents can be used to improve aircraft speed and efficiency during flight; these early contributions are still influential in modern day air traffic planning and management.

[11] To be fair, at about the same time Lionel McKenzie (USA, 1919-2010) was working on the existence problem at Rochester, and published a valid result shortly before Arrow and Debreu published theirs, but based on less primitive principles: he started from the demand functions and not from the preferences. This caused some controversy, in particular because there are reasons to think that Debreu knew of McKenzie's work and didn't acknowledge it, not even to Arrow. In order to recognize McKenzie's contribution, some

ing of the consistency of a model is obtained only after the determination conditions that are sufficient for existence of its proposed solutions.

Their agenda did not stop there. Arrow studied the issue of uniqueness and showed that the conditions for it are extremely restrictive. Debreu, on the other hand, showed that the equilibrium need not be unique or stable, although in almost all economies it is locally unique and, in fact, there are a finite number of equilibria. Besides, in joint work with H. Scarf (USA, 1930-2015), Debreu showed that Edgeworth was right in his conjecture that by increasing the number of agents, the set of allocations to which no individual or group of individuals object shrinks to only the Walrasian equilibrium allocations. Also, independent work by them, established the canonical model for the study or risk and dynamics in economics and finance.

A very prolific author in many fields, Arrow won the Nobel price in 1972 while Debreu got it eleven years later. For a while, they were coaches of the football teams of graduate students at Stanford and U.C. Berkeley, respectively. Sadly, Debreu passed away on New Year's eve, 2004, in Paris; and Arrow on February 21st, 2017, in Palo Alto, CA.

economists refer to this part of the literature as the Arrow-Debreu-McKenzie argument.

Also, not everyone considers the existence result to be a great achievement, in particular because it set the standard of mathematical formality that is followed today. According to some critic, the work itself is "the beginning of what has since become a cancerous growth in the very center of microeconomics."

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ECN200, General Equilibrium Theory

LN1: Definitions

## 1. The Economy

There is a finite number, $L \in \mathbb{N}$, of commodities, each of which can be consumed in any non-negative amount. Each commodity is denoted by $\ell = 1, \ldots, L$, used as a sub-index.

A consumer is a pair $(u : \mathbb{R}^L_+ \to \mathbb{R}, w \in \mathbb{R}^L_+)$, where $u$ represents the individual's preferences and $w$ is her endowments. The standard properties of preferences may be imposed.[1] Additionally, $u$ is said to be *smooth* if it is $\mathbf{C}^2(\mathbb{R}^L_{++})$, differentiably strictly monotone and differentiably strictly quasiconcave, and

$$\{\tilde{x} \in \mathbb{R}^L_+ \mid u(\tilde{x}) \geqslant u(x)\} \subseteq \mathbb{R}^L_{++},$$

for all $x \in \mathbb{R}^L_{++}$.[2] The reason why smoothness is useful is that it ensures that the demand functions are interior, if $w \in \mathbb{R}^L_{++}$, and differentiable.

A firm is a non-empty set $Y \subseteq \mathbb{R}^L$, which represents its technology. The standard properties of preferences may be imposed as well.

For finite numbers $I, J \in \mathbb{N}$, define the sets $\mathcal{I} = \{1, \ldots, I\}$ and $\mathcal{J} = \{1, \ldots, J\}$. Set $\mathcal{I}$ is the society, and we refer to its members by the $i = 1, \ldots, I$. Similarly, set $\mathcal{J}$ is the industry, and we use the super-index $j = 1, \ldots, J$ to denote the firm $Y^j$. If there is no production, super-index $i$ which denotes the individual $(u^i, w^i)$; if there is production individual $i$ is augmented to $[u^i, w^i, (s^{i,j})_{j \in \mathcal{J}}]$, where $s^{i,j} \geqslant 0$ represents the share of consumer $i$ in the stock of firm $j$, so $\sum_i s^{i,j} = 1$ for all $j$.

Definition 1. *An* exchange economy *is* $\{\mathcal{I}, (u^i, w^i)_{i \in \mathcal{I}}\}$. *A* standard exchange economy *is one in which each* $u^i$ *is continuous, locally-nonsatiated and quasiconcave. A* smooth exchange economy *is one in which each* $u^i$ *is smooth and each* $w^i \in \mathbb{R}^L_{++}$.

Definition 2. *A* production economy *is*

$$\{\mathcal{I}, \mathcal{J}, [u^i, w^i, (s^{i,j})_{j \in \mathcal{J}}]_{i \in \mathcal{I}}, (Y^j)_{j \in \mathcal{J}}\}.$$

*A* standard production economy *is one in which each* $u^i$ *is continuous, locally-nonsatiated and quasiconcave, and each* $Y^j$ *is closed and convex, and satisfies free-disposal and possibility of inaction.*

It is important to notice that these definitions do impose a key institution by assumption: we are describing economies of private property.

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[1] We assume that individuals have representable preferences only for convenience. For this reason, we may interchangeably say that the individual has convex preferences and that $u$ is quasiconcave.

[2] This last property is referred to as "interiority." An alternatively assumption is that for every sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}^L_{++}$, if it converges to some $x$ in $\partial\mathbb{R}^L_+$, then it is true that $\|Du(x_n)\|^{-1} Du(x_n) \cdot x_n \to 0$ and $\|Du(x_n)\| \to 0$.

In what follows, we use $x^i$ and $y^j$ to denote, respectively, individual $i$'s consumption bundle and firm $j$'s production net-put. A profile of consumption plans is denoted by $\vec{x} = (x^i)_{i \in \mathcal{I}}$ and one of production plans by $\vec{y} = (y^j)_{j \in \mathcal{J}}$.

## 2. Competitive Equilibrium

If we add a second institution, competitive markets, we obtain the definitions of competitive equilibrium. Let $p = (p_\ell)_{\ell=1}^L \in \mathbb{R}^L$ denote prices. In an exchange economy with competitive markets, the only constraint that individual $i$ faces is that her consumption cannot lie outside the set

$$\{x \in \mathbb{R}^L_+ \mid p \cdot x \leqslant p \cdot w^i\}.$$

If it is a production economy, and production levels are $(y^j)_{j \in \mathcal{J}}$, then individual $i$ faces the budget set

$$\left\{x \in \mathbb{R}^L_+ \mid p \cdot x \leqslant p \cdot w^i + \textstyle\sum_j s^{i,j} p \cdot y^j\right\}.$$

Definition 3. *On an exchange economy, a* competitive equilibrium *is* $(p, \vec{x})$ *such that:*

1. *for all* $i \in \mathcal{I}$, $x^i$ *solves*
$$\max_x \left\{u^i(x) : x \in \mathbb{R}^L_+ \text{ and } p \cdot x \leqslant p \cdot w^i\right\};$$
*and*
2. $\sum_i x^i = \sum_i w^i$.

When $I = L = 2$, a graphical representation of the economy, its equilibria and other concepts is obtained via Edgeworth boxes.

Definition 4. *In a production economy, a* competitive equilibrium *is* $(p, \vec{x}, \vec{y})$ *such that*

1. *for all* $i \in \mathcal{I}$, $x^i$ *solves*
$$\max_x \left\{u^i(x) : x \in \mathbb{R}^L_+ \text{ and } p \cdot x \leqslant p \cdot w^i + \textstyle\sum_j s^{i,j} p \cdot y^j\right\};$$
2. *for all* $j \in \mathcal{J}$, $y^j$ *solves* $\max_y \left\{p \cdot y : y \in Y^j\right\}$*; and*
3. $\sum_i x^i = \sum_i w^i + \sum_j y^j$.

Again, notice assumptions implicit in the definition: (i) it is assumed, as an institution, that there exists a complete set of markets to which all agents have unrestricted access; (ii) it is assumed, as a rule of behavior, that all agents are price takes; (iii) there are no external effects; and (iv) all commodities are privately consumed. Many results depend crucially on these assumptions.

## 3. Pareto Efficiency

In an exchange economy, an allocation is a profile $\vec{x}$ such that $\sum_i x^i = \sum_i w^i$; in a production economy, an allocation is $(\vec{x}, \vec{y})$ such that $y^j \in Y^j$ for each $j$, and $\sum_i x^i = \sum_i w^i + \sum_j y^j$.

Competitive equilibrium is the canonical non-cooperative (some people say 'individualistic') viewpoint. The simplest form of cooperative solution is the concept of Pareto efficiency:

DEFINITION 5. *Given an exchange economy, an allocation* $\vec{x}$ *is* Pareto efficient *if there does not exist another allocation* $(\hat{x}^i)_{i \in \mathcal{I}}$ *such that:*

1. *for all* $i \in \mathcal{I}$, $u^i(\hat{x}^i) \geqslant u^i(x^i)$*; and*
2. *for some* $i \in \mathcal{I}$, $u^i(\hat{x}^i) > u^i(x^i)$.

DEFINITION 6. *Given a production economy, an allocation* $(\vec{x}, \vec{y})$ *is* Pareto efficient *if there does not exist another allocation* $[(\hat{x}^i)_{i \in \mathcal{I}}, (\hat{y}^j)_{j \in \mathcal{J}}]$ *such that*

1. *for all* $i \in \mathcal{I}$, $u^i(\hat{x}^i) \geqslant u^i(x^i)$*; and*
2. *for some* $i \in \mathcal{I}$, $u^i(\hat{x}^i) > u^i(x^i)$.

It is important to notice that: (i) Pareto efficiency does away with the institutions of competitive markets (and hence prices) and private property; (ii) it does not replace the latter institutions by an alternative mechanism; and (iii) in production economies, only the welfare of consumers, and *not* the profit of the firms, matters.

## 4. THE CORE

If one maintains the institution of private property and some of the self-interest of individuals, one can refine the definition of Pareto efficiency to a "cooperative" solution for exchange economies:

DEFINITION 7. *An allocation* $x$ is in the core *of an exchange economy if there do not exist* $\mathcal{H} \subseteq \mathcal{I}$ *and* $(\hat{x}^i)_{i \in \mathcal{H}}$ *such that:*

1. $\sum_{i \in \mathcal{H}} \hat{x}^i = \sum_{i \in \mathcal{H}} w^i$;
2. *for all* $i \in \mathcal{H}$, $u^i(\hat{x}^i) \geqslant u^i(x^i)$*; and*
3. *for some* $i \in \mathcal{H}$, $u^i(\hat{x}^i) > u^i(x^i)$.

Defining the core of a production economy is possible, but requires further institutional assumptions.

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# LN2: Walras's Law

Recall that:

1. if utility function $u$ is locally nonsatiated and $x^*$ solves
$$\max_x \{u(x) \mid x \geqslant 0 \text{ and } p \cdot x \leqslant p \cdot w\}, \tag{1}$$
then $p \cdot x^* = p \cdot w$; and
2. if $u$ is strongly monotone and the consumer problem has a solution, then $p \in \mathbb{R}^L_{++}$.

Also, notice that if $p \in \mathbb{R}^L_{++}$, then the domain of Program (1) is the same if the prices are multiplied by a positive constant.

These observations imply the following result.
THEOREM 1 (WALRAS'S LAW). *In an exchange economy where $u^1$ is strongly monotone and every other $u^i$ is locally non-satiated. Suppose that $(p, x)$ is such that*

1. *for all $i$, $x^i \in \operatorname{argmax}_{B(p,w^i)} u^i(x)$; and*
2. *for all $\ell \leqslant L - 1$, $\sum_i x^i_\ell = \sum_i w^i_\ell$.*

*Then,* $(p, x)$,
$$\left(\frac{1}{p_1} p, x\right),$$
$$\left(\frac{1}{||p||} p, x\right),$$
*and*
$$\left(\frac{1}{\sum_\ell p_\ell} p, x)\right)$$
*are all competitive equilibria.*

In practice, the result says that when looking for the equilibria of an economy with monotone consumers, it suffices to guarantee that all of the markets but one clear. In technical terms, this says that the $L \times L$ system of market clearing conditions is under-determined, and is in fact an $L \times (L - 1)$ system: there is nominal price indeterminacy. So, one can drop one variable, say by letting $p_1 = 1$, and solving an $(L - 1) \times (L - 1)$ system!

The latter is a "numéraire" normalization (with the first commodity as numéraire) and confines prices to set
$$\{p \in \mathbb{R}^L_+ \mid p_1 = 1\}.$$
Alternative price normalizations are on the sphere,
$$\mathcal{S} = \{p \in \mathbb{R}^L_+ \mid ||p|| = 1\},$$

or on the $(L-1)$-dimensional simplex,

$$\Delta = \{p \in \mathbb{R}^L_+ \mid \sum_\ell p_\ell = 1\}.$$

Notice that all these sets are smaller than $\mathbb{R}^L_+$ and, in particular, that normalization on the sphere and the simplex compactifies the space of prices.

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ECN200, General Equilibrium Theory
LN3: Existence of competitive equilibrium

The following is one of the most important theorems in the history of economics: we will prove that in exchange economies where $\sum_i w^i \gg 0$ and that each $u^i$ is continuous, strictly quasi-concave and strictly monotone, there *always* exists a competitive equilibrium. Before a formal proof, some informal intuition is given.

In what follows, we denote by $\Delta$ the $(L-1)$-dimensional unit simplex, and let $\Delta^\circ = \Delta \cap \mathbb{R}^L_{++}$ and $\Delta^\partial = \Delta \setminus \Delta^\circ$.[1] The aggregate excess demand function over strictly positive prices, $Z : \Delta^\circ \to \mathbb{R}^L$, is defined by $Z(p) = \sum_i [x^i(p) - w^i]$, where

$$x^i(p) = \operatorname{argmax}_x \left\{ u^i(x) : x \geqslant 0 \text{ and } p \cdot x \leqslant p \cdot w^i \right\}.$$

## 1. An Informal Argument

Walras had in mind a price adjustment process relatively obvious: if a market exhibits excess demand (supply), its relative price should go up (resp. down). We now define a motion process reflecting Walras's idea, just as a didactic tool for the existence argument: after observing $Z(p)$, the auctioneer calls new prices in an attempt to clear markets— he makes excessively demanded goods cheaper, and excessively supplied goods more expensive.

The simplest rule,[2]

$$p \mapsto p + Z(p),$$

may violate non-negativity of prices, whereas[3]

$$p \mapsto (\max\{p_\ell + Z_\ell(p), 0\})_{\ell=1}^L$$

need not map into the simplex. So, the auctioneer[4] implements the following rule:

$$p \mapsto \gamma(p) = \left( \frac{\max\{p_\ell + Z_\ell(p), 0\}}{\sum_{\ell'} \max\{p_{\ell'} + Z_{\ell'}(p), 0\}} \right)_{\ell=1}^L,$$

which will always leave him in the simplex.[5]

In a standard economy, $\gamma$ is continuous and hence, by Brower's fixed-point theorem, there exists $p \in \Delta$ such that $\gamma(p) = p$. Ignore, for simplicity, the denominator on the definition

[1] The standard notation for the boundary would be $\partial\Delta$, but I like $\Delta^\partial$ more.

[2] "Just increase each price by its excess demand!"

[3] "OK, do the obvious thing, but don't go below zero."

[4] After thinking for a little while.

[5] Obviously, this requires $\sum_\ell \max\{p_\ell + z_\ell(p), 0\} > 0$, the proof of which is left as an exercise.

of $\gamma$, which is no problem by the homogeneity of demands. What we now have is that for each $\ell$, $p_\ell = \max\{p_\ell + Z_\ell(p), 0\}$. If $p_\ell > 0$, then $p_\ell = p_\ell + Z_\ell(p)$ and, hence, $Z_\ell(p) = 0$. Alternatively, if $p_\ell = 0$, then $Z_\ell(p) \leqslant 0$, which would be impossible under strictly monotone preferences. It follows that $[p, (x^i(p)]_{i \in \mathcal{I}}]$ is a competitive equilibrium of the given economy!

Of course, this argument is informal and has obvious problems, in particular that it requires that demands be defined even for boundary prices. A formal theorem and proof are given next.

## 2. The Formal Argument

Theorem 2 (Arrow and Debreu). *In an exchange economy, suppose that $\sum_i w^i \gg 0$ and that each $u^i$ is continuous, strictly quasi-concave and strictly monotone. Then, there exists a competitive equilibrium.*

Let us take for granted that, under the assumptions of the theorem, function $Z$ is continuous and bounded below; satisfies that $p \cdot Z(p) = 0$ for all $p \in \Delta^\circ$; and is such that $\max_\ell \{Z_\ell(p_n)\} \to \infty$ for any sequence $(p_n)_{n=1}^\infty$ in $\Delta^\circ$ such that $p_n \to p \in \Delta^\partial$.

Assuming these properties, the proof of the theorem is as follows: Define correspondence $\Gamma : \Delta \twoheadrightarrow \Delta$ by:

$$\Gamma(p) = \begin{cases} \operatorname{argmax}_\gamma \{\gamma \cdot Z(p) : \gamma \in \Delta\}, & \text{if } p \in \Delta^\circ; \\ \{\gamma \in \Delta \mid p \cdot \gamma = 0\}, & \text{if } p \in \Delta^\partial. \end{cases}$$

We first argue the following five claims:

1. *Correspondence $\Gamma$ is nonempty-, compact- and convex-valued.*

   That $\Gamma(p) \neq \varnothing$ when $p \in \Delta^\circ$ follows from Weierstrass's theorem, since $\Delta$ is compact. Now, suppose that $p \in \Delta^\partial$. By definition, there is an $l = 1, \dots, L$, for which $p_\ell = 0$. If we let $\gamma_\ell = 1$ and $\gamma_{\ell'} = 0$ for all $\ell' \neq \ell$, it is immediate that $\gamma \in \Gamma(p)$.

   That $\Gamma$ is bounded-valued is immediate, since $\Delta$ is bounded. That it is closed-valued follows by construction, since $\Delta$ is closed and limits preserve weak inequalities.

   That $\Gamma$ is convex-valued follows from convexity of $\Delta$ and linearity of the inner product.

2. *If $p \in \Delta^\circ$ and $Z(p) \neq (0, \dots, 0)$ then $\Gamma(p) \subseteq \Delta^\partial$.*

   Since $p \in \Delta^\circ$, we have that $p \cdot Z(p) = 0$. Since $Z(p) \neq 0$, there must then be an $\ell$ for which $Z_\ell(p) < 0$. If $\gamma \in \Gamma(p)$, it then follows by construction that $\gamma_\ell = 0$.

3. *If $p \in \Delta^\partial$, then $p \notin \Gamma(p)$.*

   If $p \in \Delta^\partial \subseteq \Delta$, we have that $p \neq 0$, and, hence, $p \cdot p \neq 0$.

4. $\Gamma$ *is upper hemi-continuous at all* $p \in \Delta^\circ$.

   This is immediate from the Theorem of the Maximum, since the objective function of the problem that defines $\Gamma$ on $\Delta^\circ$ is continuous.

5. $\Gamma$ *is upper hemi-continuous at all* $p \in \Delta^\partial$.

   This case is more complicated. Fix $p \in \Delta^\partial$, $(p_n)_{n=1}^\infty$ in $\Delta$ such that $p_n \to p$, and $(\gamma_n)_{n=1}^\infty$ in $\Delta$ such that $\gamma_n \in \Gamma(p_n)$ for each $n$. Since $\Delta$ is compact, by the Theorem of Bolzano-Weierstrass there exists a subsequence, $(\gamma_{n_m})_{m=1}^\infty$, and a $\gamma \in \Delta$ such that $\gamma_{n_m} \to \gamma$.

   Suppose first that $(p_{n_m})_{m=1}^\infty$ has no subsequences in $\Delta^\circ$. Since $p_{n_m} \to p$, for some $m^* \in \mathbb{N}$ we have that for all $m \geqslant m^*$, $p_{n_m} \in \Delta^\partial$ and $p_{n_m} \cdot \gamma_{n_m} = 0$, so $p \cdot \gamma = 0$.

   Alternatively, it must be true that $(p_{n_m})_{m=1}^\infty$ has a subsequence in $\Delta^\circ$, $(p_{n_{m_k}})_{k=1}^\infty$. We now argue that there exists $k^* \in \mathbb{N}$ such that, for all $k \geqslant k^*$ and all $\ell$ such that $p_\ell > 0$,

   $$Z_\ell(p_{n_{m_k}}) < \max_{\ell'} \left\{ Z_{\ell'}(p_{n_{m_k}}) \right\}.$$

   To see that this is the case, take the subsequence $(p_{n_{m_k}})_{k=1}^\infty$ in $\Delta^\circ$. Since $p_{n_{m_k}} \to p \in \Delta^\partial$, by the property of $Z$ assumed above we know that

   $$\max_{\ell} \{ Z_\ell(p_{n_{m_k}}) \} \to \infty$$

   as $k \to \infty$. For any $\ell$ such that $p_\ell > 0$, sequence $(Z_\ell(p_{n_{m_k}}))_{k=1}^\infty$ is bounded above. Thus, there exists $k^* \in \mathbb{N}$ such that

   $$Z_\ell(p_{n_{m_k}}) < \max_{\ell'} \{ Z_{\ell'}(p_{n_{m_k}}) \}$$

   for all $k \geqslant k^*$ and all $\ell$ such that $p_\ell > 0$. It follows that for all $k \geqslant k^*$ and all $\ell$ such that $p_\ell > 0$, one has that $\gamma_{\ell, n_{m_k}} = 0$ and, hence, $p_{n_{m_k}} \cdot \gamma_{n_{m_k}} = 0$. Again, this implies that $p \cdot \gamma = 0$.

   In both cases, we conclude then that $\gamma \in \Gamma(p)$.

The last two claims imply that $\Gamma$ is upper hemi-continuous. By Claim 1 and Kakutani's fixed point theorem, there exists some $p \in \Delta$ such that $p \in \Gamma(p)$. By Claim 3, $p \in \Delta^\circ$, and, hence, by Claim 2, $Z(p) = 0$.

It follows that $[p, (x^i(p))_{i \in \mathcal{I}}]$ is a competitive equilibrium.

## Appendix

To complete the proof of the theorem, we need, of course, to argue the properties of function $Z$ that were taken for granted.

First, strict quasiconcavity of all utility functions guarantees that $Z$ is a function and not a correspondence. It is continuous, as it is the sum of functions that are continuous by the Theorem of the Maximum. It is bounded below by construction, since demands are non-negative: for all $p \in \Delta^\circ$, $Z(p) \geqslant \sum_i w^i$. And that $p \cdot Z(p) = 0$ follows from the fact that each $u^i$ is strctly monotone. Now, the only missing step is to prove that $\max_\ell \{Z_\ell(p_n)\} \to \infty$ for all sequences $(p_n)_{n=1}^\infty$ in $\Delta^\circ$ such that $p_n \to p \in \Delta^\partial$.

To see that this is the case, fix one such sequence $(p_n)_{n=1}^\infty$ and suppose that it is not true that $\max_\ell \{Z_\ell(p_n)\} \to \infty$. Then, for some $x \in \mathbb{R}$ it is true that for all $n^*$, there exists $n \geqslant n^*$ such that $\max_\ell \{Z_\ell(p_n)\} \leqslant x$. Since $Z$ is bounded below, there exists $(p_{n_m})_{m=1}^\infty$ such that $(Z(p_{n_m}))_{m=1}^\infty$ is bounded. Since $\sum_i w^i \gg 0$, then for some $i$ we must have $p \cdot w^i > 0$. Fix one such $i$. Since $(Z(p_{n_m}))_{m=1}^\infty$ is bounded, then $(x^i(p_{n_m}))_{m=1}^\infty$ is bounded and, hence, has a convergent subsequence.

For notational simplicity, assume that $(x^i(p_{n_m}))_{m=1}^\infty$ itself converges to $x \in \mathbb{R}^L_+$. Let $\tilde{\ell}$ be such that $p_{\tilde{\ell}} = 0$, and let $\tilde{x} \in \mathbb{R}^L_+$ be defined as follows:

$$\tilde{x}_\ell = \begin{cases} x_\ell, & \text{if } \ell \neq \tilde{\ell}; \\ x_{\tilde{\ell}} + 1 & \text{if } \ell = \tilde{\ell}. \end{cases}$$

Since $\tilde{x} > x$, $u^i(\tilde{x}) > u^i(x)$. By continuity, there is some $\epsilon > 0$ such that, for all $x' \in B_\epsilon(\tilde{x}) \cap \mathbb{R}^L_+$ and all $x'' \in B_\epsilon(x)$, $u^i(x') > u^i(x'')$. Since $x^i(p_{n_m}) \to x$, there exists some $m_1 \in \mathbb{N}$ such that for all $m \geqslant m_1$, $x^i(p_{n_m}) \in B_\epsilon(x)$. Fix $\ell'$ such that $p_{\ell'} > 0$. Define $(x_{n_m})_{m=1}^\infty$ as follows:

$$x_{\ell,n_m} = \begin{cases} x^i_{\tilde{\ell}}(p_{n_m}) + 1, & \text{if } \ell = \tilde{\ell}; \\ x^i_{\ell'}(p_{n_m}) - \frac{\epsilon}{2}, & \text{if } \ell = \ell'; \\ x^i_\ell(p_{n_m}), & \text{otherwise.} \end{cases}$$

Since $p_{\ell',n_m} \to p_{\ell'} > 0$ and $p_{\tilde{\ell},n_m} \to p_{\tilde{\ell}} = 0$, there exists $m_2 \in \mathbb{N}$ such that for all $m \geqslant m_2$,

$$-\frac{\epsilon}{2} \cdot p_{\ell',n_m} + p_{\tilde{\ell},n_m} < 0.$$

Now, let $m \geqslant \max\{m_1, m_2\}$. Then,

$$\begin{aligned} p_{n_m} \cdot x_{n_m} &= p_{n_m} \cdot x^i(p_{n_m}) + p_{\tilde{\ell}',n_m} - \frac{\epsilon}{2} \cdot p_{\ell',n_m} \\ &< p_{n_m} \cdot x^i(p_{n_m}) \\ &\leqslant p_{n_m} \cdot w^i, \end{aligned}$$

and, nonetheless, $x^i(p_{n_m}) \in B_\epsilon(x)$ and $x_{n_m} \in B_\epsilon(\tilde{x})$, so

$$u^i(x_{n_m}) > u^i(x^i(p_{n_m})),$$

which is a contradiction.

# University of California, Davis
# ECN200, General Equilibrium Theory
# LN4: The SMD Theorem: multiplicity and instability

For the purposes of this lecture, let prices be normalized to the sphere, defined as

$$\mathcal{S} = \{p \in \mathbb{R}^L_{++} \mid ||p|| = 1\}.$$

For any $\varepsilon > 0$, we shall say that exchange economy $\{\mathcal{I}, (u^i, w^i)_{i \in \mathcal{I}}\}$ *generates* $Z : \mathcal{S} \to \mathbb{R}^L$ in

$$\mathcal{S}_\varepsilon = \{p \in \mathcal{S} \mid \forall \ell, p_\ell \geqslant \varepsilon\},$$

if, for all $p \in \mathcal{S}_\varepsilon$, $\sum_i [x^i(p) - w^i] = Z(p)$.

Theorem 2 (The Sonnenschein-Mantel-Debreu, SMD, Theorem). *Let $Z : \mathcal{S} \to \mathbb{R}^L$ be continuous and satisfy Walras's law.*[1] *For every $\varepsilon > 0$, there exists a standard exchange economy that generates $Z$ in $\mathcal{S}_\varepsilon$.*

It must be emphasized that, in the particular economy constructed for the theorem, $\mathcal{I} = \{1, \dots, L\}$, and that the implication of the theorem fails if the condition that $I < L$ is imposed.

In what follows, we prove a less ambitious result, for the case when $L = 2$:

Theorem 3 (Very weak SMD). *Let $Z : \mathcal{S} \to \mathbb{R}^2$ be continuous and satisfy Walras's law. There exist two "individual excess demand" functions $z^1, z^2 : \mathcal{S} \to \mathbb{R}^2$ such that:*

1. *each $z^i$ satisfies Walras's law: for all $p$, $p \cdot z^i(p) = 0$.*
2. *each $z^i$ satisfies WARP in $\mathcal{S}_\epsilon$: for every $\bar{p}, \hat{p} \in \mathcal{S}_\epsilon$,*

$$\left( \begin{array}{c} \hat{p} \cdot z^i(\bar{p}) \leqslant 0 \\ z^i(\hat{p}) \neq z^i(\bar{p}) \end{array} \right) \Rightarrow \bar{p} \cdot z^i(\hat{p}) > 0;$$

3. *$z^1(p) + z^2(p) = Z(p)$ for all $p \in \mathcal{S}$.*

*Proof.* Since $z$ is continuous and $\mathcal{S}_\varepsilon$ is compact, there exists $\alpha \in \mathbb{R}$ such that

$$Z(p) + \alpha p \geqslant (1, 1) \tag{$*$}$$

for all $p \in \mathcal{S}_\varepsilon$.

Define $z^1, z^2 : \mathcal{S} \to \mathbb{R}^L$ by

$$z^1(p) = [Z_1(p) + \alpha p_1][(1, 0) - p_1 p],$$

[1] That is, that for all $p \in \mathcal{S}$, $p \cdot Z(p) = 0$.

and

$$z^2(p) = [Z_2(p) + \alpha p_2][(0,1) - p_2 p].$$

By direct computation, letting $e^1 = (1,0)$ and $e^2 = (0,1)$,

$$\begin{aligned} p \cdot z^i(p) &= p \cdot \{[Z_i(p) + \alpha p_i](e^i - p_i p)\} \\ &= [Z_i(p) + \alpha p_i] p \cdot (e^i - p_i p) \\ &= [Z_i(p) + \alpha p_i](p_i - p_i p \cdot p) \\ &= [Z_i(p) + \alpha p_i](p_i - p_i \|p\|^2) \\ &= 0, \end{aligned}$$

which is to say that $z^i$ satisfies Walras's law.

On the other hand, for $p, p' \in \mathcal{S}_\varepsilon$, Eq. $(*)$ implies that

$$\begin{aligned} p \cdot z^i(p') \leqslant 0 &\Rightarrow p \cdot \{[Z_i(p') + \alpha p'_i](e^i - p'_i p')\} \leqslant 0 \\ &\Rightarrow [Z_i(p') + \alpha p'_i][p_i - p'_i(p' \cdot p)] \leqslant 0 \\ &\Rightarrow p_i - p'_i(p' \cdot p) \leqslant 0 \\ &\Rightarrow \frac{p_i}{p'_i} \leqslant p' \cdot p = \|p'\|\|p\| \cos(\vartheta) = \cos(\vartheta), \end{aligned}$$

where $\vartheta$ is the angle between $p$ and $p$. If $p = p'$, then $\vartheta = 0$ and $\cos(\vartheta) = 1$; otherwise, $\cos(\vartheta) < 1$.

This, in turn, implies WARP: fix $\bar{p}, \tilde{p} \in \mathcal{S}_\epsilon$ such, $\hat{p} \cdot z^i(\bar{p}) \leqslant 0$ and $\bar{p} \cdot z^i(\hat{p}) \leqslant 0$. Then,

$$\hat{p}_i \leqslant \bar{p}_i \leqslant \hat{p}_i.$$

If $z^i(\bar{p}) \neq z^i(\hat{p})$, it follows that $\hat{p} \neq \bar{p}$ and we further have that $\hat{p}_i < \bar{p}_i$, which implies that $\hat{p}_i < \hat{p}_i$, an obvious contradiction.

Finally, notice that

$$\begin{aligned} z^1(p) + z^2(p) &= \textstyle\sum_i [Z_i(p) + \alpha(p) p_i](e^i - p_i p) \\ &= \textstyle\sum_i \left[Z_i(p) e^i - Z_i(p) p_i p + \alpha(p) p_i e^i - \alpha(p) p_i^2 p\right] \\ &= Z(p) - p \textstyle\sum_i Z_i(p) p_i + \alpha(p) p - \alpha(p) p \|p\|^2 \\ &= Z(p), \end{aligned}$$

follows from the fact that $z$ satisfies Walras's law and $p \in \mathcal{S}$. $\square$

University of California, Davis

ECN200, General Equilibrium Theory

LN5: Local Uniqueness of Equilibrium

The SMD theorem seems rather worrying from an applied point of view. We now study how severe the problem really is, by studying the following questions: Under what conditions can one ensure that an economy has only finitely many equilibria? And that a small shock to the economy will affect equilibria by just a little? Of course, we must also study how restrictive those conditions are.

The answers to these questions are well understood, although the mathematical techniques required by these results are a little more complicated than the ones we have used so far. For this reason, before a formal discussion we introduce an intuitive argument.

Notice that in all our previous analysis individual endowments have remain fixed (to the point that we have ignored them, for example, as arguments in the demand functions). The key of the following analysis is that bad results that appear at a given economy may simply disappear with a small perturbation of individual endowments.

Graphically, consider an economy with the excess demand function of Fig. 1. In this case, the predictive power of general equilibrium theory is very limited: there is a whole continuum of equilibria and the set of equilibria need not change smoothly when one perturbs the economy. Now, at the risk of oversimplifying the problem, just assume that the endowments do not generate $Z$ but, by just a small perturbation, the aggregate excess demand function is $\hat{Z}$, as in Fig. 2. Then, both problems simply disappear! And notice what a tremendous coincidence is necessary for $Z$, and not something else like $\hat{Z}$, to be the aggregate excess demand function of the economy one is studying.

Given the oversimplification, the rest of this section presents the formal analysis. For this, fix $\mathcal{I}$ and individual, smooth preferences $(u^i)_{i\in\mathcal{I}}$. Define individual excess demands to take into account their dependence on individual endowments: $z^i : \mathbb{R}^L_{++} \times \mathbb{R}^L_{++} \to \mathbb{R}^L$, $z^i(p, w^i) = x^i(p, w^i) - w^i$, so that the aggregate excess demand function writes as

$$Z : \mathbb{R}^L_{++} \times (\mathbb{R}^L_{++})^I \to \mathbb{R}^L; Z(p, \vec{w}) = \sum_i z^i(p, w^i).$$

As before, thanks to the homogeneity of demand, we can normalize prices with a numéraire, so that $p \in \mathcal{N}_1 = \{p \in \mathbb{R}^L_{++} \mid p_1 = 1\}$, while, by Walras law, we can ignore market clearing for the numéraire, by considering simply $\tilde{Z} : \mathcal{N}_1 \times (\mathbb{R}^L_{++})^I \to \mathbb{R}^{L-1}$, defined by

$$\tilde{Z}(p, \vec{w}) = (Z_2(p, \vec{w}), \ldots, Z_L(p, \vec{w})).$$

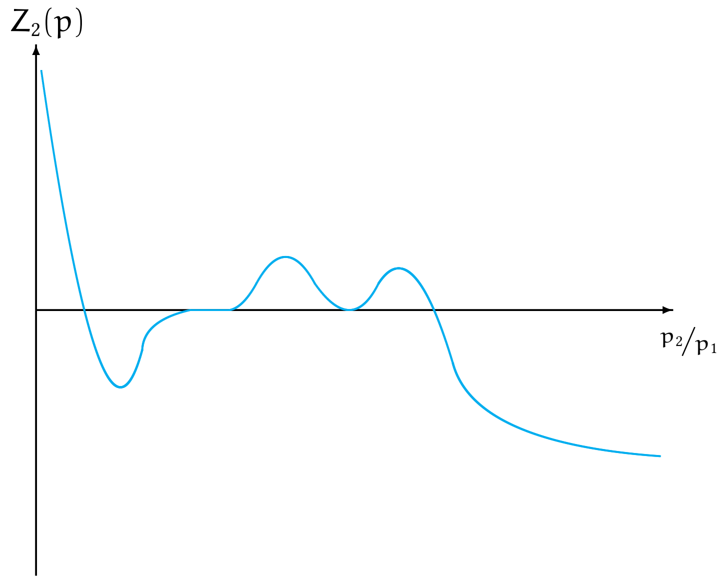


Figure 1: A "critical" excess demand function.

## 1. Finiteness of Equilibria

Given an exchange economy, we say that $p \in \mathcal{N}_1$ is a *locally unique equilibrium price vector* if:

1. $\tilde{Z}(p, \vec{w}) = 0$, and
2. there exists $\varepsilon > 0$ such that $\tilde{Z}(p', \vec{w}) \neq 0$ for every $p' \in [B_\varepsilon(p) \cap \mathcal{N}_1] \setminus \{p\}$.

We also say that an exchange economy is *regular* if $\tilde{Z}(p, \vec{w}) = 0$ implies that matrix

$$D_p \tilde{Z}(p, \vec{w}) = \left[ \begin{array}{ccc} D_{p_2} \tilde{Z}(p, \vec{w}) & \dots & D_{p_L} \tilde{Z}(p, \vec{w}) \end{array} \right]_{(L-1)\times(L-1)}$$

has rank $L - 1$.[1]

A key mathematical result is the following:

Theorem (Inverse Function Theorem). *Let* $f : \mathbb{D} \to \mathbb{R}^\eta$*, where* $\mathbb{D} \subseteq \mathbb{R}^\eta$ *is open, be of class* $\mathbb{C}^1$*. Let* $x \in \mathbb{D}$ *be such that* $\text{Rank}[Df(x)] = \eta$*. Then,*

1. *there exist* $U, V \subseteq \mathbb{R}^\eta$*, open, such that* $x \in U$*,* $f(x) \in V$ *and the restriction of* $f$ *to* $U$ *is a bijection onto* $V$*;*

[1] Notice that the matrix of the previous definition is square, so the requirement is that it be nonsingular.

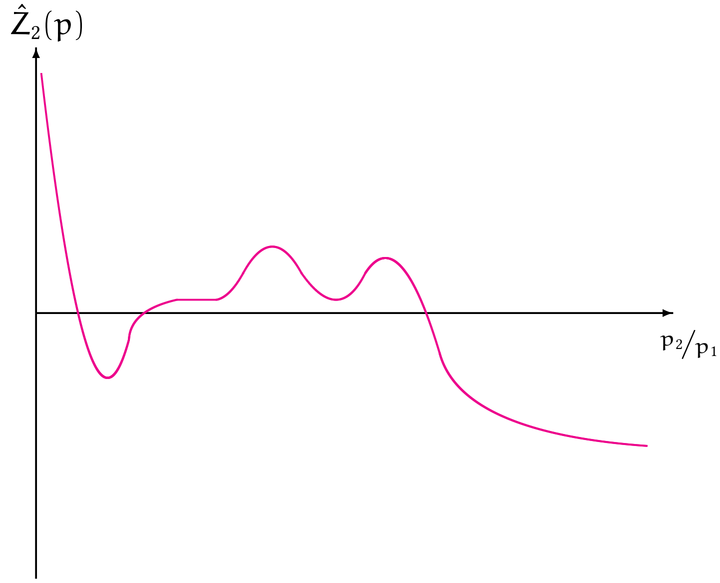


Figure 2: A small vertical shift of the excess demand function of Fig. 1.

2. *the inverse function of that restriction,* $g : V \to U$*, is* $\mathbb{C}^1$ *and for all* $y \in V$*,*

$$Dg(y) = Df(g(y))^{-1}.$$

The following is now simple:

THEOREM 1. *If a smooth exchange economy is regular, then it has finitely many equilibria (and they all are locally unique).*

*Proof.* Obviously, finiteness implies local uniqueness, although, in the proof, the causality is argued in the other way.

Define $\zeta : \mathbb{R}^{L-1}_{++} \to \mathbb{R}^L$, by $\zeta(p) = \tilde{Z}((1,p),\vec{w})$. Let $(1,p) \in \mathcal{N}_1$ be such that $\tilde{Z}((1,p),\vec{w}) = 0$. Then, $\zeta(p) = 0$ and, by regularity, $\text{Rank}[D\zeta(p)] = L - 1$. By the Inverse Function Theorem, $\zeta$ is injective in a neighborhood of $p$ in $\mathbb{R}^{L-1}$, which implies that $(1,p) \in \mathcal{N}_1$ is locally unique.

Now, suppose that $(p_n)_{n=1}^{\infty}$ in $\mathcal{N}_1$ is a sequence of distinct equilibrium prices. Then, for each $n$

$$\frac{1}{\|p_n\|} p_n \in \mathcal{S} \text{ and } Z\left(\frac{1}{\|p_n\|} p_n, \vec{w}\right) = 0.$$

By compactness of $\mathcal{S}$, there exists a subsequence $(p_{n_m})_{m=1}^{\infty}$ such that

$$\frac{p_{n_m}}{\|p_{n_m}\|} \to \bar{p} \in \{x \in \mathbb{R}^L_+ \mid \|x\| = 1\}.$$

Now, $\bar{p} \in \mathbb{R}^L_{++}$ and, by continuity of $Z$, $Z(\bar{p}, \vec{w}) = 0$.[2] Then,

$$\zeta(\frac{1}{\bar{p}_1}\bar{p}) = 0 \ \text{ and } \ \frac{1}{\bar{p}_1}\bar{p} \in \mathcal{N}_1,$$

which is impossible because $p_{n_m} \to \frac{1}{\bar{p}_1}\bar{p}$ and $\frac{1}{\bar{p}_1}\bar{p}$ is locally unique. □

## 2. Smoothness

We now show that in regular economies equilibrium prices depend smoothly on endowments. For this, the following is useful:

Theorem (Implicit Function Theorem). *Let $f : \mathbb{D} \to \mathbb{R}^{\eta}$, where $\mathbb{D} \subseteq \mathbb{R}^{\eta+\mu}$ is open, be of class $\mathbb{C}^1$. Let $x \in \mathbb{R}^{\eta}$ and $y \in \mathbb{R}^{\mu}$ be such that $(x, y) \in \mathbb{D}$ and* $\text{Rank}[D_x f(x, y)] = \eta$. *Then, there exist $U \subseteq \mathbb{R}^{\mu}$, open and such that $y \in U$, and $g : U \to \mathbb{R}^{\eta}$, of class $\mathbb{C}^1$, such that:*

1. $g(y) = x$;
2. *for all* $y' \in U$, $(g(y'), y') \in D$;
3. *for all* $y' \in U$, $f(g(y'), y') = f(x, y)$; *and*
4. *for all* $y' \in U$,
$$Dg(y') = [D_x f(g(y'), y')]^{-1} D_y f(g(y'), y').$$

Now, keeping individuals and their (smooth) preferences fixed, define the class of economies with those preferences, parameterized by individual endowment profiles in $(\mathbb{R}^L_{++})^I$.

Theorem 2. *Let $\vec{w} \in (\mathbb{R}^L_{++})^I$ give a regular economy, and let $p \in \mathcal{N}_1$ be an equilibrium price vector for that economy. There exists $W$, open and such that $\vec{w} \in W$, and there exists $\varphi : W \to \mathcal{N}_1$, of class $\mathbb{C}^1$, such that, for every $(\hat{w}^1, \ldots, \hat{w}^I) \in W$, $\tilde{Z}(\varphi(\hat{w}^1, \ldots, \hat{w}^I), (\hat{w}^1, \ldots, \hat{w}^I)) = 0$.*

*Proof.* By definition, $\tilde{Z}(p, \vec{w}) = 0$ and $D_p \tilde{Z}(p, \vec{w})$ has full rank, so the result follows immediately from the Implicit Function Theorem. □

Function $\varphi$ is a smooth, local equilibrium function.

[2] If $p \in \partial\mathbb{R}^L_{++}$, then $\|\zeta(\bar{p}_{n_m})\| \to \infty$, which is impossible because $\zeta(\bar{p}_{n_m}) = 0$ for every $m \in \mathbb{N}$.

## 3. Genericity

It is clear that regularity delivers powerful results but it remains to understand how restrictive an assumption it is. This is where the math gets a bit more complicated.

Let $\mathbb{D} \subseteq \mathbb{R}^\eta$ be open, and suppose that $f : \mathbb{D} \to \mathbb{R}^\mu$ is of class $\mathbb{C}^1$. Function $f$ is said to be *transverse to zero*, denoted by $f \pitchfork 0$, if $f(x) = 0$ implies that $\text{Rank}[Df(x)] = \mu$.

Theorem (The Transversality Theorem). *Let $\mathbb{D}_1 \subseteq \mathbb{R}^\eta$ and $\mathbb{D}_2 \subseteq \mathbb{R}^\kappa$ be open, and let $f : \mathbb{D}_1 \times \mathbb{D}_2 \to \mathbb{R}^\mu$ be of class $\mathbb{C}^r$, with $r > \max\{\eta - \mu, 0\}$. For every $y \in \mathbb{D}_2$, define $f_y : \mathbb{D}_1 \to \mathbb{R}^\mu$ by $f_y(x) = f(x, y)$. If $f \pitchfork 0$, then the set $\{y \in \mathbb{D}_2 \mid \neg(f_y \pitchfork 0)\}$ is negligible.*[3]

The following result shows that almost all economies are regular (and hence have finitely many equilibria which, at least locally, depend smoothly on parameters).

Theorem 3 (Debreu). *The subset of profiles of endowments that generate critical economies is closed and negligible.*

*Proof.* Let $W$ be such set. By the Transversality Theorem, to show that $W$ is negligible it suffices that $\tilde{Z} \pitchfork 0$. Suppose that $\tilde{Z}(p, \vec{w}) = 0$. For $\text{Rank}[D\tilde{Z}(p, \vec{w})] = L - 1$, it suffices to argue that $\text{Rank}[D_{w^1}\tilde{Z}(p, \vec{w})] = L - 1$. To see that this is the case, fix $\ell^* \in \mathcal{L}$ and consider the following perturbation to $w^1$:

$$dw^1_\ell = \begin{cases} -p_{\ell^*}, & \text{if } \ell = 1; \\ 1, & \text{if } \ell = \ell^*; \\ 0, & \text{otherwise.} \end{cases}$$

It is left as an exercise to observe that $dz_{\ell^*} = 1$ and $dz_\ell = 0$ for every $\ell \geqslant 2$, $\ell \neq \ell^*$. It follows, then, that $D_{w^1}\tilde{Z}(p, \vec{w})$ spans $e^{\ell^*}$. Since $\ell^*$ was arbitrary, $D\tilde{Z}(p, \vec{w})$ spans $\mathbb{R}^{L-1}$.

For closedness, let $(\vec{w}_n)_{n=1}^\infty$ in $W$ be such that $\vec{w}_n \to \vec{w}$. Since $\vec{w}_n \in W$ , there exists $p_n \in \mathcal{N}_1$ such that $\tilde{Z}(p_n, \vec{w}_n) = 0$ but the rank of $D_p\tilde{Z}(p_n, \vec{w}_n)$ is less that $L - 1$. Let $(\frac{1}{||p_{n_m}||}p_{n_m})_{m=1}^\infty$ be a convergent subsequence of $(\frac{1}{||p_n||}p_n)_{n=1}^\infty$, and let $\bar{p} \in \mathbb{R}^L_+$, with $||\bar{p}|| = 1$, be its limit. Since $(w_n)_{n=1}^\infty$ is convergent, it is bounded and hence, as before, $\bar{p} \in \mathbb{R}^L_{++}$.[4] By continuity, $\tilde{Z}(\frac{1}{\bar{p}_1}\bar{p}, \vec{w}) = 0$ and $\text{Rank}[D_p\tilde{Z}(\frac{1}{\bar{p}_1}\bar{p}, \vec{w})] < L - 1$, so $\bar{w} \in W$. □

[3] I.e. it has null Lebesgue measure.

[4] If $\bar{p} \in \partial\mathbb{R}^L_{++}$, then for $m$ high enough $\tilde{Z}(p_{n_m}, \vec{w}_{n_m}) \neq 0$.

University of California, Davis

ECN200, General Equilibrium Theory

LN6: The fundamental theorems of welfare economics

# 1 The first theorem

Pareto efficiency is a minimal criterion for social optimality. The first key normative result in general equilibrium theory is that, under mild assumptions, equilibrium allocations display this minimal property.

Theorem 1 (The FFTWE for production economies). *Given a standard production economy, let $(p, \vec{x}, \vec{y})$ be a competitive equilibrium. Allocation $(\vec{x}, \vec{y})$ is Pareto efficient.*

*Proof.* Suppose not: there exists an alternative profile of consumption and production plans $((\hat{x}^i)_{i\in\mathcal{I}}, (\hat{y}^j)_{j\in\mathcal{J}})$ such that

1. for all $j$, $\hat{y}^j \in Y^j$;
2. $\sum_i \hat{x}^i = \sum_i w^i + \sum_j \hat{y}^j$;
3. for all $i$, $u^i(\hat{x}^i) \geqslant u^i(x^i)$; and
4. for some $i^*$, $u^{i^*}(\hat{x}^{i^*}) > u^{i^*}(x^{i^*})$.

By 1, $p \cdot y^j \geqslant p \cdot \hat{y}^j$ for all $j$, whereas, by 3, $p \cdot \hat{x}^{i^*} > p \cdot x^{i^*}$. Suppose that for some $i$, $p \cdot \hat{x}^i < p \cdot x^i$; then, by local nonsatiation, there exists $\tilde{x} \in \mathbb{R}^L_+$ such that

$$p \cdot \tilde{x} \leqslant p \cdot x^i \leqslant p \cdot w^i + \textstyle\sum_j s^{i,j} p \cdot y^j$$

and $u^i(\tilde{x}) > u^i(x^i)$, which is impossible. So, it follows that $p \cdot \hat{x}^i \geqslant p \cdot x^i$ for all $i$.

By 2, then,

$$\begin{aligned} p \cdot \textstyle\sum_i w^i &= p \cdot \left(\textstyle\sum_i x^i - \sum_j y^j\right) \\ &= \textstyle\sum_i p \cdot x^i - \sum_j p \cdot y^j \\ &< \textstyle\sum_i p \cdot \hat{x}^i - \sum_j p \cdot \hat{y}^j \\ &= p \cdot \left(\textstyle\sum_i \hat{x}^i - \sum_j \hat{y}^j\right) \\ &= p \cdot \textstyle\sum_i w^i, \end{aligned}$$

an obvious impossibility. □

With a suitable definition of the core, we could have argued, actually, that the competitive equilibrium allocation is not simply efficient, but lies in the core of the economy. This is indeed the case for exchange economies:

THEOREM 2 (THE FFTWE FOR EXCHANGE ECONOMIES). *Given a standard exchange economy, let* $(p, \vec{x})$ *be a competitive equilibrium. Allocation* $\vec{x}$ *is a core allocation.*

COROLLARY 1. *Given a standard exchange economy, let* $(p, \vec{x})$ *be a competitive equilibrium. Allocation* $\vec{x}$ *is Pareto efficient.*

Notice that the theorem (i) does require local nonsatiation; (ii) does not use continuity or convexity, and assumes existence rather than implying it; (iii) crucially requires the implicit assumption of competitive equilibrium as we have defined it: markets are complete and all agents, firms and producers, are price takers.

On the other hand, it is necessary to understand the implication of the theorem. If one accepts the assumptions of the theorem, it implies that competitive markets deliver allocations with the minimal social property, as Smith had suggested. But is does not say more than that! It is clear that Pareto efficiency does not take into account any distributional considerations and hence many efficient allocations may be socially objectionable. In that sense, the theorem should not be understood to imply that economic policy is unnecessary if competitive markets operate. What the theorem does say is that any economic policy beyond the equilibrium outcome will make at least one individual worse off; although this result may be socially desirable, what cannot be expected is "victimless" policies.

# 2 The second theorem

The FFTWE tells us that competitive equilibrium allocations are Pareto efficient. We now study the opposite problem: given an efficient allocation, can we ensure that it is an equilibrium allocation?

So stated, the answer to the question is obviously negative: there are efficient allocations that cannot be sustained as competitive equilibrium. However, if wealth redistribution policies are allowed, *almost all* efficient allocations can be sustained by competitive trading.

THEOREM 3 (THE SFTWE FOR EXCHANGE ECONOMIES). *Given an exchange economy, let allocation* $\vec{x} \gg 0$ *be Pareto efficient. If every* $u^i$ *is continuous, strictly quasiconcave and strictly monotone, then there exists* $(p, (\hat{w}^i)_{i \in \mathcal{I}})$ *such that* $\sum_i \hat{w}^i = \sum_i w^i$ *and* $(p, \vec{x})$ *is competitive equilibrium for economy* $(\mathcal{I}, (u^i, \hat{w}^i)_{i \in \mathcal{I}})$.

*Proof.* Let $\hat{w}^i = x^i$ for all $i$. Since $\sum_i \hat{w}^i = \sum_i \hat{x}^i = \sum_i w^i \in \mathbb{R}^L_{++}$, it follows that there exists a competitive equilibrium $(p, \tilde{x})$ of $(\mathcal{I}, (u^i, \hat{w}^i)_{i\in\mathcal{I}})$. Since, $u^i(\tilde{x}^i) \geqslant u^i(x^i)$ for all $i$, and since $\vec{x}$ is efficient and $\sum_i \tilde{x}^i = \sum_i \hat{w}^i = \sum_i x^i = \sum_i w^i$, it follows that $u^i(\tilde{x}^i) = u^i(x^i)$ for all $i$. This means that $x^i$ solves

$$\max_x \left\{u^i(x) : x \in B(P, \hat{w}^i)\right\},$$

so $(p, \vec{x})$ is a competitive equilibrium for $(\mathcal{I}, (u^i, \hat{w}^i)_{i\in\mathcal{I}})$. □

In the case of production economies, the proof is more complicated but it is also very important as it uses a key result in mathematical economics: the separating hyperplane theorem:

Theorem 4 (The Separating Hyperplane Theorem). *If sets $X, Y \subseteq \mathbb{R}^A$ are disjoint and convex, then there exist $\pi \in \mathbb{R}^A \setminus \{0\}$ and $k \in \mathbb{R}$ such that: for all $x \in X$, $\pi \cdot x \geqslant k$; while for all $y \in Y$, $\pi \cdot y \leqslant k$.*

Theorem 5 (The SFTWE for Production Economies). *Given a standard production economy, let $(\vec{x}, \vec{y})$ be a Pareto efficient allocation, with $x^i \gg 0$ for all $i$. There exists*

$$(p, (m^i)_{i\in\mathcal{I}}) \in \mathbb{R}^L_+ \times (\mathbb{R}_{++})^I$$

*such that:*

1. $\sum_i m^i = p \cdot (\sum_i w^i + \sum_j y^j)$;
2. *for all $i$, $x^i$ solves the problem*

$$\max_{x\in\mathbb{R}^L_+} \left\{u^i(x) : p \cdot x \leqslant m^i\right\};$$

3. *for all $j$, $y^j$ solves the problem*

$$\max_y \left\{p \cdot y : y \in Y^j\right\}.$$

*Proof.* For each $i$, let $\mathcal{U}^i = \{x \mid u^i(x) > u^i(x^i)\}$ and define the set $\mathcal{U} = \sum_i \mathcal{U}^i$.[1] Define also the set $\mathcal{F} = \{\sum_i w^i\} + \sum_j Y^j$.

We now proceed in a series of steps:

[1] Note that we are adding sets! This operation is defined as follows: for two sets $A$ and $B$, we define

$$A + B = \{x \mid x = a + b, \text{ for some } a \in A \text{ and some } b \in B\}.$$

1. By quasiconcavity of each function $u^i$, each set $\mathcal{U}^i$ is convex, and so it follows that $\mathcal{U}$ is convex.

2. By convexity of each technology $Y^j$, we also have that set $\mathcal{F}$ is convex.

3. Since $(\vec{x}, \vec{y})$ is efficient, it follows that $\mathcal{U}$ and $\mathcal{F}$ are disjoint.

4. By the separating hyperplane theorem, there is a vector $p \in \mathbb{R}^L$, $p \neq 0$, and some constant $k$ such that $p \cdot x \geqslant k$ for all $x \in \mathcal{U}$, and $p \cdot y \leqslant k$ for all $y \in \mathcal{F}$.

5. By free-disposal of each technology $Y^j$, we have that $p > 0$.

6. For each consumer $i$, suppose that $\hat{x}^i$ is such that $u^i(\hat{x}^i) \geqslant u^i(x^i)$. By local non-satiation, we can find, for any natural number $n$, some bundle $x^i(n) \in \mathcal{U}^i$ such that $\|x^i(n) - \hat{x}^i\| < 1/n$. It follows from step 4, then, that $p \cdot \sum_i x^i(n) \geqslant k$ for every $n$. Letting $n \to \infty$, we conclude that $p \cdot \sum_i \hat{x}^i \geqslant k$.

7. In particular, the latter step implies that $p \cdot \sum_i x^i \geqslant k$ and therefore, by monotonicity and continuity of each $u^i$, that $p \gg 0$.

8. As a consequence, we also have that $u^i(x) > u^i(x^i)$ implies $p \cdot x \geqslant p \cdot x^i$, for all individuals.

9. Since $\sum_i x^i \in \mathcal{F}$, we have that, moreover,

$$p \cdot \textstyle\sum_i x^i = p \cdot \left(\sum_i w^i + \sum_j y^j\right) \leqslant k,$$

   so $p \cdot \sum_i x^i = k$.

As a consequence of the last result, $y \in Y^j$ implies that $p \cdot y \leqslant p \cdot \hat{y}^j$, since

$$\textstyle\sum_i w^i + y + \sum_{j' \neq j} y^{j'} \in \mathcal{F},$$

which implies that

$$p \cdot \left(\textstyle\sum_i w^i + y + \sum_{j' \neq j} \hat{y}^{j'}\right) \leqslant k.$$

This proves the third claim in the theorem.

To conclude the proof, define $m^i = p \cdot x^i$ for each consumer. The first implication of the theorem follows by construction, while the third part was argued above. Now, suppose that for some individual $i$ we have that for some bundle $x$, $u^i(x) > u^i(x^i)$ and $p \cdot x \leqslant m^i$ are both true. By our previous result, $p \cdot x = m^i > 0$, and, hence, by continuity of preferences, for $\epsilon \in (0, 1)$ close enough to 1 we have $u^i(\epsilon x) \geqslant u^i(x^i)$ and $p \cdot (\epsilon x) < m^i = p \cdot x^i$, which contradicts our previous results. This proves the second part of the theorem. $\square$

Notice that, unlike the FFTWE, the second fundamental theorem does imply existence of equilibrium, so the continuity and convexity assumptions are crucial. The policy implication is that policy-makers do not have to close competitive markets to attain their social goals, as long as these goals are Pareto efficient. Quite the opposite: well chosen redistribution policies and competitive markets, under the assumptions of the theorem, deliver the desired objectives. Notice too, though, that the problem of how much information a policy-maker needs, in order to figure out the correct redistribution, is not addressed by the theorem.

# University of California, Davis
# ECN200, General Equilibrium Theory
# LN6: Falsifying Competitive Equilibrium

The standard for what is to be considered scientific knowledge has been a prominent topic of debate in epistemology. One of the most influential philosophers of the last century, Karl Popper,[1] argued that scientists should actively try to prove their theories wrong, rather than merely try to verify them through inductive reasoning. The Popperian postulate sustains that scientific discovery ought to follow four steps: (i) the internal consistency of a theory must be formally checked, to verify that it contains no logical inconsistencies; (ii) the logical principles of the theory must be distinguished from its empirical implications; (iii) the theory must be compared with alternative existing theoretical knowledge that has not been refuted by empirical evidence, in order to ascertain whether it can explain phenomena that cannot be explained by the existing knowledge; (iv) finally, the theory must be submitted to tests of its empirical implications, in order for it to be corroborated (but not verified) or refuted. Interesting tests are those that are "harsh," in the sense that, *a priori*, the theory would appear likely to fail them. And if a theory fails a test, and there exists no reasonable excuse that can itself be tested, then the theory should be abandoned.

Falisificationism was brought to economics by P. Samuelson [2].[2] For him, "meaningful theorems" are hypotheses "about empirical data which could conceivably be refuted," and he proposed that the discipline should get rid of any theories that failed this criterion. For individual decision problems, Samuelson's idea gave the impetus for the development of revealed preference theory. For the competitive equilibrium model, on the other hand, the program was rather problematic: the SMD theorem was understood to have quite negative implications for the existence of testable restrictions of competitive equilibrium. Our textbook, for instance, categorizes these results as saying that "anything satisfying" the very mild restrictions of the Sonnenschein-Mantel-Debreu theorem "can actually occur." Recent research, however, has obtained more positive results regarding the empirical implications of general equilibrium theory.

D. Brown and R. Matzkin [1], searched for testable restrictions on the equilibrium manifold, i.e., the set of profiles of individual endowments and associated equilibrium prices. Given the logic of comparative statics, the equilibrium manifold is the appropriate construct to use to

[1] Born in Vienna on 28 July 1902, died on 17 September 1994. His contributions encompassed not only philosophy of science, but also mathematical logic and physics.

[2] Gary, Indiana, May 15, 1915 — Massachussets, December 13, 2009. Samuelson received the Nobel prize in 1970, two years before Arrow.

study the testable restrictions of general equilibrium: the system's exogenous variables (the individual endowments) are allowed to vary to derive restrictions on the system's endogenous variables (the equilibrium prices).

As a result of this innovation [1] describe the complete set of testable propositions of the pure exchange model on finite observations of the equilibrium manifold and prove that these tests are nonvacuous. For the case of two agents and two observations, they derive the tests in the form of a finite set of polynomial inequalities over the data alone.

To begin, suppose that there are observations on prices and quantities demanded by *one* consumer, $(p_t, x_t)_{t=1}^T$. Say that a utility function $u(x)$ *rationalizes the data* if $u(x_t) \geqslant u(x)$ for all $x$ such that $p_t \cdot x_t \geqslant p_t \cdot x$.

It is a classical result in consumer theory, due to S. Afriat, that given a data set $(p_t, x_t)_{t=1}^T$, the following conditions are equivalent:[3]

1. There exist numbers $V_t$, $\lambda_t^i > 0$, $t = 1, \dots, T$ that satisfy the "Afriat inequalities": for $t, s = 1, \dots, T$,
$$V_t \leqslant V_s + \lambda_s p_s \cdot (x_t - x_s).$$

2. There exists a concave, monotonic, continuous utility function that rationalizes the data.

Importantly, the polynomial form of the restrictions results entirely from the assumption of finiteness of data, not from any assumptions on the functional form of utility. This is important: the basic building blocks of general equilibrium models have the feature that testable restrictions will be polynomial in form. The finite polynomial form of these conditions is important because it means semi-algebraic theory can be applied to describe the data that satisfy the restrictions.

Theorem 1 (Brown and Matzkin). *Let $(p_t, (w_t^i)_{i \in \mathcal{I}})_{t=1}^T$ be given. There exists a profile of continuous, monotone and concave utility functions $(u^i)_{i \in \mathcal{I}}$ such that each $p_t$ is an equilibrium price vector for the exchange economy $\{\mathcal{I}, (u^i, w_t^i)_{i \in \mathcal{I}}\}$ if, and only if, the following is satisfied: There exist numbers $V_t^i$ and $\lambda_t^i$, and vectors $x_t^i$ such that:*

*(i) markets clear: $\sum_i x_t^i = \sum_i w_t^i$, for $t = 1, \dots, T$;*

*(ii) budget constraints are satisfied: $p_t \cdot x_t^i = p_t \cdot w_t^i$, for $t = 1, \dots, T$, and $i = 1, \dots, I$;*

*(iii) $x_t^i \geqslant 0$, $\lambda_t^i > 0$, for $t = 1, \dots, T$ and $i = 1, \dots, I$;*

[3] A more complete statement of this theorem and its proof are deferred to the appendix: see Theorem 3.

*(iv) the Afriat inequalities are satisfied for all agents: for* $t, s = 1, \ldots, T$ *and* $i = 1, \ldots, I$,

$$V_t^i - V_s^i - \lambda_s^i p_s \cdot (x_t^i - x_s^i) \leqslant 0.$$

Theorem 1 describes competitive equilibrium behavior in terms of a finite set of polynomial inequalities mediated by quantifiers. Now, for any given model and any given set of observables, either the set defined by the testable restrictions would contain all potential data, meaning the model was irrefutable for that set of observables, or it would be empty, meaning equilibrium could never be obtained, or it would contain a strict subset of all potential data. In the last case we say the model is testable, or non-vacuous, given the potential data set, and one could (theoretically) derive the testable restrictions on observable variables.

Theorem 2 (Brown and Matzkin). *The pure exchange model of competitive equilibrium is falsifiable on* $T$ *observations of prices and individual endowments* $(p_t, (w_t^i)_{i \in \mathcal{I}})_{t=1}^T$.

The proof of this theorem depends on the counterexample given by [1], showing that there exist data that do not satisfy the conditions of Theorem 1, as shown in the figures below. Suppose that one has observed the two Edgeworth boxes and prices of Fig. 1.

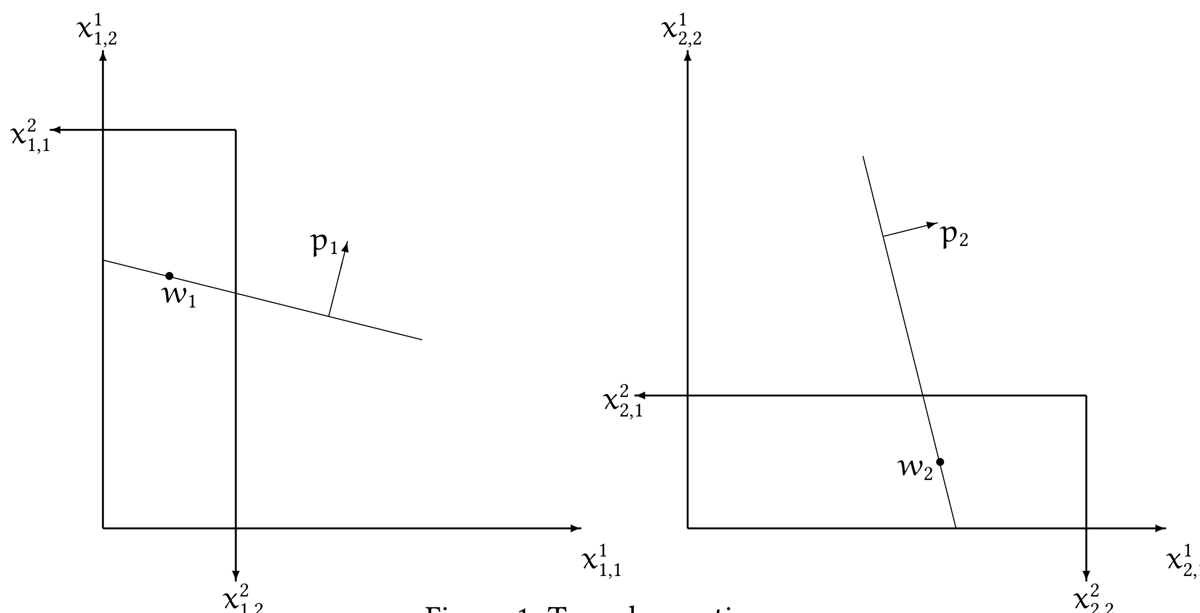


Figure 1: Two observations.

The question that has to be answered is: do there exist individual preferences such that the each observed price is competitive equilibrium for the corresponding box and the (invariant) preferences? Notice that, since we are not observing individual consumptions, it may in principle seem like revealed-preference is silent about this question. But it is not: since consumption cannot be negative, we can restrict the values that individual consumption bundles may take, even if we cannot exactly pin down their values. To do this, just superimpose the two boxes, as in Fig. 2. Whatever their exact values are, with nonnegative consumptions for individual 2, individual 1 must violate WARP! These data are then inconsistent with the general equilibrium model, and hence the model is refutable.

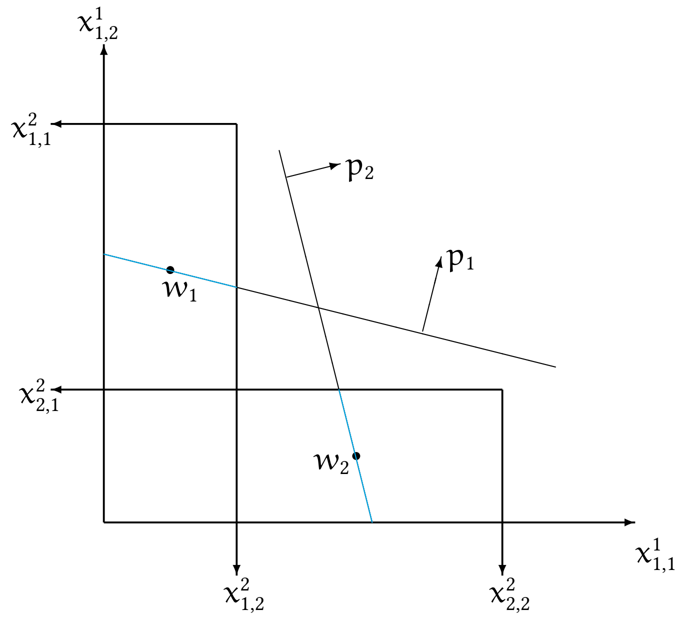


Figure 2: Impossibility of WARP for individual 1.

# Appendix

*A1: Some math*

Two mathematical results that are going to be used later are presented next. These results are to apply to finite-dimensional real spaces only.

## Systems of inequalities

The following result is usually referred to as the Theorem of the Alternative.

Theorem. *Let* $a_t \in \mathbb{R}^L$ *and* $\alpha_t \in \mathbb{R}$, *for* $t = 1, \ldots, T$, *and let S be an integer,* $1 \leqslant S \leqslant T$. *Suppose that there exists a vector* $x \in \mathbb{R}^L$ *such that* $a_t \cdot x \leqslant \alpha_t$ *for all* $t = S + 1, \ldots, T$. *Then, one and only one of the following statements is true:*

(i) *There exists a vector* $x \in \mathbb{R}^l$ *such that*

$$a_t \cdot x < \alpha_t \textit{ for all } t = 1, \ldots, S,$$

*and*

$$a_t \cdot x \leqslant \alpha_t \textit{ for all } t = S+1, \dots, T.$$

(ii) *There exists a vector* $y \in \mathbb{R}^T_+$ *such that* $y_t \neq 0$ *for some* $t \leqslant S$, *and*

$$\sum_{t=1}^{T} y_t a_t = 0 \textit{ and } \sum_{t=1}^{T} y_t \alpha_t \leqslant 0.$$

## Quantifier elimination

The statements of the theorems we've introduced contain existential quantifiers on unobserved (and even unobservable) variables of their models. Although modern computational algorithms have proven useful to deal with this kind of situation, from a purely theoretical perspective, like the one followed here, it is convenient to argue that these quantifiers can be eliminated, and to obtain as much information as possible regarding equivalent statements that are free of quantifiers. For this, we can use the classical theory of quantifier elimination introduced next.

A function $\mu : \mathbb{R}^K \to \mathbb{R}$ is a (real) *multivariate monomial* if there exists natural numbers (or zero) $\alpha_k$ such that $\mu(x) = \prod_{k=1}^{K} x_k^{\alpha_k}$. A function $\rho : \mathbb{R}^K \to \mathbb{R}$ is a (real) *multivariate polynomial* if there exist finitely many multivariate monomials, $\mu_m : \mathbb{R}^K \to \mathbb{R}$, and real numbers, $a_m \neq 0$, $m = 1, \dots, M$, such that $\rho = \sum_{m=1}^{m} a_m \mu_m$.

Define the *sign function*, sgn : $\mathbb{R} \to \{-1, 0, 1\}$, by

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0; \\ 0, & \text{if } x = 0; \\ 1, & \text{if } x > 0. \end{cases}$$

A set $A \subseteq \mathbb{R}^K$ is *semialgebraic* if it can be written as a set-theoretic expression of the form

$$A = \bigcup_{m=1}^{M} \bigcap_{n=1}^{N_m} \left\{ x \in \mathbb{R}^K \mid \operatorname{sgn}(\rho_{m,n}(x)) = s_{m,n} \right\},$$

where, for each $m = 1, \dots, M$ and each $n = 1, \dots, N_m$, function $\rho_{m,n} : \mathbb{R}^K \to \mathbb{R}$ is a multivariate polynomial and $s_{n,m} \in \{-1, 0, 1\}$. A function $\eta : A \to B$, where $A \subseteq \mathbb{R}^{K_A}$ and $B \subseteq \mathbb{R}^{K_B}$ are semialgebraic sets is a *semialgebraic map* if its graph is semialgebraic.

Theorem (Tarski and Seidenberg). *Let* $A \subseteq \mathbb{R}^K$ *be a semialgebraic set and let* $\eta : \mathbb{R}^K \text{ to} \mathbb{R}^{K'}$ *be a semialgebraic map. Then, the image of* $A$ *under* $\eta$,

$$\eta[A] = \{y \in \mathbb{R}^{K'} | \exists x \in A : \eta(x) = y\},$$

*is semialgebraic.*

This theorem gives us the following corollary, which we will later use.

Corollary 1. *Let* $A \subseteq \mathbb{R}^{K_1} \times \mathbb{R}^{K_2}$ *be a semialgebraic set and let* $\overrightarrow{A}^1$ *be its projection into* $\mathbb{R}^{K_1}$. *Then,* $\overrightarrow{A}^1$ *is semialgebraic.*

Suppose that one has observed a data set $D = \{(x_t, p_t, m_t)\}_{t=1}^T$, where $T$ is finite: for each observation, $t = 1, \dots, T$, prices are $p_t \in \mathbb{R}^L_{++}$, and individual's nominal income is $m_t > 0$, and her demand for commodities is $x_t \in B(p_t, m_t)$ with $p_t \cdot x_t = m_t$. We say that the utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$ *rationalizes the data* if $x_t$ is the unique solution to

$$\max_x \{u(x) : p_t \cdot x \leqslant m_t\}$$

for each observation $t$. We want to distinguish data sets that are rationalizable by some "standard" individual preferences from those that cannot be explained by the individually-rational behavior of a consumer who takes prices as given.

We will say that a data set $D$ satisfies the *Strong Axiom of Revealed Preferences, SARP,* if for every finite sequence of observations, $((x^k, p^k, m^k))_{k=1}^K$ in $D$ one has that the implication

$$\left( \forall k \in \{1, \dots, K-1\}, p^k \cdot x^{k+1} \leqslant m^k \text{ and } x^1 \neq x^K \right) \Rightarrow p^K \cdot x^1 > m^K$$

is true.

The following theorem says that SARP is a test of rational behavior, and that there can be no test of such behavior that is stronger than SARP.

Theorem 3 (Afriat). *Let $D$ be a finite data set. The following statements are equivalent:*

1. *There is a utility function that rationalizes $D$.*
2. *$D$ satisfies SARP.*
3. *There exist numbers $\lambda_t > 0$ and $\mu_t$, for each observation $t \in \{1, ..., T\}$, such that*

$$\mu_t \leqslant \mu_{t'} + \lambda_{t'} p_{t'} \cdot (x_t - x_{t'})$$

*for each pair of observations $t$ and $t'$, with strict inequality whenever $x_t \neq x_{t'}$.*

4. *There is a continuous, strictly concave, strictly monotone utility function that rationalizes $D$.*

*Proof.* It suffices if we show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

To see that 1 implies 2, suppose, by way of contradiction, that one can find a finite sequence

$$((x^k, p^k, m^k))_{k=1}^K$$

in $D$ such that:

(i) $p^k \cdot x^{k+1} \leqslant m^k$ for all $k = 1, ..., K-1$;

(ii) $x^1 \neq x^K$; and

(iii) $p^K \cdot x^1 \leqslant m^K$.

Given that $u$ rationalizes $D$, it follows from (i) that $u(x^{k+1}) \leqslant u(x^k)$ for each $k$ in $\{1, ..., K-1\}$, so $u(x^K) \leqslant u(x^1)$. By (iii) it then follows that $x^1$ solves the problem

$$\max_x \left\{ u(x) : p^K \cdot x \leqslant m^K \right\},$$

which is impossible by (ii), since

$$\operatorname{argmax}_x \left\{ u(x) : p^K \cdot x \leqslant m^K \right\} = \{x^K\}.$$

To see that 2 implies 3, we can give a simple proof under the extra assumption that $x_t \neq x_{t'}$ whenever $t \neq t'$, while the formal argument for the general case is deferred. Under this simplifying assumption, we want to find numbers $\mu_t$ and $\lambda_t$, for $t = 1, \ldots, T$, such that

$$\mu_t - \mu_{t'} - \lambda_{t'} p_{t'} \cdot (x_t - x_{t'}) < 0$$

for all $t$ and all $t' \neq t$, while $\lambda_t > 0$ for all $t$. Now, by the Theorem of the Alternative, if these numbers do not exist, then we can find nonnegative numbers $\alpha_{t,t'}$ and $\beta_t$, at least one of which is not zero, such that, for all $t$,

$$\sum_{t' \neq t} \alpha_{t,t'} = \sum_{t' \neq t} \alpha_{t',t} \tag{$*$}$$

and

$$-\sum_{t' \neq t} \alpha_{t,t'} p_t \cdot (x_{t'} - x_t) = \beta_t. \tag{$**$}$$

It is immediate that if all $\alpha_{t,t'}$ numbers are zero, then so must all $\beta_t$ numbers, which is impossible. It follows that there must be some $t_1$ and some $t' \neq t_1$ such that $\alpha_{t_1,t'} > 0$. Using Eq. ($**$), there must then exist some $t_2 \neq t_1$ such that

$$\alpha_{t_1,t_2} \neq 0 \text{ and } p_1 \cdot (x_{t_2} - x_{t_1}) \leqslant 0.$$

By Eq. ($*$), it follows that there must be some $t'$ such that $\alpha_{t_2,t'} > 0$, and then, by Eq. ($**$), that there must then exist some $t_3 \neq t_2$ such that

$$\alpha_{t_2,t_3} \neq 0 \text{ and } p_2 \cdot (x_{t_3} - x_{t_2}) \leqslant 0.$$

If $t_3 = t_1$, we have a violation of SARP. Otherwise, we can continue to construct a sequence $(t_1, t_2, t_3, \ldots)$ such that

$$\alpha_{t_k,t_{k+1}} \neq 0 \text{ and } p_{t_k} \cdot (x_{t_{k+1}} - x_{t_k}) \leqslant 0.$$

Since the dataset is finite, this construction must lead to some cycle, and hence to a violation of SARP.

Again, we can give a simple argument that 3 implies 4, if we defer the proof that the rationalizing function is strictly concave, and merely require that each $x_t$ be a solution of the problem

$$\max_x \{u(x) : p_t \cdot x \leqslant p_t \cdot x_t\},$$

even though there may be other solutions; again, the full proof is deferred to the Appendix. For each observation $t$, define function $\phi_t : \mathbb{R}^L \to \mathbb{R}$ by

$$\phi_t(x) = \mu_t + \lambda_t p_t \cdot (x - x_t),$$

which is concave and strictly monotone. With these functions, we can construct the utility function $u$ by letting

$$u(x) = \min_{t \in \{1,\dots,T\}} \{\phi_t(x)\},$$

which is continuous, concave and strictly monotone. By construction, $u(x_t) = \mu_t$ for each observation $t$. Now, let $x \neq x_t$ be such that $p_t \cdot x \leqslant m_t$. Then,

$$u(x) \leqslant \phi_t(x) = \mu_t + \lambda_t p_t \cdot (x - x_t) \leqslant \mu_t.$$

Now, for strong concavity, define function $h : \mathbb{R}^L \to \mathbb{R}$ by $h(x) = \sqrt{||x||^2 + 1} - 1$; this function is smooth, strictly convex and satisfies that: (i) $h(x) = 0$ for, and only for, $x = 0$; (ii) if $x \neq 0$, then $h(x) > 0$; and (iii) for each component $\ell = 1, \dots, L$, $0 \leqslant \partial h / \partial x_\ell(x) < 1$ at all $x$. Since there are only finitely many observations, there exists a strictly positive $\varepsilon$ such that

$$\mu_{t'} < \mu_t + \lambda_t p_t \cdot (x_{t'} - x_t) - \varepsilon h(x_{t'} - x_t)$$

whenever $x_{t'} \neq x_t$, while $\mu_{t'} = \mu_t$ whenever $x_{t'} = x_t$. Now, for each observation $t$, define function $\phi_t : \mathbb{R}^L \to \mathbb{R}$ by

$$\phi_t(x) = \mu_t + \lambda_t p_t \cdot (x - x_t) - \varepsilon h(x - x_t),$$

which is strictly concave. Also, note that

$$\frac{\partial \phi_t}{\partial x_\ell}(x) = \lambda_t p_{t,\ell} - \varepsilon \frac{\partial h}{\partial x_\ell}(x - x_t) > \lambda_t p_{t,\ell} - \varepsilon,$$

for each component $\ell$, so we can take $\varepsilon$ small enough to ensure that all $\phi_t$ functions are strictly monotone. With these functions, we can construct the utility function $u$ by letting

$$u(x) = \min_{t \in \{1,\dots,T\}} \{\phi_t(x)\},$$

which is continuous, strictly concave and strictly monotone. By construction, $u(x_t) = \mu_t$ for each observation $t$. Now, let $x \neq x_t$ be such that $p_t \cdot x \leqslant m_t$. Then,

$$u(x) \leqslant \phi_t(x) = \mu_t + \lambda_t p_t \cdot (x - x_t) - \varepsilon h(x - x_t) < \mu_t.$$

That 4 implies 1 is obvious. $\square$

To conclude, let us see an alternative, constructive proof that statement 2 implies statement 3 in Theorem 3.

*Proof.* Fix $\varepsilon > 0$. Define the "directly revealed preference" binary relation $\succsim^D$ on $\{x_t\}_{t=1}^T$ by saying that $x_t \succsim^D x_{t'}$ if (and only if) $p_t \cdot x_{t'} \leqslant m_t$, and the "revealed preference" relation $\succsim^I$ on the same domain, by saying that $x_t \succsim^I x_{t'}$ if there is a finite sequence $(x^k)_{k=1}^K$ such that

$$x_t \succsim^D x^1 \succsim^D x^2 \succsim^D \ldots \succsim^D x^K \succsim^D x_{t'}.$$

It is immediate that $\succsim^I$ es reflexive and transitive.

Now, given $I \subseteq \{1, ..., T\}$, define

$$\max(I) = \{t \in I \mid (t' \in I \text{ and } x_{t'} \succsim^I x_t) \Rightarrow x_t \succsim^I x_{t'}\}.$$

Note that if $I \neq \varnothing$ then $\max(I) \neq \varnothing$.[4]

Now, consider the following algorithm, which runs in finite time: Given $D$,

1. Let $I = \{1, ..., T\}$ and $B = \varnothing$.
2. Take $m \in \max(I)$.
3. Define $E = \{t \in I \mid x_t \succsim^I x_m\}$.
4. If $B = \varnothing$, let $\mu_m = \lambda_m = 1$ and go to 6.
5. Let
$$\mu_m = \min_{t \in E} \left\{ \min_{t' \in B} \{\min\{\mu_{t'} + \lambda_{t'} p_{t'} \cdot (x_t - x_{t'}) - \varepsilon, \mu_{t'} - \varepsilon\}\} \right\}$$
and
$$\lambda_m = \max_{t \in E} \left\{ \max_{t' \in B} \left\{ \max \left\{ \frac{\mu_{t'} - \mu_m + \varepsilon}{p_t \cdot (x_{t'} - x_t)}, 1 \right\} \right\} \right\}$$
6. For all $t \in E$, let $\mu_t = \mu_m$ and $\lambda_t = \lambda_m$.
7. Let $I = I \setminus E$ and $B = B \cup E$. If $I \neq \varnothing$, go to 2. (If $I = \varnothing$, stop.)

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[4] To see this, denumerate $I = \{t_1, \ldots, t_{\#I}\}$, and consider the following construction: let $m = t_1$ and $b_{t_0} = x_{t_1}$, and for each $i \in \{1, \ldots, \#I\}$, proceed consecutively as follows:

(1) if $x_{t_i} \succsim^I b_{t_{i-1}}$, then define $b_{t_i} = x_{t_i}$ and $m = t_i$;

(2) otherwise, if it is not true that $x_{t_i} \succsim^I b_{t_{i-1}}$, then let $b_{t_i} = b_{t_{i-1}}$ and and $m = t_{i-1}$.

We want to show that, at the end of this construction, $m = \max(I)$. Notice first that, by reflexivity and transitivity of $\succsim^I$, $b_{t_{\#I}} \succsim^I b_{t_i}$ for all $i$ in $\{1, \ldots, \#I\}$. Note also that $x_m = b_{t_{\#I}}$, by construction. Now, suppose that for some $t_i \in I$, $x_{t_i} \succsim^I x_m$. Then, $x_{t_i} \succsim^I b_{t_{\#I}}$, and, given that $b_{t_{\#I}} \succsim^I b_{t_{i-1}}$, by transitivity $x_{t_i} \succsim^I b_{t_{i-1}}$, which implies that $b_{t_i} = x_{t_i}$ and $b_{t_{\#I}} \succsim^I b_{t_i} = x_{t_i}$. It follows that $x_m = b_{t_{\#I}} \succsim^I b_{t_i} = x_{t_i}$ and, hence, that $m \in \max(I)$.

We want to show that the output of the algorithm, $\{(\lambda_t, \mu_t)\}_{t=1}^{T}$ satisfies the conditions of statement 3. The fact that each $\lambda_t > 0$ follows by construction. Now, to obtain the inequality in the statement, it suffices to show that after each pass of the algorithm by step 6, the values of $\mu$ and $\lambda$ computed so far satisfy the inequality. We consider four cases, and take $\varepsilon > 0$ to be small enough:

*Case 1:* $t \in B$ *and* $t' \in E$. Note that by step 5 of the algorithm, for each $t \in B$ and each $t' \in E$ one must have that $p_{t'} \cdot (x_t - x_{t'}) > 0$, for otherwise $p_{t'} \cdot x_t \leqslant p_{t'} \cdot x_{t'} = m_{t'}$, which would imply that $x_{t'} \succsim^D x_t$ and, hence, that $x_{t'} \succsim^I x_t$; this would in turn imply that in a previous pass of the algorithm, when $t$ entered $E$, $t'$ should have entered too, which contradicts the premise that now $t \in B$ and $t' \in E$. Now, then, with $t \in B$ and $t' \in E$ we have that

$$\lambda_{t'} = \lambda_m \geqslant \max_{t'' \in B} \left\{ \max \left\{ \frac{\mu_{t''} - \mu_m + \varepsilon}{p_{t'} \cdot (x_{t''} - x_{t'})}, 1 \right\} \right\} \geqslant \frac{\mu_t - \mu_m + \varepsilon}{p_{t'} \cdot (x_t - x_{t'})}$$

from where $\lambda_{t'} p_{t'} \cdot (x_t - x_{t'}) \geqslant \mu_t - \mu_m + \varepsilon$ and then, by step 6, $\mu_t = \mu_m$ which implies the result.

*Case 2:* $t \in B$ *and* $t' \in E$. By construction,

$$\mu_{t'} = \mu_m \leqslant \min_{t'' \in B} \{\min\{\mu_{t''} + \lambda_{t''} p_{t''} \cdot (x_{t'} - x_{t''}) - \varepsilon, \mu_{t''} - \varepsilon\}\} \leqslant \mu_t + \lambda_t p_t \cdot (x_{t'} - x_t) - \varepsilon.$$

*Case 3:* $t, t' \in E$ *such that* $x_t \neq x_{t'}$. By definition, $x_t \succsim^I x_m \succsim^I x_{t'}$, which implies by SARP that $p_{t'} \cdot x_t > m_{t'}$ and, hence, that $p_{t'} \cdot (x_t - x_{t'}) > 0$. Since $\mu_t = \mu_{t'}$, it is immediate that

$$\mu_{t'} < \mu_t + \lambda_t p_t \cdot (x_{t'} - x_t)$$

and then, for small enough $\varepsilon$,

$$\mu_{t'} \leqslant \mu_t + \lambda_t p_t \cdot (x_{t'} - x_t) - \varepsilon.$$

*Case 4:* $t, t' \in E$ *such that* $x_t = x_{t'}$. This case is immediate, by construction. □

## References


[1] Brown, D. and R. Matzkin, "Testable restrictions on the equilibrium manifold," *Econometrica*, 64, 1249-1262, 1996.

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# University of California, Davis
# ECN200, General Equilibrium Theory
# LN8: Core Convergence

Recall that every competitive equilibrium allocation is in the core, but there may be core allocations that are not equilibria. Now, it is intuitive to think that the presence of more agents in the economy "shrinks" its core, since there are more coalitions that can object a given allocation.[1]

Debreu and Herbert Scarf[2] studied this problem: (i) is it true that this reduction takes place? and (ii) if we push the number of agents to infinity, will non-equilibrium allocations remain in the core? The respective answers are Yes and No, which is why we normally think that competitive markets work well in economies with many agents: when there are many agents, their cooperative behavior can be decentralized via competitive markets.

## 1. Informal Argument

To see the intuition of this idea, consider and Edgeworth box economy of Fig. 1. Point $a$ represents a core allocation which is not equilibrium. Notice, however, that both agents would prefer a situation in which 1 consumes in $c$ and 2 in $b$. Could they agree on implementing that situation? No, since there would be excess demand for commodity 1 and excess supply commodity 2.

Now, if you bring to this economy an agent who is willing to supply commodity 1 in exchange for commodity 2, the latter change may be possible. For this, simply note that a replica of consumer 1 offered to consume in $c$ does exactly that: he brings the required amount of commodity 1 and consumes the excess of commodity 2. With these carefully chosen trades, allocation $(c, c, b)$ would be feasible for a coalition $\{1, \text{replica of } 1, 2\}$ of a larger economy, and it would be more desirable for its members than $(a, a, a)$. In an informal sense, the presence of the replica of 1 eliminates allocation $a$ from the core.

[1] We are using the term "shrink" loosely, since the presence of more agents changes the dimension of the allocation space, so comparing the sizes of the cores will require some refinement of the argument.

[2] July 25, 1930 – November 15, 2015.

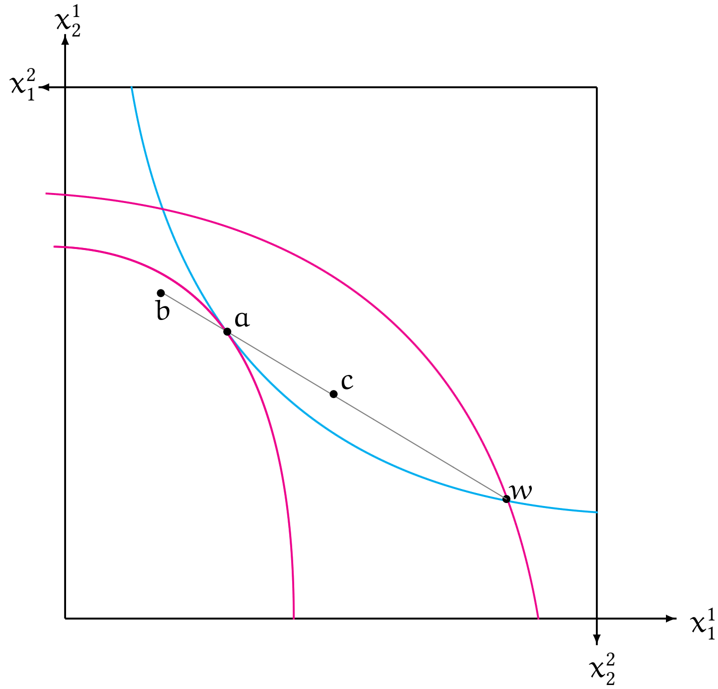


Figure 1: Core shrinkage upon replication of an economy

## 2. Formal Argument

Fix a standard exchange economy, $\mathcal{E}$. For each natural number $N$, let $\mathcal{E}^N$ be the N-fold replica of $\mathcal{E}$:[3]

$$\mathcal{E}^N = \{\mathcal{I} \times \{1, \ldots, N\}, (u^{i,n}, w^{i,n})_{(i,n)\in\mathcal{I}\times\{1,\ldots,N\}}\},$$

where for all $(i, n)$, $(u^{i,n}, w^{i,n}) = (u^i, w^i)$.

Theorem 1. *If* $(p, (x^i)_{i\in\mathcal{I}})$ *is a competitive equilibrium for* $\mathcal{E}$*, then*

$$(p, (x^{i,n})_{(i,n)\in\mathcal{I}\times\{1,\ldots,N\}}),$$

*with* $x^{i,n} = x^i$ *for every* $(i, n)$*, is an equilibrium for* $\mathcal{E}^N$*.*

*Proof.* This is left as an exercise. □

Given $\mathcal{E}^N$, a feasible allocation $(x^{i,n})_{(i,n)\in\mathcal{I}\times\{1,\ldots,N\}}$ is said to have the *equal-treatment property* if $x^{i,n} = x^{i,n'}$ for all $i$ and all $n, n'$.

Theorem 2. *Suppose that each* $u^i$ *is strictly quasiconcave, and that* $(p, x)$ *is a competitive equilibrium for* $\mathcal{E}^N$*. Then,* $x$ *has the equal-treatment property.*

[3] Enlarging the economy using replicæ will take care of the dimensionality issue mentioned before.

*Proof.* Suppose not: let $\hat{\imath}, \hat{n}, \tilde{n}$ be such that $x^{\hat{\imath},\hat{n}} \neq x^{\hat{\imath},\tilde{n}}$. Let $(\hat{x}^{i,n})_{(i,n)\in\mathcal{I}\times\{1,\ldots,N\}}$ be defined as follows:

$$\hat{x}^{i,n} = \begin{cases} \frac{1}{2}(x^{\hat{\imath},\hat{n}} + x^{\hat{\imath},\tilde{n}}), & \text{if } (i,n) \in \{(\hat{\imath},\hat{n}), (\hat{\imath},\tilde{n})\}; \\ x^{i,n}, & \text{otherwise}; \end{cases}$$

which is feasible. Since $(p, x)$ is a competitive equilibrium, $u^i(x^{\hat{\imath},\hat{n}}) = u^i(x^{\hat{\imath},\tilde{n}})$ and, so, by strong quasiconcavity, $u^i(\hat{x}^{\hat{\imath},\hat{n}}) > u^i(x^{\hat{\imath},\hat{n}})$ and $u^i(\hat{x}^{\hat{\imath},\hat{n}}) > u^i(x^{\hat{\imath},\tilde{n}})$. By construction, for all $(i,n) \in \mathcal{I} \times \{1,\ldots,N\}$, $u^i(\hat{x}^{i,n}) \geqslant u^i(x^{i,n})$. This contradicts the FFTWE. □

These last two theorems define a one-to-one correspondence between equilibria of $\mathcal{E}$ and equilibria of $\mathcal{E}^N$.

THEOREM 3. *Suppose that each $u^i$ is strongly quasiconcave. If $x$ is a core allocation for $\mathcal{E}^N$, then it has the equal-treatment property.*

*Proof.* Suppose not: let $\hat{\imath}$, $\hat{n}$ and $\tilde{n}$ be such that $x^{\hat{\imath},\hat{n}} \neq x^{\hat{\imath},\tilde{n}}$. For each $i \in \mathcal{I}$, let

$$n^i \in \text{argmin}_{n\in\{1,\ldots,N\}}\{u^i(x^{1,n})\}.$$

Construct $\mathcal{H} = \{(i, n^i)\}_{i\in\mathcal{I}}$ and

$$(\hat{x}^{(i,n^i)})_{i\in\mathcal{I}} = \left(\frac{1}{N}\sum_n x^{i,n}\right)_{i\in\mathcal{I}}.$$

By construction, $\mathcal{H} \subseteq \mathcal{I} \times \{1,\ldots,N\}$, $\mathcal{H} \neq \varnothing$, whereas

$$\sum_{h\in\mathcal{H}} \hat{x}^h = \frac{1}{N}\sum_{i,n} x^{i,n} = \frac{1}{N}\sum_{i,n} w^{i,n} = \sum_i w^i = \sum_{h\in\mathcal{H}} w^h.$$

By strict quasiconcavity, $u^h(\hat{x}^h) \geqslant u^h(x^h)$ for every $h \in \mathcal{H}$ , whereas $u^{\hat{\imath}}(\hat{x}^{\hat{\imath},n^{\hat{\imath}}}) > u^{\hat{\imath}}(x^{\hat{\imath},n^{\hat{\imath}}})$, which is impossible. □

Denote by $W$ the set of all equilibrium allocations for $\mathcal{E}$ and let $C_N$ be the "dimension-free" core of $\mathcal{E}^N$:

$$C^N = \{(x^i)_{i\in\mathcal{I}} \in (\mathbb{R}^L_+)^I \mid (x^{i,n} = x^i)_{(i,n)\in\mathcal{I}\times\{1,\ldots,N\}} \text{ is in the core of } \mathcal{E}^N\}.$$

THEOREM 4 (THE DEBREU-SCARF THEOREM). *Suppose that each $u^i$ is monotone and and strictly quasiconcave, and $w^i \in \mathbb{R}^L_{++}$ for all $i$. Then,*

$$\bigcap_{N=1}^{\infty} C^N = W.$$

*Proof.* That $W \subseteq \cap_{N=1}^{\infty} C^N$ is left as an exercise.

For the opposite inclusion, let $x \in \cap_{N=1}^{\infty} C^N$ and let $P$ be the convex hull of

$$\bigcup_i \left\{ z \in \mathbb{R}^L \mid u^i(w^i + z) > u^i(x^i) \right\}.$$

By monotonicity, $P$ is non-empty.

Our first claim is that $0 \notin P$. To see why, suppose otherwise: let $\alpha \in \mathbb{R}_+^I$ be such that $\sum_i \alpha^i z^i = 0$ for $(z^i)_{i \in \mathcal{I}}$ such that $u^i(w^i + z^i) > u^i(x^i)$ for all $i \in \mathcal{I}$. Then, for each $m \in \mathbb{N}$, let

$$a_m^i = \min \left\{ n \in \mathbb{N} \mid n \geqslant m\alpha^i \right\};$$

also, let $\mathcal{H}^1 = \{i \in \mathcal{I} \mid \alpha^i > 0\} \neq \varnothing$ and define, for all $i \in \mathcal{H}^1$,

$$z_m^i = \frac{m\alpha^i}{a_m^i} z^i.$$

By construction, for all $i \in \mathcal{H}^1$,

$$w^i + z_m^i \in \prod_\ell (w_\ell^i, w_\ell^i + z_\ell^i]$$

and $\lim_{m \to \infty}(w^i + z_m^i) = w^i + z^i$. The latter implies that there exists $M \in \mathbb{N}$ such that for all $i \in \mathcal{H}^1$, $u^i(w^i + z_M^i) > u^i(x^i)$.

Now, construct $\mathcal{H} \subseteq \mathcal{I} \times \{1, \ldots, M\}$ by picking, for each $i \in \mathcal{I}^1$, $a_M^i$ replicas of $i$ and allocating to each one of these replicas $\hat{x}^h = w^i + z_M^i$. It is immediate that for all $h \in \mathcal{H}$, $u^h(\hat{x}^h) > u^h(x^h)$, and

$$\sum_{h \in \mathcal{H}} \hat{x}^h = \sum_{i \in \mathcal{H}^1} a_M^i \left( w^i + \frac{M\alpha^i}{a_M^i} z^i \right) = \sum_{h \in \mathcal{H}} w^h + \sum_i \alpha^i z^i = 0,$$

which contradicts the fact that $(x^{i,n} = x^i)_{(i,n) \in \mathcal{I} \times \{1,\ldots,M\}}$ is in the core of $\mathcal{E}^M$.

Since $x$ is in the core of the original economy, $C^1$, we know that $\sum_i x^i = \sum_i w^i$. Since $0 \notin P$ and $P$ is convex, by the separating hyperplane theorem there exists $p \in \mathbb{R}^L \setminus \{0\}$ such that for all $z \in P$, $p \cdot z \geqslant 0$. By monotonicity of all preferences, $p \in \mathbb{R}_{++}^L$ and it is immediate that for all $i \in \mathcal{I}$, if $u^i(\hat{x}) > u^i(x^i)$, then $p \cdot \hat{x} \geqslant p \cdot w^i$. Moreover, as in the proof of the SFTWE, by continuity, for all $i$, if $u^i(\hat{x}) \geqslant u^i(x^i)$, then $p \cdot \hat{x} \geqslant p \cdot w^i$, and, in particular, $p \cdot x^i \geqslant p \cdot w^i$.

Now, since $\sum_i x^i = \sum_i w^i$, it follows that for every $i$, $p \cdot x^i = p \cdot w^i$, and if $u^i(\hat{x}) \geqslant u^i(x^i)$, then $p \cdot \hat{x} \geqslant p \cdot w^i$.

Again as in the proof of the SFTWE, since each $w^i \in \mathbb{R}_{++}^L$, if $u^i(\hat{x}) > u^i(x^i)$, then $p \cdot \hat{x} > p \cdot w^i$. It follows that $(p, x)$ is a competitive equilibrium for economy $\mathcal{E}$. □

University of California, Davis

ECN200, General Equilibrium Theory

LN11: Public Goods and Lindahl Equilibrium

A particular instance of externalities occurs when a commodity that is non-rival and non-exclusive. Such commodities are known as *public goods*, and their treatment as if they were "private" commodities leads again to Pareto inefficient equilibrium allocations.

Suppose that there exist $L+1$ commodities. Individual preferences are represented by functions of the form $u^i : \mathbb{R}_+^{L+1} \to \mathbb{R}$. Denoting by $x$ a bundle of the first $L$ commodities, and by $y$ the consumption of the last commodity, we write the utility of individual $i$ as $u^i(x, y)$. What makes the last commodity special is that if there is a profile $\vec{y} = (y^1, \ldots, y^I)$ of individual demands for it, then each person's consumption equals $\sum_i y^i$.

Suppose also that the first $L$ commodities are available in private endowments, $w^i \in \mathbb{R}_+^L$, while the last one has to be produced: there exist a firm that produces $Y = f(X)$ units of that commodity if it uses a bundle $X$ of the other commodities as input. Individual $i$ is assumed to own a share $s^i$ of the firm's equity.

Al functions $u^i$ are assumed to be of class $C^2$, differentiably strictly monotone, and differentiably strictly quasi-concave. Technology $f : \mathbb{R}_+^L \to \mathbb{R}_+$ is assumed to be of class $C^2$, differentiably monotone and differentiably concave.

## 1. Competitive Equilibrium

Denoting by $p$ the vector of prices of the first $L$ commodities and by $q$ the price of the last commodity, individual $i$'s budget constraint is given by

$$p \cdot x^i + q y^i \leqslant p \cdot w^i + s^i (qY - p \cdot X),$$

where $x^i$ and $y^i$ are, respectively, her private demands for the first $L$ goods and the last good. Since the last good is public, her utility level is $u^i(x^i, \sum_j y^j)$ though. In the language of the previous note, the last commodity imposes an externality.

Again, the definition of competitive equilibrium requires an assumption on how each individual sees the determination of the others' demands for the public good.

Definition 1. *A competitive, or Nash-Walras, equilibrium is an array* $(p, q, \vec{x}, \vec{y}, X, Y)$*, where* $\vec{x} = (x^1, \ldots, x^I)$ *and* $\vec{y} = (y^1, \ldots, y^I)$ *are, respectively, profiles of consumption bundles of the first* $L$ *goods and of the last good, such that:*

*i. for each* $i$, $(x^i, y^i)$ *solves the problem*

$$\max_{x,y} \left\{u^i\left(x, y + \textstyle\sum_{j \neq i} y^j\right) : p \cdot x + qy \leqslant p \cdot w^i + s^i(qY - p \cdot X)\right\};$$

*ii. for the firm,* $(X, Y)$ *solves the problem*

$$\max_{\hat{X}, \hat{Y}} \left\{q\hat{Y} - p \cdot \hat{X} : \hat{Y} = f(\hat{x})\right\};$$

*iii. markets clear:* $\sum_i x^i + X = \sum_i w^i$ *and* $\sum_i y^i = Y$.

The first-order conditions for competitive equilibrium require that for some $\vec{\lambda} \gg 0$,

$$D_x u^i(x^i, Y) = \lambda^i p \ \text{ and } \ \partial_y u^i(x^i, Y) = \lambda^i q, \tag{1}$$

for all $i$, while

$$qDF(X) = p \tag{2}$$

for the firm.

Since $D_x u^i \gg 0$ and $\partial_y u^i > 0$ for all $i$, Eqs. (1) and (2) imply that $p \gg 0$, $q > 0$ and $\lambda^i > 0$ for all $i$.

## 2. Inefficiency of Competitive Equilibrium Allocations

Define an allocation to be $(\vec{x}, y, X, Y)$, and say that it is feasible if $\sum_i x^i + X \leqslant \sum_i w^i$ and $y \leqslant Y = f(X)$.

As in the case of externalities, efficiency of allocation $(\vec{x}, y, X, Y)$ requires that for some $\vec{\mu} = (1, \mu^2, \ldots, \mu^I) \gg 0$, some $\nu > 0$ and some $\delta \gg 0$,

$$\mu^i D_x u^i(x^i, Y) = \delta \tag{3}$$

for all $i$, while

$$\sum_i \mu^i \partial_y u^i(x^i, Y) = \nu \ \text{ and } \ \delta = \nu Df(X). \tag{4}$$

Suppose that the competitive equilibrium allocation is Pareto efficient. From Eqs. (1) and (3), $\mu^i \lambda^i p = \delta \gg 0$. Using (1) again,

$$\lambda^i = \frac{1}{q} \partial_y u^i(x^i, Y),$$

so that

$$\mu^i \frac{p}{q} \partial_y u^i(x^i, Y) = \delta. \tag{5}$$

Adding Eq. (5) across individuals,

$$\left[\sum_i \mu^i \partial_y u^i(x^i, Y)\right] p = Iq\delta.$$

From Eq. (4), the latter means that $\nu p = Iq\delta$, while, from Eqs. (2) and (4),

$$\nu p = \nu q Df(X) = q\delta.$$

Since $q \neq 0$ and $\delta \neq 0$, the equality $Iq\delta = q\delta$ requires that $I = 1$. Except on the trivial case of a one-individual economy, the equilibrium allocation is thus inefficient.

## 3. Lindahl's Solution

Suppose that $I \geqslant 2$ and "someone" recognizes the inefficiency of the equilibrium allocation and proposes an alternative institutional arrangement:

1. each individual recognizes that the amount of public good available to her is the total supplied by the firm;
2. over that total quantity, each individual pays a personalized price, to contribute to its funding; and
3. the total collected from those personalized prices is paid to the firm.

Under this arrangement, the concept of equilibrium is different. Let us denote by $q^i$ the price paid, per unit of the total amount of public good available to the society, by individual $i$. Denote by $y$ such total amount.

Definition 2. *A Lindahl equilibrium is an array* $(p, \vec{q}, \vec{x}, y, X, Y)$, *where* $\vec{x} = (x^1, \dots, x^I)$, $\vec{q} = (q^1, \dots, q^I)$ *and* $y \in \mathbb{R}_+$, *such that*

*i. for each* $i$, $(x^i, y)$ *solves the problem*

$$\max_{\hat{x}, \hat{y}} \left\{u^i(\hat{x}, \hat{y}) : p \cdot \hat{x} + q^i \hat{y} \leqslant p \cdot w^i + s^i[qY - p \cdot X]\right\}$$

*where* $q = \sum_i q^i$;

*ii. for the firm,* $(X, Y)$ *solves the problem*

$$\max_{\hat{X}, \hat{Y}} \left\{\sum_i q^i \hat{Y} - p \cdot \hat{X} : \hat{Y} = f(\hat{X})\right\};$$

*iii. markets clear:* $\sum_i x^i + X = \sum_i w^i$ *and* $y = Y$.

The importance of this concept is that it is an extension of the definition of competitive equilibrium that retains all its properties. The FFTWE and SFTWE, for instance, apply with no modification. If the institutional arrangement that underlies this concept of equilibrium corresponds to real life, then the equilibrium allocation is Pareto efficient, assuming, of course, that all other premises of the definition apply.

## 4. A (Very Brief) Introduction to Implementation

The concept of Lindahl equilibrium is theoretically useful but the institutional arrangement it assumes is likely untenable. Game theory has studied the problem of how to design an institutional setting (mechanism) such that the individually rational interaction in that setting induces socially desirable outcomes.

For the case at hand, a mechanism (or game form) consists of a set of individual message (or strategy) spaces and an outcome function that determines an allocation of commodities as a function of the messages. If the message space of individual $i$ is $M^i$, one denotes $M = \times_i M^i$, so that $\vec{m} \in M$ represents a profile of strategies. The mechanism defines an allocation function $(x^1(\vec{m}, \ldots, x^I(\vec{m}), y(\vec{m}), X(\vec{m}), Y(\vec{m}))$. The mechanism is feasible if its allocation function gives a feasible allocation for each profile of messages.

Note that, given the economy, the mechanism defines a *game* with individual *strategy spaces* $M^i$ and *payoff functions* $v^i(\vec{m}) = v^i(m^i, m^{\neg i}) = u^i(x^i(\vec{m}), y(\vec{m}))$. For this game, a *Nash equilibrium* is a profile of strategies $\vec{m}$ such that each $i$ solves

$$\max_m \left\{ v^i(m, m^{\neg i}) : m \in M^i \right\}.$$

The mechanism *implements (in Nash equilibrium)* the set of allocations given by the allocation function at the Nash equilibria of the game.

## 5. Implementing the Set of Lindahl Equilibrium Allocations

For the sake of simplicity, let us assume that:

1. there is only one private good, so $L = 1$;
2. there are only three individuals, so $I = 3$;
3. the technology to produce the public good satisfies constant returns to scale, so the production function is $f(X) = \varphi X$, for some number $\varphi > 0$.

Also, for the sake of simplicity, we will ignore all non-negative constraints in what follows.

Under these assumptions, consider the following mechanism. For each consumer, the message space is $M^i = \mathbb{R}$ and the allocation function is constructed as follows:

1. the supply of public good is $Y(\vec{m}) = m^1 + m^2 + m^3$;
2. usage of the private good by the firm is $X(\vec{m}) = Y(\vec{m})/\varphi$;
3. three personalized prices are defined as follows:

$$q^1(\vec{m}) = \frac{1}{3\varphi} + m^2 - m^3, q^2(\vec{m}) = \frac{1}{3\varphi} + m^3 - m^1, \text{ and } q^3(\vec{m}) = \frac{1}{3\varphi} + m^1 - m^2;$$

4. the allocation of the private good is given by $x^i(\vec{m}) = w^i - q^i(\vec{m})Y(\vec{m})$;
5. and the demand for the public good is $y(m) = Y(m)$.

To see that the mechanism is feasible, note that $q^1(\vec{m})+q^2(\vec{m})+q^3(\vec{m}) = 1/\varphi$, so that

$$\textstyle\sum_i x^i(\vec{m}) + X(\vec{m}) = \sum_i w^i - \frac{1}{\varphi}Y(\vec{m}) + \frac{1}{\varphi}Y(\vec{m}) = \sum_i w^i,$$

while

$$\varphi X(\vec{m}) = Y(\vec{m}) = y(\vec{m}).$$

Importantly, note that each $q^i(\vec{m})$ is independent from $m^i$, so that, abusing notation slightly, we just write $q^i(m^{\neg i})$.

Theorem 1 (Walker, Groves-Ledyard). *The mechanism above implements the set of Lindahl equilibrium allocations of the economy.*

*Proof.* The more interesting part of the theorem is that every Nash equilibrium allocation of the game induces a Lindahl equilibrium allocation. To see that this is the case, suppose that $\vec{m}$ is a Nash equilibrium of the game. Then, $m^1$ must solve

$$\max_m \left\{v^1(m, m^2, m^3) = u^1(w^1 - q^1(m^2, m^3)(m + m^2 + m^3), m + m^2 + m^3)\right\},$$

or, equivalently,

$$\max_y \left\{u^1(w^1 - q^1(m^2, m^3)y, y)\right\},$$

which is the problem that defines individual rationality in the definition of Lindahl equilibrium at prices $q^1 = q^1(m^2, m^3)$. The same is true, mutatis mutandis, for the other two individuals.

That $[X(\vec{m}), Y(\vec{m})]$ maximizes the profits of the firm in the definition of Lindahl equilibrium follows from the fact that $q^1(\vec{m}) + q^2(\vec{m}) + q^3(\vec{m}) = 1/\varphi$, while market clearing is given by the feasibility of the mechanism.

The other direction, that every Lindahl equilibrium allocation corresponds to a Nash equilibrium, is straightforward once one notes that the system

$$\begin{aligned} m^1 + m^2 + m^3 &= y \\ m^1 - m^2 \qquad\quad &= q^3 - \tfrac{1}{3\varphi} \\ -m^1 \qquad\quad + m^3 &= q^2 - \tfrac{1}{3\varphi} \\ + m^2 - m^3 &= q^1 - \tfrac{1}{3\varphi} \end{aligned}$$

has one (and only one) solution given $(q^1, q^2, q^3, y)$. $\square$

University of California, Davis

ECN200, General Equilibrium Theory

LN9: Externalities, Equilibrium and Efficiency

The FFTWE makes only one explicit assumption—that all consumers have locally non-satiated preferences. Implicitly, though, the theory makes others: that anonymous and perfectly competitive markets are functioning for each one of the commodities; and that each individual's consumption affects only her well-being. We now concentrate on the latter assumption and study the extreme extent to which the conclusion of the theorem depends on it.

## 1. Nash-Walras Equilibrium

Consider an exchange economy where individual preferences are defined over $\mathbb{R}_+^{LI}$ and can be represented by utility functions $u^i : \mathbb{R}_+^{LI} \to \mathbb{R}$, so that each individual's utility depends (potentially) on the whole allocation of consumption bundles, $\vec{x} = (x^1, \dots, x^I)$. To keep the notation short, one usually denotes $u^i(\vec{x}) = u^i(x^i, x^{\neg i})$, where

$$x^{\neg i} = (x^1, \dots, x^{i-1}, x^{i+1}, \dots x^I)$$

is the sub-profile of consumption bundles of all the members of society other than $i$.

In this case, the definition of competitive equilibrium must take a position on how each individual understands the determination of the others' consumption bundles. The most common definition is to assume that she takes those other bundles as given.

Definition 1. *A competitive, or Nash-Walras, equilibrium for this economy is a pair consisting of a vector of prices and an allocation, $(p, \vec{x})$, such that*

*i. for each $i$, bundle $x^i$ solves the problem*

$$\max_x \left\{ u^i(x, x^{\neg i}) : p \cdot x \leqslant p \cdot w^i \right\};$$

*ii. markets clear: $\sum_i x^i = \sum_i w^i$.*

The definition of Pareto efficiency is as before: a feasible allocation is Pareto efficient if there does not exist another feasible allocation where at least one individual is better off and none is worse off.

The main message of this note is that the conclusion of the FFTWE fails in the presence of externalities, and quite badly: in almost all economies with externalities, all equilibrium allocations are inefficient.

## 2. The General Case

Consider again the general case, and suppose that $(p, \vec{x}, \vec{\lambda})$ is a competitive equilibrium for the economy with endowments $\vec{w}$. If the allocation is interior, the first-order conditions for individual rationality require that

$$D_{x^i} u^i(\vec{x}) = \lambda^i p \tag{1}$$

for each $i$.

On the other hand, if $\vec{x}$ is Pareto efficient it must solve

$$\max_{\hat{x}} \left\{ u^1(\hat{x}) \mid u^i(\hat{x}) \geqslant u^i(\vec{x}), \text{ for all } i \geqslant 2, \text{ and } \textstyle\sum_i \hat{x}^i = \sum_i w^i \right\}.$$

Writing the Lagrangean of this problem as

$$\mathcal{L}(\hat{x}, \mu, \nu) = u^1(\hat{x}) + \sum_{i \geqslant 2} \mu^i [u^i(\hat{x}) - u^i(\vec{x})] + \nu \cdot \sum_i (w^i - \hat{x}^i),$$

and assuming again interiority, we can find the necessary first-order conditions for efficiency: that for each $i$

$$\sum_{i'} \mu^{i'} D_{x^i} u^{i'}(\vec{x}) = \nu, \tag{2}$$

where $\mu^1 = 1$ is chosen to simplify the notation.

If the competitive equilibrium allocation is efficient, we can use Eq. (1) to re-write Eq. (2) as

$$\mu^i \lambda^i p + \sum_{i' \neq i} \mu^{i'} D_{x^i} u^{i'}(\vec{x}) = \nu \tag{3}$$

for each $i$.

In the case of no externalities, $D_{x^i} u^{i'}(x) = 0$ when $i \neq i'$, and Eq. (3) has obvious solutions in $\vec{\mu}$ and $\nu$. Under non-trivial externalities, the system is extremely unlikely to have a solution, as long as $L \geqslant 2$ and $I \geqslant 2$, since it has $LI$ equations and only $L + I - 1$ variables.

## 3. Separability: A simple case for illustration

Unlike before, a full analysis of the failure of the FFTWE would require that we formalize the idea of "almost all profiles of preferences *and endowments*," which is not easy. Instead, we illustrate the result and (part of) the required technique in a simple setting: a two-person exchange economy with $L$ commodities, where consumption of the first commodity affects not just the person who is demanding it, but also the other individual (in a separable manner).

That is, suppose that each individual's utility is $u^i : \mathbb{R}^{L+1} \to \mathbb{R}$, given by

$$u^i(x^i, x_1^{\neg i}) = v^i(x^i) + \alpha^i x_1^{\neg i},$$

where $\neg i$ is used to denote the person other than $i$, and $v^i : \mathbb{R}^L_+$ satisfies all the assumption of smooth exchange economies. For this economy, the following function yields the extended approach to equilibrium analysis: with prices normalized to $\mathcal{N}_1$, define

$$F(p, x^1, x^2, \lambda^1, \lambda^2, w^1, w^2) = \begin{pmatrix} Dv^1(x^1) - \lambda^1 p \\ p \cdot (w^1 - x^1) \\ Dv^2(x^2) - \lambda^2 p \\ p \cdot (w^2 - x^2) \\ \tilde{x}^1 + \tilde{x}^2 - \tilde{w}^1 - \tilde{w}^2 \end{pmatrix}.$$

Since the external effects are separable, it follows that the roots of $F$ are the competitive equilibria of the economy, at the corresponding endowments.

Our goal is to prove that generically on preferences and endowments, all competitive equilibrium allocations of the economy are Pareto inefficient. Before proving this, we need to introduce a technical lemma.

Lemma 1. *Let $(p, x^1, x^2, \lambda^1, \lambda^2)$ be a competitive equilibrium for the economy with endowments $(w^1, w^2)$, and suppose that the allocation $(x^1, x^2) \gg 0$ is Pareto efficient. Then, it must be true that $\alpha^1 \lambda^2 = \alpha^2 \lambda^1$.*

*Proof.* Allocation $(x^1, x^2) \gg 0$ is Pareto efficient only if it solves the problem

$$\max_{\hat{x}^1, \hat{x}^2} \left\{ u^1(\hat{x}^1, \hat{x}^2_1) \mid u^2(\hat{x}^2, \hat{x}^1_1) \geqslant u^2(x^2, x^1_1) \text{ and } \hat{x}^1 + \hat{x}^2 = w^1 + w^2 \right\}.$$

A necessary condition for this is that for some $\mu > 0$ and $\nu \gg 0$,

$$Dv^1(x^1) + \mu \begin{pmatrix} \alpha^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha^1 \\ 0 \end{pmatrix} + \mu Dv^2(x^2) = \nu, \tag{4}$$

which implies that

$$\frac{\partial v^1}{\partial x^1_1}(x^1) + \mu\alpha^2 = \mu \frac{\partial v^2}{\partial x^2_1}(x^2) + \alpha^1 \text{ and } \frac{\partial v^1}{\partial x^1_2}(x^1) = \mu \frac{\partial v^2}{\partial x^2_2}(x^2). \tag{5}$$

(This simply means that the marginal rates of substitution, properly measured to consider external effects, should be equalized across the two individuals.)

At an interior competitive equilibrium, $Dv^1(x^1) = \lambda^1 p$ and $Dv^2(x^2) = \lambda^2 p$. Substitution into Eq. (5) gives

$$\lambda^1 + \mu\alpha^2 = \mu\lambda^2 + \alpha^1 \text{ and } \lambda^1 = \mu\lambda^2$$

using the fact that $p_1 = 1$. □

Theorem 1. *Except on a negligible set of profiles $(w^1, \alpha^1, w^2, \alpha^2)$ all competitive equilibrium allocations of the economy are Pareto inefficient.*

*Proof.* Define the function

$$G(p, x^1, x^2, \lambda^1, \lambda^2, w^1, w^2, \alpha^1, \alpha^2) = \begin{pmatrix} F(p, x^1, x^2, \lambda^1, \lambda^2, w^1, w^2, \alpha^1, \alpha^2) \\ \alpha^1\lambda^2 - \alpha^2\lambda^1 \end{pmatrix},$$

and note that, for some row vector $\delta$,

$$DG = \begin{pmatrix} D_{x,\lambda,p,w}F & D_\alpha F \\ \delta & (\lambda^2, -\lambda^1) \end{pmatrix} = \begin{pmatrix} D_{x,\lambda,p,w}F & 0 \\ \delta & (\lambda^2, -\lambda^1) \end{pmatrix},$$

where the arguments of functions $F$ and $G$ are left implicit.

We already know that when $F = 0$ matrix $D_{x,\lambda,p,w}F$ has full row rank. This implies, since $\lambda^2 \neq 0$, that so does matrix $DG$, when $G = 0$. In other words, $G \pitchfork 0$.

By the transversal density theorem, we have that $G(\cdot, w^1, \alpha^1, w^2, \alpha^2) \pitchfork 0$ except on a negligible subset on $(w^1, \alpha^1, w^2, \alpha^2)$. This means that, generically, when $G = 0$ matrix $D_{x,\lambda,p}G$

has full row rank. But this is impossible since this latter matrix has one more row than it has columns, so transversality requires that, generically, the equality $G = 0$ be impossible.

Explicitly, the previous result means that for almost all values $(w^1, \alpha^1, w^2, \alpha^2)$, if

$$F(p, x^1, x^2, \lambda^1, \lambda^2, w^1, w^2) = 0,$$

then $\alpha^1\lambda^2 \neq \alpha^2\lambda^1$. By the previous lemma, the latter implies the result. $\square$

## 4. Solutions

For the sake of simplicity, maintain the setting and assumption of the previous section.

### *4.1. The old approach: Pigouvian taxation*

Imagine that the government introduces the following fiscal policy: for each unit of commodity 1 that she buys, consumer $i$ is levied a tax of $\tau^i$; and there is a lump-sum transfer of $m^i$ given to each $i$.

Definition 2. *Given the fiscal policy, a competitive equilibrium for this economy is a pair consisting of a vector of prices and an allocation,* $(p, x^1, x^2)$ *such that*

*i. for each* $i$*, bundle* $x^i$ *solves the problem*

$$\max_x \left\{ u^i(x, x^{\neg i}) : p \cdot x + \tau^i x_1 \leqslant p \cdot w^i + m^i \right\};$$

*ii. markets clear:* $\sum_i x^i = \sum_i w^i$*; and*

*iii. the fiscal policy is balanced:* $m^1 + m^2 = \tau^1 x_1^1 + \tau^2 x_1^2$.

Theorem 2. *Suppose that allocation* $(\hat{x}^1, \hat{x}^2) \gg 0$ *is Pareto efficient. There exists a fiscal policy for which a competitive equilibrium allocation is* $(\hat{x}^1, \hat{x}^2)$.

*Proof.* Since the allocation is interior and efficient, it satisfies Eq. (4). Let the fiscal policy be taxes $\tau^1 = -\mu\alpha^2$ and $\tau^2 = -\alpha^1$, and transfers $m^1$ and $m^2$ yet to be determined; let prices be $p = \nu$.

For individual rationality, at the right transfers, it suffices to show that $\hat{x}^i$ satisfies the first-order conditions of the optimization problem, given $\hat{x}^{\neg i}$, prices $p$ and tax $\tau^i$. By direct computation, those conditions are that

$$Dv^i(x^i) = \lambda^i \left( p + \begin{pmatrix} \tau^i \\ 0 \end{pmatrix} \right) = \lambda^i \left( \nu + \begin{pmatrix} \alpha^{\neg i} \\ 0 \end{pmatrix} \right)$$

which occurs at $x^i = \hat{x}^i$, by Eq. (4), with $\lambda^1 = 1$ and $\lambda^2 = 1/\mu$. Of course, the transfers have to be chosen such that bundle $\hat{x}^i$ is just affordable for $i$:

$$m^i = p \cdot \hat{x}^i + \tau^i \hat{x}^i_1 - p \cdot w^i.$$

This construction guarantees individual rationality, whereas market clearing is given by feasibility of the allocation. It only remains to show that the policy is fiscally balanced. Indeed,

$$m^1 + m^2 = \tau^1 \hat{x}^1_1 + \tau^2 \hat{x}^2_1 - p \cdot (w^1 + w^2 - \hat{x}^1 - \hat{x}^2) = \tau^1 \hat{x}^1_1 + \tau^2 \hat{x}^2_1.$$

by market clearing. □

### 4.2. *The Coasian solution: property rights*

Alternatively, suppose that the government creates two more "goods". For each unit of commodity 1 that individual $i$ wants to consume, she must buy one unit of consumption permits from individual $\neg i$. The properties are non-substitutable and each $i$ owns an infinite amount of permits that she can sell to $\neg i$. The price of the consumption permits is determined competitively in markets, so the government's only role is the enforcement of the requirement of permits.

Definition 3. *Given the allocation of property rights, a competitive equilibrium with consumption permits for this economy is a pair consisting of a vector of prices and an allocation,* $[(p, q^1, q^2), (x^1, z^1), (x^2, z^2)]$ *such that*

*i. for each* $i$, *bundle* $(x^i, z^i, Z^i)$ *solves the problem*

$$\max_{x,z,Z} \left\{ u^i(x, Z) : p \cdot x + q^{\neg i} z \leqslant p \cdot w^i + q^i Z \ \textit{and} \ x_1 = z \right\};$$

*ii. goods markets clear:* $\sum_i x^i = \sum_i w^i$*; and*

iii. *permits markets clear:* $z^1 = Z^2$ *and* $z^2 = Z^1$.

THEOREM 3. *Suppose that both individuals have locally non-satiated preference. If*

$$[(p, q^1, q^2), (x^1, z^1), (x^2, z^2)]$$

*is a competitive equilibrium given the property rights, then allocation* $(x^1, x^2)$ *is efficient.*

*Proof.* Let $(\hat{x}^1, \hat{x}^2)$ be Pareto superior to $(x^1, x^2)$, even though $[(p, q^1, q^2), (x^1, z^1), (x^2, z^2)]$ is a competitive equilibrium. By local non-satiation, for both individuals

$$p \cdot \hat{x}^i + q^{\neg i}\hat{x}_1^i \geqslant p \cdot w^i + q^i \hat{x}_1^{\neg i},$$

and the inequality is strict for at least one of them. Adding,

$$p \cdot (\hat{x}^i + \hat{x}^2) + q^2\hat{x}_1^1 + q^1\hat{x}_1^2 > p \cdot (w^1 + w^2) + q^1\hat{x}_1^2 + q^2\hat{x}_1^1,$$

or

$$p \cdot (\hat{x}^i + \hat{x}^2) > p \cdot (w^1 + w^2),$$

which means that the superior allocation $(\hat{x}^1, \hat{x}^2)$ cannot be feasible. $\square$