



## Recent advances on testability in economic equilibrium models

Andrés Carvajal\*

University of California Davis, United States of America  
EPGE-FGV, Brazil

### ARTICLE INFO

Manuscript handled by Editor S Takahashi

#### Keywords:

Refutability  
Testability  
Revealed preferences  
Empirical implications

### ABSTRACT

The revealed preference program started by Paul Samuelson brought a Popperian view of what constituted true scientific discovery to economic theory.<sup>1</sup> In his *Foundations*, Samuelson states that “meaningful theorems” are hypotheses “about empirical data which could conceivably be refuted”.<sup>2</sup> He worried that by appealing to unobservable fundamentals such as preferences, the body of economic theory built upon the premise of rational individual choice lacked a scientific foundation.

While the program quickly succeeded in providing an empirical counterpart for the individual choice problem, its application to social outcomes, namely market interactions and games, proved more difficult, and only in the 1990s did the profession start to develop the testable implications of equilibrium concepts in these contexts. Carvajal et al. (2004) covered the early literature that followed the seminal contributions of Brown and Matzkin (1996) and Sprumont (2000). This survey is an update on the state of that literature covering the results of the last two decades. As the reader will see, the *Journal of Mathematical Economics* has played a key role in developing these ideas.

The reader is encouraged to read the previous survey published by this journal twenty years ago, Carvajal et al. (2004). Many of the results covered here are developments of others that were surveyed there, and the choice of notation here is largely to maintain consistency with the one used there. The motivation of both surveys is, of course, the same: economic models ought to generate testable implications, and neither the general equilibrium model nor the idea of equilibrium in strategic settings should be exempt from that principle.

Carvajal et al. (2004) arranged its exposition in two main lines. First it presented the literature on the extent to which individual rationality imposes structure on the equilibrium manifold of an exchange economy. The canonical results of Brown and Matzkin (1996) on the testable implications of rationality on market data were presented, followed by the literature on the identification of preferences from the equilibrium manifold.<sup>3</sup> The motivation for this literature were the seemingly negative implications of the celebrated Sonnenschein–Mantel–Debreu (SMD) theorem.

The second line of results in Carvajal et al. (2004) dealt with the empirical implications of game-theoretical equilibrium concepts.<sup>4</sup> While there is no analogous in game theory to the *smd* theorem, the literature

was motivated by the Popperian paradigm that Samuelson invoked for economic theory. The main results covered there were Sprumont (2000) and Ray and Zhou (2001): the hypothesis of Nash equilibrium is refutable in (finite) games under the assumption that the analyst has observed a comprehensive data set consisting of all possible equilibrium results in all possible restricted domains of strategic interaction.

This new survey builds upon the previous one and is organized, broadly speaking, along the same two lines. As a form of motivation, we first review the old negative view that the profession had derived from the SMD theorem on the testable restrictions of competitive equilibrium theory, and provide a result that has the same spirit for Nash equilibrium in games. After that, we present recent results divided in four sections: (1) the classical competitive equilibrium model without any market failures; (2) the equilibrium model with externalities and public goods; (3) bargaining and imperfect competition games; and (4) some other game-theoretical settings. This division of topics may seem arbitrary but it follows a presentational purpose. The results of Section 2 follow the logic of Brown and Matzkin (1996) on how to extend revealed preference theorems to settings where individual

\* Correspondence to: University of California Davis, United States of America.  
E-mail address: [acarvajal@ucdavis.edu](mailto:acarvajal@ucdavis.edu).

<sup>1</sup> Samuelson (1938)

<sup>2</sup> See Samuelson (1947), p. 4.

<sup>3</sup> Kübler et al. (2002), Chiappori et al. (2004) and Balasko (2004) were the main results at that point. After that, Carvajal and Riasco (2005) and Carvajal and Riasco (2008) reconcile the earlier results and provide a more general argument.

<sup>4</sup> See Section 4 there.

choices are not observed. These first set of results is made straightforward by the lack of strategic interaction that characterizes the perfect competitive equilibrium setting. Section 3 lifts this assumption and covers results where the revealed preference arguments are extended to allow for strategic interactions in competitive markets. The reason why the purely game-theoretic results are split in two sections is that the results of Section 4 are methodologically close to the ones in Section 3, while the results in Section 5 follow a different logic. The former rely on the first-order characterization of individual rationality, the latter appeal to other types of argument.

For a reader who is completely new to the literature on revealed preferences, it is advisable to review other material beforehand. Many of the results presented here are easier to understand once one is familiar with the basic theory of revealed preferences in individual settings, which is comprehensively surveyed by Chambers and Echenique (2016).<sup>5</sup> The so-called Afriat inequalities of individual revealed preference analysis are particularly amenable to their extension to non-individual settings, and many of the results presented in Sections 2, 3, and 4 will be easier to understand once the reader has a good grasp of the individual case.<sup>6</sup> Nishimura et al. (2017) offer a detailed treatment of Afriat inequalities in settings that go beyond the individual consumer problem where they were initially conceived. The reader will also benefit from the survey on that topic that is part of this special issue of *JME*, Dzielwulski et al. (2024).

Another word of advice may be useful here. For the reader who is interested only on general equilibrium theory without any market failure, it is enough to read Section 1.1 here, Section 2.2 in Carvajal et al. (2004), and then Section 2 here. If externalities and public goods are of interest, Sections 2.5 and 2.6 in Carvajal et al. (2004) should precede Section 3 here. For the reader interested only in game-theoretical applications, the road map starts from Section 1.2; after that, Sections 4 in Carvajal et al. (2004) and 5.4 here study settings in which the analyst has comprehensive data sets; if only partial observation is assumed Section 4 and the rest of Section 5 cover those results—the former includes analysis that relies on first-order conditions, while the latter groups other kinds of logic.

## 1. Motivation

### 1.1. Markets: the SMD theorem

It is well known that the individual rationality of a consumer who takes prices as given imposes strong restrictions on her demand function. The SMD theorem establishes that little of that structure is preserved upon aggregation: essentially every function mapping prices into quantities can be the aggregate excess demand function of a competitive economy, at least for prices that do not become arbitrarily small. To be precise, take any continuous function that maps prices into quantities, is homogeneous of degree 0 and satisfies Walras's law. The theorem states that, at least for prices are bounded below strictly above zero, such an arbitrary function is the excess demand of some economy where there are at least as many consumers as there are commodities, each consumer being represented by a continuous, strictly monotone and strictly quasi-concave utility function.

This result has significant positive implications for the theory of general competitive equilibrium.<sup>7</sup> In addition, it was understood to imply that all the empirical implications of individual rationality disappear at the aggregate level. The theorem itself and the literature that followed were even dubbed the “anything goes” theory.

<sup>5</sup> Readers interested in further results in this literature that cannot be covered in this survey will enjoy the excellent coverage of Chambers and Echenique (2016).

<sup>6</sup> At the very least, see Section 2.1 in Carvajal et al. (2004).

<sup>7</sup> See Toda and Walsh (2024) for a discussion on local uniqueness of equilibrium that is motivated by the SMD theorem.

This pessimistic view of the empirical content of general equilibrium theory was proved to be untenable by Brown and Matzkin (1996), which is surveyed in detail in Carvajal (2003). Intuitively, the flaw was in the interpretation of the result for empirical purposes and is twofold. First, the SMD theorem deals with the aggregate excess demand of the economy, which is an unobservable object under the null hypothesis of competitive equilibrium—except at equilibrium prices, of course, but at those prices the excess demand is defined to be zero. Second, this function captures the response of an endogenous variable (demand) to changes in another endogenous variable (prices).

From an empirical perspective, the testability exercise is based on the response of an observable endogenous variable to the observed perturbation of exogenous variables. The SMD theorem is not such an exercise, and an analyst must rely, for example, on the response of prices to changes in the aggregate wealth of the economy and its distribution, as pointed out by Brown and Kannan (2008). The interested reader is encouraged to go over Section 2.2 in Carvajal et al. (2004) before proceeding to the extensions of Brown and Matzkin (1996) presented here in Sections 2 and 3.

### 1.2. Games: the weakness of individual rationality

The application of revealed preference methods to games has followed two approaches. One assumes that the analyst has observed all the endogenous information resulting from the interaction between players. In simultaneous-move games, this means that they have observed, for instance, all the Nash equilibria of the game; in sequential games, the analyst has observed the strategies of all players, including what they would do if they found themselves off the equilibrium path. The second approach dispenses with the assumption that the observed information is comprehensive: in one-shot games, one equilibrium has been observed; in sequential games, perhaps only the actions over the equilibrium path.<sup>8</sup>

Carvajal et al. (2004) covered in some detail the literature on testability of equilibrium in non-cooperative sequential games.<sup>9</sup> Some literature on cooperative games has been developed, including Agatsuma (2016), which we covered in Section 2.3.1, and de Clippel and Rozen (2022). In what follows, we concentrate on non-cooperative, simultaneous-move games.

When the analyst assumes that everything that is endogenous has been observed, the revealed preference approach is likelier to deliver testable implications, not only because of the wealth of information available for testing but also because the behavioral hypothesis becomes stronger: everything that has been observed must be consistent with the premise, and everything that has not been observed must be inconsistent with it. One can reject a hypothesis when it is impossible to explain all the data available, and also, when explaining all those data would require the inclusion of more observations. In that sense, the second approach, which allows the available data to be partial, is more challenging.

To see the extent of the difference, consider the following setting. Two players,  $a$  and  $b$ , play a game where each simultaneously chooses a number in the interval  $[0, 1]$ . The analyst does not observe the game's payoffs but knows that the pairs  $\{(x_a^r, x_b^r) \mid r = 1, \dots, R\}$  have been played. The hypothesis to be tested is Nash equilibrium.

The weaker version of the hypothesis is that there exist two functions  $U_a, U_b : [0, 1]^2 \rightarrow R$  such that  $(x_a^r, x_b^r)$  is a Nash equilibrium of the game with those preferences, for every  $r$ . The stronger version requires the same, but also that no  $(x_a, x_b)$  not observed in  $\{(x_a^r, x_b^r)\}$  be an equilibrium. Put more explicitly:

<sup>8</sup> That the comprehensiveness of the observed data makes a difference for testing a theory is not unique to multi-agent settings: see Nishimura et al. (2017) and De Clippel and Rozen (2021).

<sup>9</sup> The main result after that review, Ray and Snyder (2013), had already been described there.

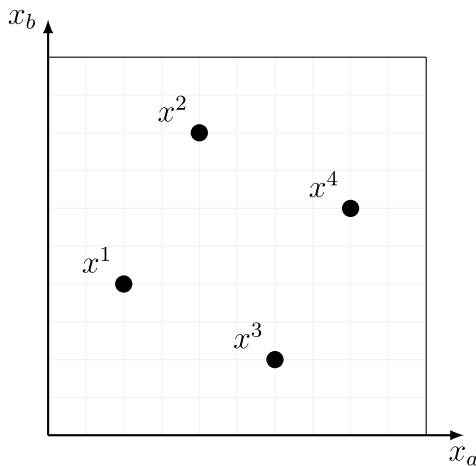


Fig. 1. Four observations in a  $[0, 1] \times [0, 1]$  game.

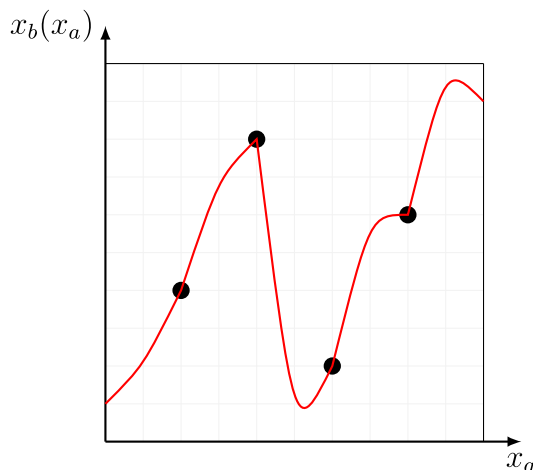


Fig. 2. Best response function for player  $b$ .

**Definition.** The data set  $\{(x_a^r, x_b^r)\}$  is consistent with Nash equilibrium if

$$\forall x \in [0, 1], U_a(x, x_b^r) \leq U_a(x_a^r, x_b^r) \wedge U_b(x_a^r, x) \leq U_b(x_a^r, x_b^r), \tag{1}$$

for all  $r$ . If, in addition, it is true that

$$\begin{aligned} [\forall x \in [0, 1], U_a(x, x_b) \leq U_a(x_a, x_b) \wedge U_b(x_a, x) \leq U_b(x_a, x_b)] \\ \Rightarrow \exists r : (x_a^r, x_b^r) = (x_a, x_b), \end{aligned} \tag{2}$$

the data set consists of all the Nash equilibria of some game.

The first version of the hypothesis is in the spirit of revealed preference analysis. It tests whether each player's choices were individually rational, for some given criterion, every time they were observed to choose. If that hypothesis is rejected, observing new data will not change the conclusion. Unfortunately, the latter is not true of the second version. Under this hypothesis, the analyst may reject because of the second requirement, in which case introducing more observations may suffice to change the conclusion.

On the other hand, the power of the two hypotheses is very different. For the stronger version, Zhou (2006) offers a characterization.

**Theorem 1 (Zhou).** Let the data set  $\{(x_a^r, x_b^r)\}$  be given and suppose that

$$x_a^1 < x_a^2 < \dots < x_a^R, \tag{3}$$

while there is a permutation  $\{\rho(1), \rho(2), \dots, \rho(R)\}$  of  $\{1, \dots, R\}$  such that

$$x_b^{\rho(1)} < x_b^{\rho(2)} < \dots < x_b^{\rho(R)}. \tag{4}$$

The following two statements are equivalent:

1. The data set consists of all the Nash equilibria of some game with continuous utility functions that are quasi-concave in each player's own actions.
2. There exist no  $r, s = 1, \dots, R$  such that

$$\begin{aligned} (a) \min\{x_a^{\rho(s)}, x_a^{\rho(s+1)}\} \leq \min\{x_a^r, x_a^{r+1}\} \leq \max\{x_a^r, x_a^{r+1}\} \leq \\ \max\{x_a^{\rho(s)}, x_a^{\rho(s+1)}\}, \text{ while} \\ (b) \min\{x_b^r, x_b^{r+1}\} \leq \min\{x_b^{\rho(s)}, x_b^{\rho(s+1)}\} \leq \max\{x_b^{\rho(s)}, x_b^{\rho(s+1)}\} \leq \\ \max\{x_b^r, x_b^{r+1}\}. \end{aligned}$$

The intuition of why the second statement is necessary for the first one can be seen graphically. Suppose there are four observations, as in Fig. 1. Note that the data satisfies Eq. (3), while the permutation  $\{\rho(1) = 3, \rho(2) = 1, \rho(3) = 4, \rho(4) = 2\}$  forces Eq. (4).

Note that

$$x_a^1 = \min\{x_a^{\rho(1)}, x_a^{\rho(2)}\} < x_a^2 = \min\{x_a^2, x_a^3\} < x_a^3 = \max\{x_a^2, x_a^3\}$$

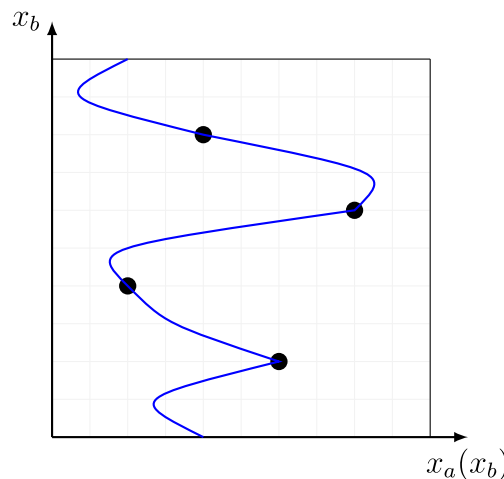


Fig. 3. Best response function for player  $a$ .

$$= \{x_a^{\rho(1)}, x_a^{\rho(2)}\},$$

while

$$\begin{aligned} x_b^3 = \min\{x_b^1, x_b^3\} = \min\{x_b^{\rho(1)}, x_b^{\rho(2)}\} < x_b^1 = \max\{x_b^{\rho(1)}, x_b^{\rho(2)}\} \\ \leq x_b^2 = \max\{x_b^2, x_b^3\}, \end{aligned}$$

so the two conditions in the second statement of Theorem 1 are satisfied. This observation implies that the first statement in that same theorem cannot be true: there do not exist continuous functions  $U_a$  and  $U_b$  that are quasi-concave in each player's own actions and for which Eqs. (1) and (2) are true. To see why this is the case, suppose that those two functions exist. Each would give rise to an upper hemi-continuous and convex-valued best response correspondence for the respective player. Assuming for simplicity that these are functions, Figs. 2 and 3 hypothesize their graphs.<sup>10</sup>

Two conclusions arise if the two best-response graphs are overlapped, as in Fig. 4. The first one is that the two graphs intersect at the four points observed in the data set. This result is the requirement that the data observed by the analyst must be Nash equilibria of the game. The second one is that the best responses must intersect at least once

<sup>10</sup> To be sure, note that Fig. 2 is read as the variable in the vertical axis being a function of the one in the horizontal axis, but in Fig. 3 the dependence is the opposite.

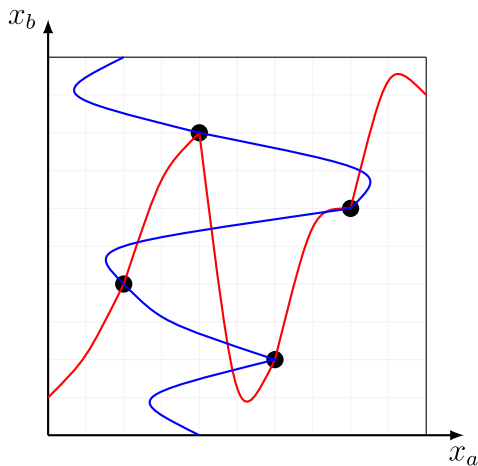


Fig. 4. Possible Nash equilibria of the game.

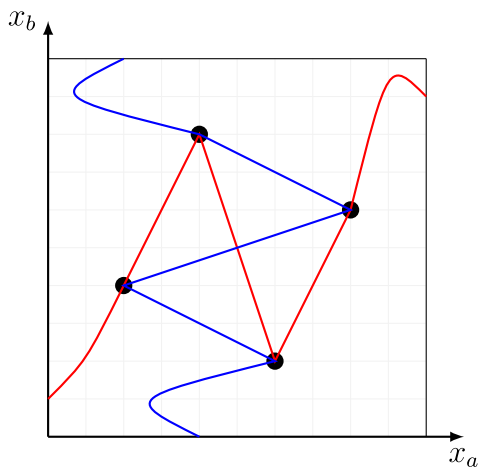


Fig. 5. Possible Nash equilibria of the game.

more. No matter how we draw them, this happens as long as the two functions represent well-defined, continuous best response functions, as in Fig. 5 for example: some intersections can be avoided, but not all. Any extra intersection would be an unobserved equilibrium that allows the analyst to reject the hypothesis that all the equilibria have been observed.

If only the weaker hypothesis that all observed pairs are Nash equilibria is of interest, the data set of Fig. 1 yields no rejection. In fact, Carvajal (2003) shows that essentially no data set does.<sup>11</sup>

**Theorem 2 (Carvajal).** *Let the data set  $\{(x_a^r, x_b^r)\}$  be given and suppose that for all pair of distinct observations  $r, s$ ,  $x_a^r \neq x_a^s$  and  $x_b^r \neq x_b^s$ . Then, the data set is consistent with Nash equilibrium, with continuous utility functions that are strictly quasi-concave in each player's own actions.*

This result is an immediate consequence of a stronger and more general one. Suppose there are  $N$  agents, all allowed to choose actions from  $[0, 1]$ . Suppose that for each observation, in addition to the choice  $x_n^r$  of each agent, the analyst observes a lower and an upper bound that constrained the agent's choice. These bounds are assumed to be exogenous and are denoted by  $\alpha_n^r$  and  $\beta_n^r$ , respectively.

<sup>11</sup> The following result is what I referred to in the introduction as an analogous in game theory of the SMD theorem.

**Definition.** The data set  $\{(\alpha_n^r, x_n^r, \beta_n^r)\}$  is consistent with Nash equilibrium if there exist functions  $(U_n)$  defined on  $[0, 1]^N$  such that

$$\forall x \in [\alpha_n^r, \beta_n^r], U_n(x, x_{-n}^r) \leq U_n(x_n^r, x_{-n}^r) \tag{5}$$

for all  $n$  and  $r$ , where the habitual notation  $x_{-n} = (x_1, \dots, x_{n-1}, x_{n+1}, \dots, x_N)$  is being used.<sup>12</sup>

**Theorem 3 (Carvajal).** *Let the data set  $\{(\alpha_n^r, x_n^r, \beta_n^r)\}$  be given and assume that  $\alpha_n^r \leq x_n^r \leq \beta_n^r$  for all  $n$  and  $r$ . The following two statements are equivalent:*

1. The data set is consistent with Nash equilibrium and the utility functions can be chosen to be continuous and strictly concave in the players' own actions.
2. For all  $r, s$  and all  $n$ ,

- (a)  $x_{-n}^r = x_{-n}^s$ ,  $\alpha_n^s \leq x_n^r \leq \beta_n^s$ , and  $\alpha_n^r \leq x_n^s \leq \beta_n^r$  together imply  $x_n^r = x_n^s$ ;
- (b)  $x_{-n}^r = x_{-n}^s$ ,  $\alpha_n^r < x_n^r$ , and  $x_n^r \geq \beta_n^s$  together imply  $x_n^s = \beta_n^s$ ; and
- (c)  $x_{-n}^r = x_{-n}^s$ ,  $x_n^r < \beta_n^r$  and  $\alpha_n^s \geq x_n^r$  together imply  $x_n^s = \alpha_n^s$ .

In the second statement, the second and third conditions exploit the monotonicity that one-dimensional concave functions satisfy on each side of their local maxima. The first condition is an extension of WARP. Importantly, the three conditions only apply when the remaining players have kept their choices invariant. This requirement is the only premise under which, in the current setting, monotonicity and WARP are implied, because it is the only way to guarantee that agent  $n$  is maximizing the same objective function  $U_n(\cdot, x_{-n}^r) = U_n(\cdot, x_{-n}^s)$  in observations  $r, s$ .

An immediate implication of this theorem is that any data set such that for any pair of observations  $r, s$ , there exist two distinct players  $n, m$  such that  $x_n^r \neq x_n^s$  and  $x_m^s \neq x_m^r$  is consistent with Nash equilibrium, with preferences satisfying continuity and strict concavity in own action.<sup>13</sup>

## 2. Market interactions

### 2.1. Exchange economies

Consider a pure exchange economy with  $N$  agents, each with preferences representable by a non-satiated utility function  $U_n$  over  $L$  commodities. Each agent has an initial endowment  $\omega_n \in \mathbb{R}_{++}^L$ , so a feasible allocation is a profile of consumption vectors  $(x_n)_{n=1}^N$ , with  $x_n \in \mathbb{R}_+^L$ , such that  $\sum_{n=1}^N x_n = \sum_{n=1}^N \omega_n$ . A competitive equilibrium is a feasible allocation and prices such that each individual's observed bundle maximizes her utility subject to the budget constraint implied by the observed prices and individual endowment.<sup>14</sup>

Let a data set consisting of  $R$  observations,  $\{p^r, (\omega_n^r)\}_{r=1}^R$ , be given.<sup>15</sup> These data are consistent with the competitive equilibrium model if there are utility functions such that each  $p^r$  is an equilibrium price vector for the exchange economy with the preferences represented by those utility functions and the endowments  $(\omega_n^r)$ .

<sup>12</sup> Note that condition (5) generalizes Eq. (1). In words, this condition requires that for each  $r$ , the profile of actions  $(x_n^r)$  be a Nash equilibrium of the game where the agents have preferences  $(U_n)$  and are restricted to play over the domain  $\times_n [\alpha_n^r, \beta_n^r]$ .

<sup>13</sup> In fact, Carvajal (2003) shows that differentiability, concavity in all actions, and even local uniqueness of equilibrium impose no further implications than the three conditions in the second statement of the theorem.

<sup>14</sup> For the rest of the paper,  $(x_n)$  will be understood to mean  $(x_n)_{n=1}^N$ , and expressions like  $\sum_{n=1}^N x_n$  will be written simply as  $\sum_n x_n$ .

<sup>15</sup> In what follows, sets like  $\{p^r\}_{r=1}^R$  will be written simply as  $\{p^r\}$ .

**Definition.** The data set  $\{p^r, (\omega_n^r)\}$  is consistent with competitive equilibrium if there exist utility functions  $(U_n)$  for which, for each  $r$ , there exist individual demands  $(x_n^r)$  such that: (a) each  $x_n^r$  maximizes  $U_n(x)$  subject to the budget constraint  $p^r \cdot x \leq p^r \cdot \omega_n^r$ , and (b) all markets clear:  $\sum_n x_n^r = \sum_n \omega_n^r$ .

Brown and Matzkin (1996) characterize the class of data sets that are consistent with competitive equilibrium and for which the utility functions can be restricted to be continuous, monotone, and quasi-concave.

The canonical result is the conjunction of Theorems 1 and 2 in Brown and Matzkin (1996).<sup>16</sup> It is based on the well-known Afriat characterization of individual rationality of a competitive consumer and consists of a finite set of polynomial inequalities.<sup>17</sup>

Being a non-linear system, however, the implementation of the Brown–Matzkin test is potentially challenging for large data sets: Cherchye et al. (2011) shows that the rationalizability problem is NP-complete, and proposes an alternative characterization that replaces the non-linear inequalities in the system by an equivalent system of linear inequalities in which one of the variables can only take the values 0 and 1.

The critical step in the “mixed integer programming” approach of Cherchye et al. (2011) is to provide an extra characterization of the Afriat inequalities. For the following result, suppose that there is only one individual,<sup>18</sup> and assume that individual consumption bundles  $\{x^r\}$  have been observed. The third statement in the following theorem introduces the ancillary variable  $\rho^{r,s} \in \{0, 1\}$ , and is thus an integer programming characterization.

**Theorem 4 (Afriat, Varian, Cherchye et al.).** Let the individual data set  $\{p^r, x^r\}$ , with  $p^r \gg 0$  at all  $r$ , be given. The following five statements are equivalent:

1. There exists a non-satiated utility function  $U$  such that each  $x^r$  maximizes  $U(x)$  subject to the budget constraint  $p^r \cdot x \leq p^r \cdot x^r$ .
2. The data satisfy the generalized axiom of revealed preference, GARP:  $p^r \cdot x^r \geq p^r \cdot x^s$ ,  $p^s \cdot x^s \geq p^s \cdot x^t$ , ..., and  $p^u \cdot x^u \geq p^u \cdot x^v$  together imply  $p^v \cdot x^v \leq p^v \cdot x^r$ .
3. There exist integer numbers  $\rho^{r,s} \in \{0, 1\}$ ,  $r, s = 1, \dots, R$ ,  $s \neq r$ , that satisfy the consumer system integer inequalities, csi:

$$\begin{aligned} p^r \cdot x^r - p^r \cdot x^s &< \rho^{r,s} \bar{W} \\ p^r \cdot x^r - p^r \cdot x^s &\leq (1 - \rho^{s,r}) \bar{W} \\ \rho^{r,s} + \rho^{s,t} &\leq 1 + \rho^{r,t} \end{aligned} \tag{6}$$

for all distinct  $r, s, t$ , where  $\bar{W} = \max_{r \in \{1, \dots, R\}} \{p^r \cdot x^r + 1\}$ .

4. There exist numbers  $u^r, \lambda^r > 0$  that satisfy the Afriat inequalities:

$$u^s - u^r + \lambda^r p^r \cdot (x^s - x^r) \leq 0,$$

for all  $r, s$ .

5. The utility function  $U$  in statement 1 can be chosen to be continuous, monotone, and concave.

**Proof.** In order to highlight the contribution of Cherchye et al. (2011), we prove the equivalence between statements 2 and 3.

We first argue that statement 2 implies 3 by constructing a solution to csi. Define  $\rho^{r,s} = 1$  if  $p^r \cdot x^t \leq p^r \cdot x^r$ ,  $p^r \cdot x^u \leq p^r \cdot x^t$ , ..., and  $p^v \cdot x^s \leq p^v \cdot x^v$  are possible. If no such sequence exists, say that  $\rho^{r,s} = 0$ .

Note first that if  $p^r \cdot x^r - p^r \cdot x^s \geq 0$ , then, by construction,  $\rho^{r,s} = 1$ , and, by definition of  $\bar{W}$ ,

$$p^r \cdot x^r - p^r \cdot x^s < \rho^{r,s} \cdot \bar{W}. \tag{7}$$

If, alternatively,  $p^r \cdot x^r - p^r \cdot x^s < 0$ , the latter equation is obvious.

Suppose, on the other hand, that  $\rho^{s,r} = 1$ . This means that there is a sequence for which  $p^s \cdot x^t \leq p^s \cdot x^s$ ,  $p^r \cdot x^u \leq p^r \cdot x^t$ , ..., and  $p^v \cdot x^r \leq p^v \cdot x^v$ . By GARP, then  $p^r \cdot x^r - p^r \cdot x^s \leq 0$ , which implies that

$$p^r \cdot x^r - p^r \cdot x^s \leq (1 - \rho^{s,r}) \cdot \bar{W}. \tag{8}$$

If  $\rho^{s,r} = 0$ , the same inequality is, again, obvious.

These last two observations, Eqs. (7) and (8), yield the first two inequalities in (6). As for the third inequality, note that it suffices to show that if  $\rho^{r,s} = \rho^{s,t} = 1$ , then  $\rho^{r,t} = 1$ . To see this, suppose there exist sequences such that  $p^r \cdot x^u \leq p^r \cdot x^r$ ,  $p^u \cdot x^v \leq p^u \cdot x^u$ , ..., and  $p^{w'} \cdot x^s \leq p^{w'} \cdot x^{w'}$ , while  $p^s \cdot x^{u'} \leq p^s \cdot x^s$ ,  $p^{v'} \cdot x^{u'} \leq p^{v'} \cdot x^{u'}$ , ..., and  $p^{w''} \cdot x^t \leq p^{w''} \cdot x^{w''}$ . Concatenating these sequences as  $(r, u, v, \dots, w, s, u', v', \dots, w', t)$ , we get, immediately, that  $\rho^{r,t} = 1$

To show that statement 3 implies statement 2, take a solution  $(\rho^{r,s})$  to the csi system. Suppose that  $p^r \cdot x^r \geq p^r \cdot x^s$ ,  $p^s \cdot x^s \geq p^s \cdot x^t$ , ..., and  $p^u \cdot x^u \geq p^u \cdot x^v$ . The first inequality in Eq. (6) implies  $\rho^{r,s} = \rho^{s,t} = \dots = \rho^{u,v} = 1$ . Using the third inequality recursively, this implies that  $\rho^{r,v} = 1$ . By the second inequality,  $p^v \cdot x^v \leq p^v \cdot x^r$ .  $\square$

This result implies the following extension of the Brown–Matzkin theorem:

**Theorem 5 (Brown and Matzkin, Cherchye et al.).** Let the data set  $\{p^r, (\omega_n^r)\}_{r=1}^R$  be given. The following three statements are equivalent:

1. There exists a profile of continuous, monotone, and concave utility functions that rationalize the data.
2. There exist numbers  $u_n^r, \lambda_n^r > 0$ , and vectors  $x_n^r \geq 0$  such that all markets clear, and the budget constraints and Afriat inequalities are satisfied:

- (a)  $\sum_n x_n^r = \sum_n \omega_n^r$ , for all  $r$ ;
- (b)  $p^r \cdot x_n^r = p^r \cdot \omega_n^r$ , for all  $r$  and  $n$ ;
- (c)  $u_n^s - u_n^r - \lambda_n^r p^r \cdot (x_n^s - x_n^r) \leq 0$ , for all  $r, s$  and  $n$ .

3. There exist integer numbers  $\rho_n^{r,s} \in \{0, 1\}$  and vectors  $x_n^r \geq 0$  such that conditions (2a) and (2b) are satisfied, as well as the csi inequalities:

$$\begin{aligned} p^r \cdot x_n^r - p^r \cdot x_n^s &\leq \min \{ \rho_n^{r,s}, 1 - \rho_n^{s,r} \} \cdot \bar{W}_n \text{ and } \rho_n^{r,s} + \rho_n^{s,t} \leq 1 + \rho_n^{r,t}, \\ \text{for all } r, s, t \text{ and } n, \text{ with } \bar{W}_n &= \max_r \{ p^r \cdot x_n^r + 1 \}. \end{aligned}$$

What changes in this new approach is how an analyst determines the existence of a solution to the rationalizability systems. In Brown and Matzkin (1996), the process invokes the Tarski-Seidenberg quantifier elimination theorem to eliminate, successively, the unobservable variables that appear in the system, which runs in doubly exponential time.<sup>19</sup> The MIP approach of Cherchye et al. (2011) makes no use of the Tarski-Seidenberg theorem, which immediately makes it more amenable to generalization and is an accepted approach to handle NP-complete problems. In fact, Cherchye et al. (2011) apply their test of a data set of US regions, with  $N = 8$  and  $R = 12$ . This data set would be large for the Tarski-Seidenberg approach, but Cherchye et al. (2011) implements it efficiently for the original hypothesis of equilibrium and some extensions of it.

Of course, the refutability conclusion that follows from the example of a non-rationalizable data set given by Brown and Matzkin (1996) continues to apply in the context of Cherchye et al. (2011): the system in statement 3 of Theorem 5 has no solution for the data that give rise to Figure 1 in Carvajal et al. (2004). Instead of repeating that example here, we extend it to the case of a production economy, which will be helpful later on.

<sup>16</sup> They appeared as Theorems 3 and 4 in Carvajal et al. (2004).

<sup>17</sup> See Brown and Kannan (2008) for two algorithms that implement these inequalities.

<sup>18</sup> So the sub-index  $n$  is omitted.

<sup>19</sup> The algorithms in Brown and Kannan (2008) provide an efficiency gain, as they are only exponential.

## 2.2. Production economies

The central insight of [Brown and Matzkin \(1996\)](#) is that there exists a tension between the principle of individual rationality and the requirement of market clearing in the presence of non-negativity constraints on consumption. Given the observed Edgeworth boxes and the individuals' budget constraints, also observed, finding feasible allocations where all agents satisfy GARP may be impossible. That tension was argued for the case of exchange economies, which leaves the question of whether the competitive equilibrium hypothesis is refutable in a production economy. The question is valid since production implies that neither the consumption Edgeworth boxes nor the individual constraints are observed unless each firm's production plan is available in the data set.

Suppose that in addition to prices and profiles of individual endowments, one observes the individual shares on the only firm in the economy,  $(\theta_n)$ . Say that the data is *rationalizable* if there exist a profile of individual utility functions  $(U_n)$  and an (aggregate) technology set  $F \subseteq \mathbb{R}^L$  such that, for each observation  $r$ , the observed prices  $p^r$  are competitive equilibrium prices for the corresponding endowments, the preferences and the technology.

**Definition.** The data set  $\{p^r, (\omega_n^r, \theta_n^r)\}$  is *consistent with competitive equilibrium in an aggregate production economy* if there exist utility functions  $(U_n)$  and a technology set  $F \subseteq \mathbb{R}^L$  for which, for each  $r$ , there exist individual consumptions  $(x_n^r)$  and a production plan  $X^r$  such that: (a) each  $x_n^r$  maximizes  $U_n(x)$  subject to the budget constraint  $p^r \cdot x \leq p^r \cdot (\omega_n^r + \theta_n^r X^r)$ ; (b)  $X^r$  maximizes  $p^r \cdot X$  subject to the technological constraint  $X \in F$ ; and (c)  $\sum_n x_n^r + X^r = \sum_n \omega_n^r$ .

### 2.2.1. Constant returns to scale

Suppose one has observed an individual production data set  $\{p^r, X^r\}$ . [Varian \(1984\)](#) proves that there exists a non-empty, convex technology,  $F$ , displaying constant returns to scale and free disposal and such that  $p^r \cdot X \leq p^r \cdot X^r$ , for all  $X \in F$

(9)

if, and only if,  $p^r \cdot X^s \leq p^r \cdot X^r = 0$  for all observations  $r$  and  $s$ . The following result is immediate:

**Theorem 6.** Let the data set  $\{p^r, (\omega_n^r)\}$  be given. The following two statements are equivalent:

1. The data is consistent with competitive equilibrium in an aggregate production economy, with utility functions that are continuous, monotone, and concave, and a non-empty, convex technology displaying constant returns to scale and free disposal.
2. There exist numbers  $u_n^r, \lambda_n^r > 0$ , and vectors  $x_n^r \geq 0$  and  $X^r$  such that:

- (a)  $\sum_n x_n^r = \sum_n \omega_n^r + X^r$ , for all  $r$ ;
- (b)  $p^r \cdot x_n^r = p^r \cdot \omega_n^r$ , for all  $r$  and  $n$ ;
- (c)  $u_n^s - u_n^r - \lambda_n^r p^r \cdot (x_n^s - x_n^r) \leq 0$ , for all  $r, s$  and  $n$ ; and
- (d)  $p^r \cdot X^s \leq p^r \cdot X^r = 0$  for all  $r$  and  $s$ .

The system in the second statement is the extension of the system proposed by [Brown and Matzkin \(1996\)](#), though it can also be written in terms of the MIP approach of [Cherchye et al. \(2011\)](#). The market clearing condition is modified to account for the firm's output, and the axioms for profit maximization proposed by [Varian \(1984\)](#) are included. More subtly, note that the budget constraints of the individuals remain unchanged because we assume that technology  $F$  displays constant returns to scale, so  $p^r \cdot Y^r = 0$  at all  $r$ . This is why the observation of property shares  $(\theta_n)$  is immaterial for this result.

Suppose that one has observed the data set that gives rise to the two Edgeworth boxes in [Fig. 6](#). Since maximized profits are null, the budget constraints are observed to be  $p^r \cdot x_n^r \leq p^r \cdot \omega_n^r$  and  $p^s \cdot x_n^s \leq p^s \cdot \omega_n^s$ . Since  $\{(p^r, x_1^r), (p^s, x_1^s)\}$  must satisfy WARP, it follows that (at least) one of the

two bundles must be unaffordable at the prices of the other observation. With no loss of generality, we can assume that  $x_1^r$  must lie in the thicker portion of that observation's budget line in [Fig. 7](#) to guarantee that  $p^s \cdot x_1^r > p^s \cdot \omega_1^s$ .

The non-negativity constraint of individual 2 implies that the most she can offer to individual 1 at observation 1 is given by point C in [Fig. 8](#). The rest of individual 1's consumption at that observation must come from the firm's output, which means that  $X^r$  must be on the thicker portion of the  $p^r \cdot x = 0$  locus in [Fig. 8](#). But, then,  $p^s \cdot X^r > 0$ , which contradicts condition (2d) of the rationalizability system. The data set is thus impossible to rationalize.

### 2.2.2. Unobserved profits

Under constant returns to scale, profit maximization implies zero profits for the firm, and individual budgets are observed. If only no free lunch is assumed, individual budgets are not observed, so a non-rationalizable example is more challenging and interesting.

First, one needs to revise [Theorem 6](#), as condition (2d) in that system is neither necessary nor sufficient for the existence of the rationalizing technology. To do this, one needs to strengthen the results of [Varian \(1984\)](#) as follows: there exists a no free lunch technology  $F$  such that Eq. (9) holds at all  $r$  if, and only if,  $p^r \cdot X^s \leq p^r \cdot X^r$  at all  $r$  and  $s$ , and for some  $\rho \in \mathbb{R}_{++}^L$ , one has  $\rho \cdot X^r \leq 0$  at all  $r$ .<sup>20</sup> Then, one observes that, remarkably, the same data set used by [Brown and Matzkin \(1996\)](#), and in the exercise with constant returns to scale above, will fail to be rationalizable in the present setting. For the sake of definiteness, consider a specific numeric example: at one observation,  $\omega_1^r = (9, 1)$ ,  $\omega_2^r = (1, 1)$ , and  $p^r = (100, 1)$ ; at another,  $\omega_1^s = (1, 9)$ ,  $\omega_2^s = (1, 1)$ , and  $p^s = (1, 100)$ ; the observed ownership of the firm is  $\theta_1 = 1$ , and  $\theta_2 = 0$ .

If these data are rationalizable, then there exist vectors  $(x_1^r, x_2^r, X^r)$  and  $(x_1^s, x_2^s, X^s)$  such that:

$$p^r \cdot X^r =: \pi^r \geq p^r \cdot X^s, \quad (10)$$

$$p^s \cdot X^s =: \pi^s \geq p^s \cdot X^r, \quad (11)$$

$$(p^s \cdot x_1^r \leq p^s \cdot \omega_1^s + \pi^s \wedge p^r \cdot x_1^s \leq p^r \cdot \omega_1^r + \pi^r) \Rightarrow x_1^r = x_1^s, \quad (12)$$

$$p^r \cdot x_1^r = p^r \cdot \omega_1^r + \pi^r, \quad (13)$$

$$p^s \cdot x_1^s = p^s \cdot \omega_1^s + \pi^s, \quad (14)$$

$$p^r \cdot x_2^r = p^r \cdot \omega_2^r, \quad (15)$$

$$p^s \cdot x_2^s = p^s \cdot \omega_2^s, \quad (16)$$

$$x_1^r + x_2^r = \omega_1^r + \omega_2^r + X^r, \quad (17)$$

$$x_1^s + x_2^s = \omega_1^s + \omega_2^s + X^s, \quad (18)$$

$$x_1^r, x_2^r \geq 0, \quad (19)$$

$$x_1^s, x_2^s \geq 0, \quad (20)$$

$$X_1^r > 0 \Rightarrow X_2^r < 0. \quad (21)$$

Eqs. (10) and (11) are WAPM, which the firm's decisions must obey. Eq. (12) is WARP for individual 1. Eqs. (13) to (16) are the budget

<sup>20</sup> The proof of this equivalence is as follows. For necessity, the first condition is obvious. Now, suppose that for all  $\rho \in \mathbb{R}_{++}^L$ , there is some  $r$  such that  $\rho \cdot X^r > 0$ . Then, the system  $\rho \cdot X^1 \leq 0, \dots, \rho \cdot X^R \leq 0, \rho \cdot (-e^1) < 0, \dots, \rho \cdot (-e^L) < 0$ , where each  $e^\ell$  is the  $\ell$ th canonical vector in  $\mathbb{R}^L$ , has no solution (on  $\rho$ ). It follows from Farkas's lemma that there exist  $(\alpha^r) \in \mathbb{R}_+^R$  and  $(\beta_\ell) \in \mathbb{R}_+^L \setminus \{0\}$  such that  $\sum_r \alpha^r X^r = \sum_\ell \beta_\ell e^\ell > 0$ . Then,  $(\alpha^r) > 0$  and, by convexity,  $X = (\sum_r \alpha^r)^{-1} \sum_r \alpha^r X^r \in F$ . But  $X > 0$ , which contradicts no free lunch.

For sufficiency, as in [Varian \(1984\)](#), let  $F$  be the convex hull of set  $\cup_r (\{X^r\} - \mathbb{R}_+^L)$ , which is nonempty, closed and convex. That Eq. (9) holds follows from [Theorem 2](#) in [Varian \(1984\)](#). To see that  $F$  satisfies no free lunch, suppose that  $X \in F$  and  $X > 0$ . By construction, there exists  $(\tilde{X}^r) \in \times_r (\{X^r\} - \mathbb{R}_+^L)$  and  $(\alpha^r) \in \mathbb{R}_+^R$  such that  $\sum_r \alpha^r = 1$  and  $\sum_r \alpha^r \tilde{X}^r = X$ . Since  $\rho \in \mathbb{R}_{++}^L$ ,  $0 < \rho \cdot X = \rho \cdot \sum_r \alpha^r \tilde{X}^r \leq \rho \cdot \sum_r \alpha^r X^r = \sum_r \alpha^r (\rho \cdot X^r) \leq 0$ .

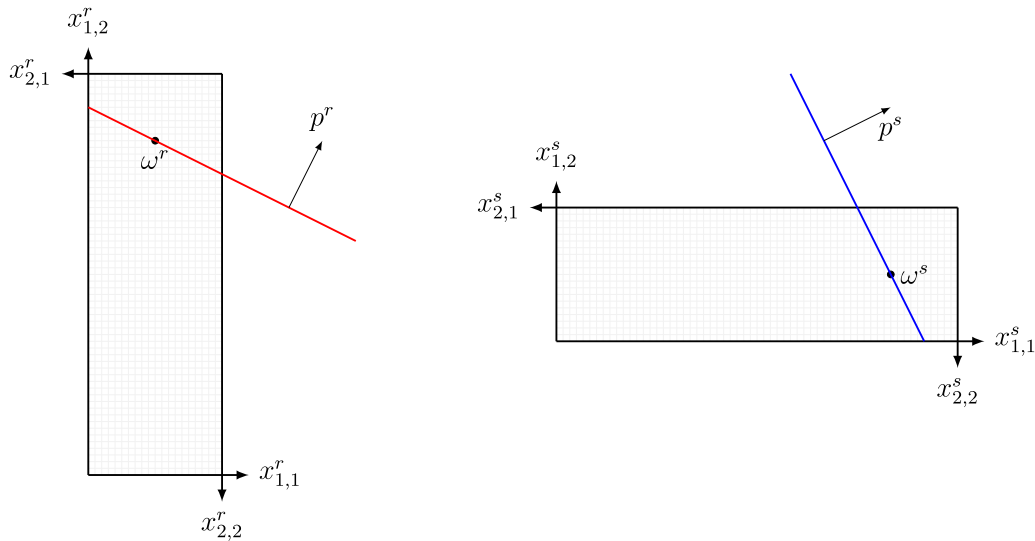


Fig. 6. Two observations.

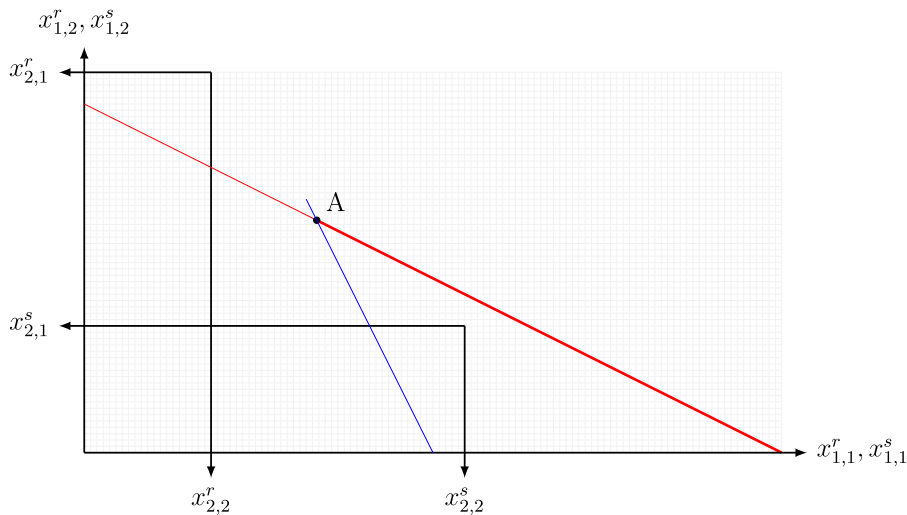


Fig. 7. Overlapping the two observations: WARP for individual 1.

constraints, binding, for both individuals at both observations. Eqs. (17) and (18) are the market clearing conditions, while (19) and (20) are the non-negativity constraints in consumption. Critically, Eq. (21) imposes no-free-lunch on the firm’s technology.

After some algebra that we relegate to an Appendix, the following claims are obtained:

**Claim 1.**  $x_1^r \geq x_1^s \geq -2$  and  $x_2^s \geq x_2^r \geq -2$ .

**Claim 2.**  $p^s \cdot x_1^r < p^s \cdot \omega_1^s + \pi^s$  and  $p^r \cdot x_1^s < p^r \cdot \omega_1^r + \pi^r$ .

**Claim 3.**  $x_1^r \neq x_1^s$ .

Claim 1 finds bounds on the firm’s output. With these bounds in place, Claims 2 and 3 imply that WARP is inconsistent with the rest of the conditions and, hence, that the data are not consistent with competitive equilibrium under the assumptions imposed on the preferences and technology.

### 2.3. Welfare analysis

#### 2.3.1. General mechanisms

So far, the analysis has focused on testing equilibrium hypotheses under market mechanisms. The literature has also considered testing other hypotheses, chiefly efficiency, under non-market mechanisms.

In the context of allocating  $L$  private commodities only, suppose that one observes the individual endowments  $(\omega_n^r)$  and individual consumptions  $(x_n^r)$  at each observation.

**Definition.** The data set  $\{(\omega_n^r, x_n^r)\}$  is consistent with efficiency and individual rationality if there exist a profile of preferences  $(U_n)$  and profiles of Pareto weights  $\{(\mu_n^r)\} \gg 0$  such that each observed  $(x_n^r)$  maximizes the weighted social utility  $\sum_n \mu_n^r U_n(x_n^r)$  subject to the individual rationality constraints that  $U_n(x_n^r) \geq U_n(\omega_n^r)$  for all  $n$ , and  $\sum_n x_n^r = \sum_n \omega_n^r$ .

As this hypothesis amounts to a single optimization problem, it can be characterized through suitable Afriat inequalities, as demonstrated by Bachmann (2006):

**Theorem 7 (Bachmann).** Let a data set  $\{(\omega_n^r, x_n^r)\}$  be given. The following two statements are equivalent:

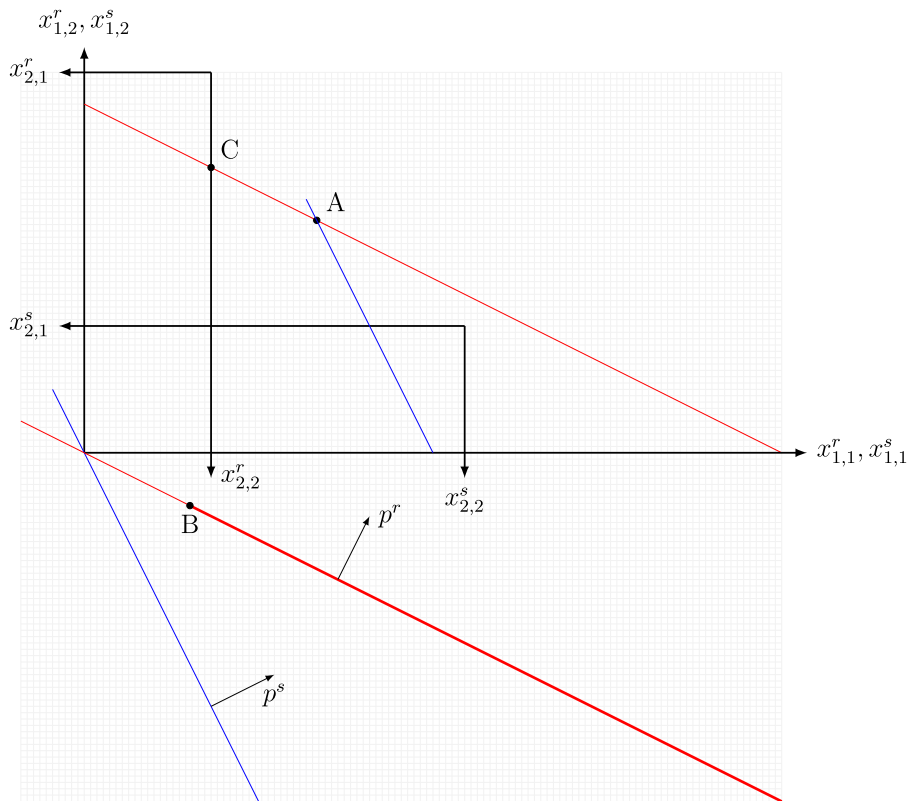


Fig. 8. Conflict between WARP and profit maximization. (Here, the distance from (0,0) to B is the same as from C to A.)

1. The data set is consistent with efficiency and individual rationality, with individual preferences that are differentiable, strictly monotone, and concave.<sup>21</sup>
2. There exist numbers  $u_n^r, v_n^r, \mu_n^r > 0$  and vectors  $\delta_n^r, \lambda^r \gg 0$  that solve the following system:

- (a)  $u_n^s - u_n^r - \frac{1}{\mu_n^r} \lambda^r \cdot (x_n^s - x_n^r) \leq 0$ , for all  $r, s$  and  $n$ ;
- (b)  $v_n^s - v_n^r - \delta_n^r \cdot (\omega_n^s - \omega_n^r) \leq 0$ , for all  $r, s$  and  $n$ ;
- (c)  $v_n^s - u_n^r - \frac{1}{\mu_n^r} \lambda^r \cdot (\omega_n^s - x_n^r) \leq 0$  and  $u_n^r - v_n^s - \delta_n^s \cdot (x_n^r - \omega_n^s) \leq 0$ , for all  $r, s$  and  $n$ ;
- (d)  $\delta_n^s = \delta_n^r$ , if  $\omega_n^s = \omega_n^r$ , for all  $r, s$  and  $n$ ;
- (e)  $\delta_n^s = \frac{1}{\mu_n^r} \lambda^r$ , if  $\omega_n^s = x_n^r$ , for all  $r, s$  and  $n$ ;
- (f)  $u_n^r \geq v_n^r$ , for all  $r$  and  $n$ ; and
- (g)  $\sum_n x_n = \sum_n \omega_n^r$  at all  $r$ .

In this setting, the numbers  $u_n^r$  and  $v_n^r$  represent, respectively, agent  $n$ 's utility levels at  $x_n^r$  and  $\omega_n^r$ ;  $\lambda^r$  is the vector of Lagrange multipliers for the aggregate feasibility constraints;  $\delta_n^r$  is the gradient of  $n$ 's utility function at  $\omega_n^r$ ; and  $\mu_n^r$  is the sum of the agent's Pareto weight and her individual rationality's Lagrange multiplier. The fact that the allocation maximizes the weighted sum of utilities implies that the product of  $\mu_n^r$  and the gradient of  $n$ 's utility function at  $x_n^r$  equals the vector of Lagrange multipliers for the aggregate feasibility constraints.

Assuming that there is one commodity that enters linearly in all the agents' utility functions, Agatsuma (2016) refines this analysis to the case of the core of the economy. That is, suppose that the data set is of the form  $\{(\omega_n^r, \kappa_n^r, x_n^r, y_n^r)\}$ , where in addition to the individual consumption and endowments of  $L$  goods, the analyst also observes the endowments,  $\kappa_n^r$ , and holdings,  $y_n^r$ , of a numéraire commodity. The

assumption is that the preferences of individual  $n$  are represented by the mapping  $(x, y) \mapsto U_n(x) + y$ . Under this assumption, the economy is one of transferable utility (TU). Denoting by  $\mathcal{P}$  the collection of all non-empty subsets of agents, with typical elements  $C, D$ , say that the data set is consistent with the core of a TU economy if there are preferences such no coalition of individuals can be made better off by isolating themselves from the rest of the society and reallocating their aggregate resources internally.

**Definition.** The data set  $\{(\omega_n^r, \kappa_n^r, x_n^r, y_n^r)\}$  is consistent with the core of a TU economy if there exist functions  $(U_n)$  such that for every observation  $r$  and every coalition  $C \in \mathcal{P}$ ,

$$\sum_{n \in C} [U_n(x_n^r) + y_n^r] \geq \max_{(x_n)_{n \in C}} \{ \sum_{n \in C} U_n(x_n) : \sum_{n \in C} (x_n - \omega_n^r) \leq 0 \} + \sum_{n \in C} \kappa_n^r. \quad (22)$$

An extension of Theorem 7 is as follows.

**Theorem 8 (Agatsuma).** Let a data set  $\{(\omega_n^r, \kappa_n^r, x_n^r, y_n^r)\}$  be given. The following two statements are equivalent:

1. The data set is consistent with the core of a TU economy, with individual functions that are differentiable, strictly monotone, and concave.
2. There exist numbers  $u_n^r, \tilde{u}_{C,n}^r > 0$ , vectors  $\delta_n^r, \tilde{\delta}_{C,n}^r \gg 0$ , and vectors  $\tilde{x}_{C,n}^r \geq 0$  that solve the following system:

- (a)  $u_n^s - u_n^r - \delta_n^r \cdot (x_n^s - x_n^r) \leq 0$ , for all  $r, s$  and  $n$ ;
- (b)  $\tilde{u}_{D,n}^s - \tilde{u}_{C,n}^r - \tilde{\delta}_{C,n}^r \cdot (\tilde{x}_{D,n}^s - \tilde{x}_{C,n}^r) \leq 0$ , for all  $r, s, C, D$  and  $n \in C \cap D$ ;
- (c)  $\tilde{u}_{C,n}^s - u_n^r - \delta_n^r \cdot (\tilde{x}_{C,n}^s - x_n^r) \leq 0$  and  $u_n^r - \tilde{u}_{C,n}^s - \tilde{\delta}_{C,n}^s \cdot (x_n^r - \tilde{x}_{C,n}^s) \leq 0$ , for all  $r, s, C$  and  $n \in C$ ;
- (d)  $\tilde{\delta}_{D,n}^s = \tilde{\delta}_{C,n}^r$ , if  $\tilde{x}_{D,n}^s = \tilde{x}_{C,n}^r$ , for all  $r, s, C, D$  and  $n \in C \cap D$ ;
- (e)  $\tilde{\delta}_{C,n}^s = \delta_n^r$ , if  $\tilde{x}_{C,n}^s = x_n^r$ , for all  $r, s, C$  and  $n \in C$ ;

<sup>21</sup> Bachmann (2006) requires strict concavity, which implies that the inequalities below be strict when the terms in parenthesis are non-zero. We weaken the requirement to just concavity, for ease of presentation.

- (f)  $\sum_{n \in C} \bar{x}_{C,n}^r \leq \sum_{n \in C} \omega_n^r$ , for all  $r$  and  $C$ ; and
- (g)  $\sum_{n \in C} (\bar{u}_{C,n}^r + \kappa_n^r) \leq \sum_{n \in C} (u_n^r + y_n^r)$ , for all  $r$ .

The intuition for the new variables appearing in this theorem is as follows.  $(\bar{x}_{C,n}^r)_{n \in C}$  is a solution to the maximization problem on the right-hand side of Eq. (22). Condition (2f) guarantees the feasibility of this solution.  $\bar{u}_{C,n}^r$  and  $\bar{\delta}_{C,n}^r$  are, respectively, the value and gradient of  $U_n$  at the point  $\bar{x}_{C,n}^r$ . Individual by individual, conditions (2a) to (2e) are the Afriat inequalities corresponding to the observed points  $\{x_n^r\}$  and the counter-factual points  $\{\bar{x}_{C,n}^r\}$ . The remaining condition, (2g), tests Eq. (22) itself.

### 2.3.2. Market allocations

Chambers and Echenique (2021) studies a related question. Suppose that the data  $\{p^r, (x_n^r)\}$  is generated by competitive behavior in a market, and one wants to use the information obtained from that data to test the hypothesis that, for the same underlying preferences, the allocation  $(\bar{x}_n)$  is efficient.<sup>22</sup> Chambers and Echenique (2021) answer this question using the revealed-preferences relations that can be inferred from the data set. We can obtain the following equivalent result using the Afriat inequalities approach in the same fashion as Bachmann (2006).

**Theorem 9.** *Let the data set  $\{p^r, (x_n^r)\}$  and the profile of consumption bundles  $(\bar{x}_n) \gg 0$  be given. The following two statements are equivalent:*

1. *There exists a profile of differentiable, strictly monotone, and concave utility functions  $(U_n)$  for which: (a) each  $x_n^r$  maximizes  $U_n(x)$  subject to the budget constraint  $p^r \cdot x \leq p^r \cdot x_n^r$ , and (b) the profile  $(\bar{x}_n)$  is efficient in the economy with preferences  $(U_n)$  and aggregate endowment  $\sum_n \bar{x}_n$ .*
2. *There exist numbers  $u_n^r, v_n, \lambda_n^r, \mu_n > 0$  and vectors  $\delta_n \gg 0$  that solve the following system:*

- (a)  $u_n^s - u_n^r - \lambda_n^r p^r \cdot (x_n^s - x_n^r) \leq 0$ , for all  $r, s$  and  $n$ ;
- (b)  $v_n - u_n^r - \lambda_n^r p^r \cdot (\bar{x}_n - x_n^r) \leq 0$  and  $u_n^r - v_n - \delta_n \cdot (x_n^r - \bar{x}_n) \leq 0$ , for all  $r$  and  $n$ ;
- (c)  $\lambda_n^r p^r = \lambda_n^s p^s$  if  $x_n^r = x_n^s$ , for all  $r, s$  and  $n$ ;
- (d)  $\lambda_n^r p^r = \delta_n$  if  $x_n^r = \bar{x}_n$ , for all  $r$  and  $n$ ; and
- (e)  $\mu_n \delta_n = \mu_m \delta_m$ , for all  $n, m$ .

**Proof.** For necessity, since preferences are smooth, we can let  $\lambda_n^r$  be  $n$ 's Lagrange multiplier at observation  $r$ , which satisfies  $\lambda_n^r p^r = \nabla U_n(x_n^r)$ , while  $\delta_n = \nabla U_n(\bar{x}_n)$ . The first three conditions in the system of statement 2 follow from strict concavity of  $U_n$ , and the next two are by definition. Only the last condition, (2e), requires an argument: since the utility functions are concave and the allocation is efficient, it must maximize the function  $\sum_n \mu_n U_n(x_n)$ , subject to  $\sum_n x_n = \sum_n \bar{x}_n$ , for some profile  $(\mu_n) \gg 0$  of Pareto weights. The first-order conditions of this program imply the condition.

For sufficiency, as usual, let the individual utility functions be given by

$$U_n(x) = \min \left\{ u_n^r + \lambda_n^r p^r \cdot (x - x_n^r), v_n + \delta_n (x - \bar{x}_n) \right\},$$

which are continuous, concave, and strictly monotone. By the conditions, they further satisfy that  $U_n(x_n^r) = u_n^r$  and  $U_n(\bar{x}_n) = v_n$ , for all  $r$  and  $n$ .

Notice that if  $p^r \cdot x \leq p^r \cdot x_n^r$ , then  $U_n(x) \leq u_n^r + \lambda_n^r p^r \cdot (x - x_n^r) \leq u_n^r = U_n(x_n^r)$ , since  $\lambda_n^r > 0$ . This inequality gives us requirement (1a) in the first statement. As for requirement (1b), notice that if  $(x_r)$  is Pareto superior,

$$\sum_n \mu_n v_n = \sum_n \mu_n U_n(\bar{x}_n) < \sum_n \mu_n U_n(x_n) \leq \sum_n \mu_n [v_n + \delta_n (x_n - \bar{x}_n)]$$

<sup>22</sup> In the sense that there is no Pareto superior allocation  $(x_n)$  with  $\sum_n x_n = \sum_n \bar{x}_n$ .

It follows that  $\sum_n \mu_n \delta_n (x_n - \bar{x}_n) > 0$ , which, using condition (2e), implies that  $\mu_1 \delta_1 \sum_n (x_n - \bar{x}_n) > 0$ . Then,  $\sum_n x_n \neq \sum_n \bar{x}_n$ , making the superior allocation unfeasible.

Of course, these utility functions are neither strictly monotone nor differentiable, but small local perturbations and convolutions can be used to guarantee these conditions.<sup>23</sup>  $\square$

## 3. Public economics

In this section, I consider the setting of competitive economies with externalities or public goods. The interactions between the agents in these economies are game theoretical, so the results lie at the intersection between general equilibrium theory and game theory.

### 3.1. Externalities

An application of this theory that has received attention in the literature is in the case of public economics. Carvajal et al. (2004) already reported some negative results on extending the Brown–Matzkin test to economies with externalities (see Theorem 9 there). These results were refined by Carvajal (2010) to obtain a refutable hypothesis by requiring minimal degrees of functional separability between each agent's consumption decisions and the external effects imposed by other agents.

Suppose there are  $N \geq 2$  consumers and  $L + 1$  commodities, the last of which is an externality in the sense that each individual is affected not only by their consumption of it but also by the consumption of each of the other consumers. Formally, the utility of individual  $n$  is  $U_n(x_n, y_n, y_{-n})$  with the usual meaning:  $x_n$  is the individual's consumption of the first  $L$  commodities,  $y_n$  is her consumption of the externality, and  $y_{-n} = (y_m)_{m \neq n}$  is the profile of the other agents' demands of the externality.

Staying, for simplicity, in the context of an exchange economy, let the endowments of the externality be  $(\kappa_n)$ , all of which are assumed to be strictly positive. Denoting by  $q$  the price of the last commodity, a pair  $(p, q)$  is a vector of Nash-Walras equilibrium prices if there exist individual demands for all the commodities that maximize each individual's utility, given the prices and the demands of others, and clear all markets. In this setting, the data set is consistent with competitive equilibrium if there exist utility functions such that each  $(p^r, q^r)$  is a vector of Nash-Walras equilibrium prices for the economy with those preferences and endowments  $(w_n^r, \kappa_n^r)$ .

**Definition.** The data set  $\{p^r, q^r, (\omega_n^r, \kappa_n^r)\}$  is consistent with competitive equilibrium in an economy with externalities if there exist utility functions  $(U_n)$  for which, for each  $r$ , there exist individual demands  $(x_n^r, y_n^r)$  such that: (a) each  $(x_n^r, y_n^r)$  maximizes  $U_n(x, y, y_{-n})$  subject to the budget constraint  $p^r \cdot x + q^r y \leq p^r \cdot \omega_n^r + q^r \cdot \kappa_n^r$ ; and (b)  $\sum_n (x_n^r, y_n^r) = \sum_n (\omega_n^r, \kappa_n^r)$ .

Theorem 9 in Carvajal et al. (2004) showed that this hypothesis of rationalizability is non-refutable unless the data set further includes some information on the individual demands for the externality, or further conditions are imposed. Carvajal (2010) first shows that, surprisingly, any data set is rationalizable by a profile of continuous, concave, strictly monotone preferences that satisfy strategic complementarity.<sup>24</sup> Importantly, though, refutability is restored if one restricts attention to utility functions that are weakly separable in the sense that they can be written as  $U_n(V_n(x_n), y_n, y_{-n})$ , for a strictly monotone function  $V_n$  and a function  $U_n$  that is strictly monotone in its first two arguments.

**Theorem 10 (Carvajal).** *Let the data set  $\{p^r, q^r, (\omega_n^r, \kappa_n^r)\}$  be given.*

<sup>23</sup> See Chiappori and Rochet (1987) for how to get differentiable functions, and Matzkin and Richter (1991) for how to make them strictly concave.

<sup>24</sup> See Proposition 2 in Carvajal (2010).

1. If the set is consistent with competitive equilibrium in an economy with externalities and the utility functions are weakly separable and satisfy differentiability, concavity, and strict monotonicity in own consumption, then there exist numbers  $y_n^r, \lambda_n^r, \mu_n^r > 0$  and vectors  $x_n^r \gg 0$  such that:  $p^r \cdot x_n^r + q^r y_n^r = p^r \cdot \omega_n^r + q^r \kappa_n^r$ ,

$$u_n^s - u_n^r - \frac{\mu_n^r}{\lambda_n^r} (v_n^s - v_n^r) - \mu_n^r q^r (y_n^s - y_n^r) \leq 0, \text{ whenever } y_{-n}^s = y_{-n}^r, \quad (23)$$

and

$$v_n^s - v_n^r - \lambda_n^r p^r \cdot (x_n^s - x_n^r) \leq 0, \quad (24)$$

for every  $r, s$  and  $n$ , while  $\sum_n (x_n^r, y_n^r) = \sum_n (\omega_n^r, \kappa_n^r)$  for every  $r$ .

2. If the system of equations in the previous statement has a solution, the data set is consistent with competitive equilibrium in an economy with externalities and the utility functions can be constructed to satisfy weak separability, concavity, and strict monotonicity.

This theorem makes the problem of externalities clear: the presence of external effects weakens the hypothesis of individually rational choice, as evidenced by Eq. (23)—this “overall” Afriat inequality is necessary only when the consumption of the externality by every other agent in the economy remains unchanged. Eq. (24), on the other hand, shows how functional separability restores the power of the hypothesis: this Afriat inequality, which tests the rationality of the individual consumption choices on the goods other than the externality, is unconditional on the other agents’ choices.

Deb (2009) follows a different approach. Keeping the context and notation we just introduced,<sup>25</sup> a profile of utility functions ( $U_n$ ) is said to have a potential if there exists a function  $V(x_1, \dots, x_N, y_1, \dots, y_N)$  such that

$$U_n(x_n, y_n, y_{-n}) \geq U_n(x_n', y_n', y_{-n}) \Leftrightarrow V(x_n, x_{-n}, y_n, y_{-n}) \geq V(x_n', x_{-n}, y_n', y_{-n})$$

for all  $n$ , all pairs of consumption bundles  $(x_n, y_n)$  and  $(x_n', y_n')$ , and all profiles of demands by other consumers  $(x_{-n}, y_{-n})$ . The usefulness of this assumption is that if such a potential function exists and is concave, an allocation of the economy with those preferences and some given endowments is a Nash-Walras equilibrium if it maximizes the potential function, given the feasibility constraints implied by the endowments.<sup>26</sup> The potential function is not observable, but its existence implies an “aggregate” version of the Afriat inequalities.<sup>27</sup>

**Theorem 11 (Deb).** Let the data set  $\{p^r, q^r, (\omega_n^r, \kappa_n^r)\}$  be given. The following two statements are equivalent:

1. The data set is consistent with competitive equilibrium in an economy with externalities and the utility functions have a satisfy differentiable, concave, and strongly monotone potential.
2. There exist numbers  $v^r, \lambda_n^r > 0$  such that:  $p^r \cdot x_n^r + q^r y_n^r = p^r \cdot \omega_n^r + q^r \kappa_n^r$  for every  $n$  and  $r$ , while

$$v^s - v^r - \sum_n \lambda_n^r [p^r \cdot (x_n^s - x_n^r) + q^r (y_n^s - y_n^r)] \leq 0 \quad (25)$$

for every  $r, s$ , and  $\sum_n (x_n^r, y_n^r) = \sum_n (\omega_n^r, \kappa_n^r)$  for every  $r$ .

<sup>25</sup> Which is less general than the one studied by Deb (2009).

<sup>26</sup> See Slade (1994) and Monderer and Shapley (1996) for the earliest development of the theory of potential games.

<sup>27</sup> In general, suppose there are  $N$  agents, each of them allowed to choose actions from  $X_n$  and having payoffs  $U_n : X \rightarrow \mathbb{R}$ , where  $X = \prod_n X_n$ . This game has a potential if there exists a function  $V : X \rightarrow \mathbb{R}$  such that for all  $n$  and all  $x_{-n} \in \prod_{m \neq n} X_m$ ,

$$V(x, x_{-n}) \geq V(x', x_{-n}) \Leftrightarrow U_n(x, x_{-n}) \geq U_n(x', x_{-n}),$$

for all  $x, x' \in X_n$ . A data set  $\{(x_n^r)\}$  is consistent with Nash equilibrium in a game that admits a concave potential if, and only if, there exist numbers  $\{v^r\}$  and vectors  $\{(\delta_n^r)\}$  such that  $v^s - v^r - \sum_n \delta_n^r \cdot (x_n^s - x_n^r) \leq 0$  for all  $r$  and  $s$ . The reason why this observation is particularly interesting in the externalities setting considered by Deb (2009) is the structure imposed on the vectors  $\{(\delta_n^r)\}$  by the observation of prices.

As mentioned before, requiring the existence of a potential restores rationalizability since the same potential function must be maximized at all observations. This feature materializes itself in the previous theorem in Eq. (25), which, unlike Eqs. (23) and (24), holds for the aggregate of the economy and not individual by individual.<sup>28</sup>

### 3.2. Public goods

Importantly, when the external effects in the economy come from the existence of public goods, the competitive equilibrium hypothesis remains refutable, as demonstrated by Carvajal (2010). Moreover, the test can be written in MIP form (Carvajal and Song, 2018).

Maintaining the notation of the previous section, we now interpret the last commodity as a public good, so each individual’s utility is  $U_n(x_n, \sum_m y_m)$ . For the problem to be interesting, we assume that there is a firm with technology  $F \subseteq \mathbb{R}^{L+1}$  and observed ownership structure  $(\theta_n)$ . Suppose that the data set consists of all prices, individual endowments, and the firm’s production plans  $(X^r, Y^r)$ . The data set is consistent with competitive equilibrium provision of public goods if there exist a profile of utility functions and a technology such that at each observation, the observed prices and production choices are part of a Nash-Walras equilibrium given the corresponding endowments, the individual preferences, and the technology.

**Definition.** The data set  $\{p^r, q^r, (\omega_n^r, \kappa_n^r, \theta_n), X^r, Y^r\}$  is consistent with competitive equilibrium provision of public goods if there exist utility functions  $(U_n)$  and a technology  $F$  such that, for all  $r$ : (a)  $(X^r, Y^r)$  maximizes  $p^r \cdot X + q^r Y$  subject to the technological constraint that  $(X, Y) \in F$ ; and (b) there exists a profile of individual demands  $(x_n^r, y_n^r)$  such that  $\sum_n (x_n^r, y_n^r) = \sum_n (\omega_n^r, \kappa_n^r) + (X^r, Y^r)$  and each  $(x_n^r, y_n^r)$  maximizes  $U_n(x, y + \sum_{m \neq n} y_m^r)$ , subject to the constraint that  $p^r \cdot x + q^r y \leq p^r \cdot \omega_n^r + q^r \kappa_n^r + \theta_n(p^r \cdot X^r + q^r Y^r)$ .

Focusing on the case of constant returns to scale in production, we can simplify notation by writing the nominal wealth of each individual as  $W_n^r = p^r \cdot \omega_n^r + q^r \cdot \kappa_n^r$ , which is observed. Also, denote  $\bar{W}_n = \max_r \{W_n^r\} + 1$ .

**Theorem 12 (Carvajal, Carvajal and Song).** Let the data set  $\{p^r, q^r, (\omega_n^r, \kappa_n^r, \theta_n), X^r, Y^r\}$  be given. The following three statements are equivalent:

1. The data set is consistent with competitive equilibrium provision of public goods, with a profile of continuous and monotone preferences and a constant returns to scale technology.
2. There exists a solution with vectors  $(x_n^r, y_n^r) \geq 0$  and numbers  $u_n^r, \lambda_n^r > 0$  to the following system:

- (a)  $\sum_n x_n^r = \sum_n \omega_n^r + X^r$  and  $\sum_n y_n^r = \sum_n \kappa_n^r + Y^r$ , for all  $r$ ;
- (b)  $p^r \cdot X^r + q^r Y^r = 0$  and  $p^r \cdot X^s + q^r Y^s \leq 0$ , for all  $r$  and  $s$ ;
- (c)  $p^r \cdot x_n^r + q^r \cdot y_n^r = W_n^r$ , for all  $r$  and  $n$ ; and
- (d)  $u_n^s \leq u_n^r + \lambda_n^r [p^r \cdot (x_n^s - x_n^r) + q^r \max\{\sum_m y_m^s - \sum_m y_m^r, -y_n^r\}]$ , for all  $r, s$  and  $n$ .

3. There exists a solution with vectors  $(x_n^r, y_n^r) \geq 0$  and numbers  $\rho_n^{r,s}, \alpha_n^{r,s}, \beta_n^{r,s} \in \{0, 1\}$  to the system consisting of conditions (2a), (2b), (2c), and the following conditions instead of (2d):

- (e)  $p^r \cdot (x_n^s - x_n^r) + q^r \sum_m (y_m^s - y_m^r) + [\rho_n^{r,s} - 2(\alpha_n^{r,s} - 1)] \cdot \bar{W}_n > 0$ , for all  $r, s$  and  $n$ ;
- (f)  $p^r \cdot (x_n^s - x_n^r) + q^r \sum_m (y_m^s - y_m^r) + [1 - \rho_n^{r,s} - 2(\alpha_n^{r,s} - 1)] \cdot \bar{W}_n \geq 0$ , for all  $r, s$  and  $n$ ;
- (g)  $p^r \cdot (x_n^s - x_n^r) + q^r y_n^s + [\rho_n^{r,s} - 2(\beta_n^{r,s} - 1)] \cdot \bar{W}_n > 0$ , for all  $r, s$  and  $n$ ;

<sup>28</sup> The fact that the numbers  $v^r$  are not indexed by  $n$  is not a typo: these are values of the potential function, not the individual utility functions.

- (h)  $p^r \cdot (x_n^s - x_n^r) + q^r y_n^r + [1 - \rho_n^{s,r} - 2(\beta_n^{r,s} - 1)] \cdot \bar{W}_n \geq 0$ , for all  $r, s$  and  $n$ ;
- (i)  $\alpha_n^{r,s} + \beta_n^{r,s} = 1$ , for all  $r, s$  and  $n$ ;
- (j)  $\rho_n^{r,s} + \rho_n^{s,t} \leq 1 + \rho_n^{r,t}$ , for all  $r, s, t$  and  $n$ ;

In this theorem, statement 2 comes from Carvajal (2010). The first three conditions are straightforward: market clearing, profit maximization, and Walras's law. The fourth condition is an extension of the standard Afriat inequalities to the case of public goods, which uses the results of Forges and Minelli (2009). Note that this system is non-linear not just because of the products of (unobservable) variables, as in Brown and Matzkin (1996) but also because of the presence of the max operator in condition (2d).

Statement 3 extends the idea of Cherchye et al. (2011) to use the dichotomic variable  $\rho_n^{r,s}$ , but also introduces two more such variables,  $\alpha_n^{r,s}$  and  $\beta_n^{r,s}$ , to deal with the max operator. The resulting system is a linear MIP.

Carvajal and Song (2018) further shows that if individual demands for the public good are part of the data set, the test can be strengthened to the hypothesis that there are "warm-glow effects" in the sense that each individual's utility function is of the form  $U_n(x_n, y_n, \sum_m y_m)$ . Under the assumption that there is only one private good, if the data set also contains information on the individual demands for this commodity, Deb et al. (2014) obtain testable implications for even more general functions that allow for other-regarding preferences and donations among individuals:  $U_n(x_n, (x_m), y_n, (y_m))$ .

### 3.3. Welfare

Before, we considered tests of competitive behavior in economies with public goods and efficiency in markets without external effects. Carvajal et al. (2004) already covered earlier literature on testing the hypothesis efficiency in public good economies, namely the work of Snyder (1999). In the same fashion as other early literature, those results relied on the Tasrki-Seidenberg theorem and thus obtained algorithms that were difficult to implement. As with their test of competitive behavior, Carvajal and Song (2018) derived an MIP test for the hypothesis that the allocations of a public goods economy are efficient.

Recall the setting of Section 3.2, and assume again that the analyst has observed all prices and individual endowments, and also all the production plans of the firm.

**Definition.** The data set  $\{p^r, q^r, (\omega_n^r, \kappa_n^r), X^r, Y^r\}$  is consistent with the efficient provision of the public good if there exist a profile of preferences  $(U_n)$ , a technology  $F$ , and profiles of individual demands and Lindahl prices  $\{(q_n^r, x_n^r)\}$  such that: (a) individual consumption plans are rational in the sense that  $(x_n^r, \sum_m \kappa_m^r + Y^r)$  maximizes  $U_n(x, y)$  subject to the budget constraint that

$$p^r \cdot x + q_n^r y \leq p^r \cdot \omega_n^r + q^r \kappa_n^r + \theta_n(p^r \cdot X^r + q^r Y^r);$$

- (b) each of the firm's production plans  $(X^r, Y^r)$  maximizes  $p \cdot X + qY$  over  $(X, Y) \in F$ ; (c) private markets clear:  $\sum_n x_n^r = \sum_n \omega_n^r + X^r$ ; and (d) the Lindahl prices fund the public good:  $\sum_n q_n^r = q^r$ .

As before, in the case of constant returns to scale in production, we simplify notation by writing  $W_n^r = p^r \cdot \omega_n^r + q^r \cdot \kappa_n^r$  and  $\bar{W}_n = \max_r \{W_n^r\} + 1$ . Also, the observed aggregate endowment of the public good at observation  $r$  is denoted  $K^r = \sum_n \kappa_n^r$ .

**Theorem 13 (Carvajal and Song).** Let the data set  $\{p^r, q^r, (\omega_n^r, \kappa_n^r), X^r, Y^r\}$  be given. The following two statements are equivalent:

1. The data set is consistent with efficient provision of the public good, with each utility function being continuous, monotone, and concave, and the technology displaying constant returns to scale.

2. There exist vectors  $x_n^r \geq 0$ , numbers  $q_n^r \geq 0$ , and integer numbers  $\rho_n^{r,s} \in \{0, 1\}$ , such that:

- (a)  $p^r \cdot x_n^r + q_n^r(K^r + Y^r) = W_n^r$ , for all  $r$  and  $n$ ;
- (b) for each pair of distinct observations,

$$W_n^r - [p^r \cdot x_n^s + q_n^r(K^s + Y^s)] < \rho_n^{r,s} \cdot \bar{W}_n$$

and

$$W_n^r - [p^r \cdot x_n^s + q_n^r(K^s + Y^s)] \leq (1 - \rho_n^{s,r}) \cdot \bar{W}_n,$$

for all  $r, s$  and  $n$ ;

- (c)  $\rho_n^{r,s} + \rho_n^{s,t} \leq 1 + \rho_n^{r,t}$ , for all distinct  $r, s, t$ ;
- (d)  $p^r \cdot X^s + q^r Y^s \leq p^r \cdot X^r + q^r Y^r = 0$ , for all  $r, s$ ; and
- (e)  $\sum_n x_n^r = \sum_r \omega_n^r + X^r$  and  $\sum_n q_n^r = q^r$ .

As in the case of competitive equilibrium, this test can be extended to the case of an economy with warm-glow effects (Carvajal and Song, 2018).

### 3.4. Household behavior

The testable implications of general equilibrium models in public economics are closely related to those of the collective model of the household first introduced by Chiappori (1988) and Chiappori (1992).<sup>29</sup> The previous survey, Carvajal et al. (2004), covered the first testability results in this literature, namely those in Chiappori (1988) and Snyder (2000).

A significant development is Cherchye et al. (2007). Suppose the household has two agents,  $n = a, b$ . The analyst has observed a data set consisting of the prices and the household consumption of all commodities.<sup>30</sup> The analyst does not observe how the consumption is allocated within the household between its three possible uses: private consumption by each member,  $x_n$  for  $n = a, b$ , and public consumption,  $x_h$ . Each individual is assumed to have preferences over these allocations,  $U_n(x_a, x_b, x_h)$ .

**Definition.** The data set  $\{p^r, x^r\}$  is consistent with efficient household decisions if there exist preferences  $(U_a, U_b)$ , allocations  $(x_a^r, x_b^r, x_h^r)$  and Pareto weights  $\mu^r > 0$  such that each  $(x_a^r, x_b^r, x_h^r)$  maximizes the household utility function  $U_a(x_a, x_b, x_h) + \mu^r U_b(x_a, x_b, x_h)$  subject to the budget constraint  $p^r \cdot (x_a + x_b + x_h) \leq p^r \cdot x^r$ .

Note that in this definition, the Pareto weights are neither observed nor assumed invariant across observations.

The main result in Cherchye et al. (2007) is a characterization of consistency with efficient household decisions. As in the literature of the efficient provision of public goods, it uses the fact that intra-household efficiency implies the existence of Lindahl prices that apply inside the household.

**Theorem 14 (Cherchye et al.).** Let the data set  $\{p^r, x^r\}$  be given. The following two statements are equivalent:

1. The data set is consistent with efficient household decisions, with continuous and concave utility functions.
2. There exists a solution with vectors  $x_a^r, x_b^r, x_h^r, p_{n,a}^r, p_{n,b}^r, p_{n,h}^r \geq 0$  and numbers  $u_n^r, \lambda_n^r > 0$ , for  $n = a, b$ , to the following system:

- (a)  $x_a^r + x_b^r + x_h^r = x^r$ , for all  $r$ ;
- (b)  $p_{a,a}^r + p_{b,a}^r = p_{a,b}^r + p_{b,b}^r = p_{a,h}^r + p_{b,h}^r = p^r$ , for all  $r$ ; and

<sup>29</sup> At the empirical level, see Blundell et al. (1993) and Browning and Chiappori (1998). For general surveys of this literature, see Vermeulen (2002) and Donni (2016). This model continues to be the keystone of household economics; for very recent applications, see, for example, Amarante et al. (2023), Vu (2023) and Zhao and Qu (2024).

<sup>30</sup> Prices are assumed to be strictly positive.

$$(c) \quad u_n^s - u_n^r - \lambda^r [p_{n,a}^r \cdot (x_a^s - x_a^r) + p_{n,b}^r \cdot (x_b^s - x_b^r) + p_{n,h}^r \cdot (x_h^s - x_h^r)] \leq 0, \text{ for all } r, s \text{ and } n = a, b.$$

In the second statement,  $x_m^r$  is the allocation of commodities to  $m = a, b, h$  at observation  $r$ , while  $p_{n,m}^r$  is the personalized price that agent  $n = a, b$  pays for each unit of household consumption allocated to use  $m = a, b, h$ . Condition (2b) is the requirement that, for each use  $m = a, b, h$ , the sum of the two personalized prices paid in the household must fund the observed market prices.

The Afriat inequalities in condition (2c) are a good opportunity to understand the connection between the approach that uses a social welfare function (Bachmann, 2006) and the one that uses Lindahl prices (Snyder, 1999; Carvajal, 2010) when studying efficiency hypotheses. Consider the “planner problem”

$$\max_{x_a, x_b, x_n} \{U_a(x_a, x_b, x_n) + \mu^r U_b(x_a, x_b, x_n) : p^r \cdot (x_a + x_b + x_n) \leq M^r\}.$$

The first-order conditions are that for some number  $\gamma^r > 0$ ,

$$\nabla_{x_m} U_a(x_a^r, x_b^r, x_n^r) + \mu^r \nabla_{x_m} U_b(x_a^r, x_b^r, x_n^r) = \gamma^r p^r, \tag{26}$$

for  $m = a, b, h$ . By concavity of both utility functions, we further have that

$$U_n(x_a^s, x_b^s, x_n^s) \leq U_n(x_a^r, x_b^r, x_n^r) + \sum_{m=a,b,h} \nabla_{x_m} U_n(x_a^r, x_b^r, x_n^r) \cdot (x_m^s - x_m^r) \tag{27}$$

for both  $n = a, b$ .

If we now define Lindahl prices  $p_{a,m}^r = 1/\gamma^r \nabla_{x_m} U_a(x_a^r, x_b^r, x_n^r)$  and  $p_{b,m}^r = \mu^r/\gamma^r \nabla_{x_m} U_b(x_a^r, x_b^r, x_n^r)$ , the first-order conditions (26) yield condition (2b) in Theorem 14. If, in addition, we define the numbers  $\lambda_a^r = \gamma^r$ ,  $\lambda_b^r = \mu^r/\gamma^r$ ,  $u_n^r = U_n(x_a^r, x_b^r, x_n^r)$ , then the inequalities in Eq. (27) yield the Afriat inequalities in condition (2c) of the theorem.

Notice that condition (2c) defines a non-linear system, which again brings computational difficulties. Notably, Cherchye et al. (2007) provides a quantifier-free sufficient condition. They also show that, as long as  $L, R \geq 3$ , the quantifier-free equivalent of the system defined by (2c) is non-tautological.

#### 4. Strategic interactions and first-order conditions

##### 4.1. Bargaining

Another setting that has generated interest is the Nash bargaining solution, NBS henceforth. In a one-commodity world, suppose that the agents  $n = 1, \dots, N$  bargain over an exogenously given budget. An analyst has observed a data set where at each observation  $r$ , agent  $n$  was received  $x_n^r$  from the bargain, and her outside option was  $\omega_n^r < x_n^r$ .

**Definition.** The data set  $\{(x_n^r, \omega_n^r)\}$  is consistent with the NBS if there exist utility functions  $U_n$  such that, at each  $r$ , allocation  $(x_n^r)$  maximizes

$$\prod_n [U_n(x_n) - U_n(\omega_n^r)] \tag{28}$$

over all possible profiles  $(x_n)$  such that  $\sum_n x_n = \sum_n x_n^r$  and  $U_n(x_n) \geq U_n(\omega_n^r)$  for all  $n$ .

Carvajal and González (2014) and Chambers and Echenique (2014) follow Chiappori et al. (2012) in developing the testable implications of this hypothesis.

**Theorem 15** (Chiappori et al., Carvajal and González, Chambers and Echenique). Let the data set  $\{(x_n^r, \omega_n^r)\}$ , with  $x_n^r > \omega_n^r$  for all  $n$  at all  $r$ , be given. The following two statements are equivalent:

1. The data set is consistent with the NBS and each utility function can be chosen to be differentiable, strictly increasing and strictly concave.
2. There exist numbers  $u_n^r, v_n^r, \delta_n^r, \lambda^r > 0$  that solve the following system:

$$(a) \quad u_n^s - u_n^r - \lambda^r (u_n^r - v_n^r) \cdot (x_n^s - x_n^r) \leq 0, \text{ with strict inequality if } x_n^s \neq x_n^r, \text{ for all } r, s \text{ and } n;$$

$$(b) \quad v_n^s - v_n^r - \delta_n^r \cdot (\omega_n^s - \omega_n^r) \leq 0, \text{ with strict inequality if } \omega_n^s \neq \omega_n^r, \text{ for all } r, s \text{ and } n;$$

$$(c) \quad v_n^s - u_n^r - \lambda^r (u_n^r - v_n^r) \cdot (\omega_n^s - x_n^r) \leq 0 \text{ and } u_n^r - v_n^s - \delta_n^s \cdot (x_n^r - \omega_n^s) \leq 0, \text{ with strict inequality if } \omega_n^s \neq x_n^r, \text{ for all } r, s \text{ and } n; \text{ and}$$

$$(d) \quad u_n^r > v_n^r, \text{ for all } r \text{ and } n.$$

To see the insight of the theorem, let  $u_n^r = U_n(x_n^r)$ ,  $v_n^r = U_n(\omega_n^r)$ , and  $\delta_n^r = U_n'(\omega_n^r)$ . The first-order conditions of the Nash bargaining problem (28) are that, for all  $n$ ,

$$U_1'(x_1^r) \cdot \prod_{m \neq 1} (u_m^r - v_m^r) = U_n'(x_n^r) \cdot \prod_{m \neq n} (u_m^r - v_m^r).$$

This implies that for all agents  $n, m$ ,

$$\frac{U_n'(x_n^r)}{u_n^r - v_n^r} = \frac{U_m'(x_m^r)}{u_m^r - v_m^r}$$

so, if we define  $\lambda^r = U_1'(x_1^r)/(u_1^r - v_1^r)$ , we get that  $U_n'(x_n^r) = \lambda^r (u_n^r - v_n^r)$ . With this construction, the first three conditions of the second statement are simply the Afriat inequalities for each individual’s utility function. The fourth condition is the Nash problem’s participation constraint.

To see that the existence of a solution to the system of the second statement is an actual restriction, consider the comparative statics of the NBS on the aggregate to be distributed through bargaining. For simplicity, suppose there are only two agents,  $n = a, b$ , and denote  $X = x_a + x_b$ . From the first-order condition

$$U_a'(x_a)[U_b(X - x_a) - v_b] = U_b'(X - x_a)[U_a(x_a) - v_a],$$

Carvajal and González (2014) shows that over the manifold of solutions to the problem

$$dx_a = \frac{\{U_b'(x_b)[U_a(x_a) - v_a] - U_a'(x_a)U_b'(x_b)\} dX - U_b'(x_b)dv_a + U_a'(x_a)dv_b}{U_a'(x_a)[u_b(x_b) - v_b] - 2U_a'(x_a)U_b'(x_b) + U_b'(x_b)[U_a(x_a) - v_a]}. \tag{29}$$

Since  $U_a$  and  $U_b$  are strictly increasing and strictly concave,  $U_a(x_a) > v_a$ , and  $U_b(X - x_a) > v_b$ , it follows that the denominator in this expression is negative. For the same reasons, the term that multiplies  $dX$  is negative, while the terms that multiply  $dv_a = U_a'(\omega_a)d\omega_a$  and  $dv_b = U_b'(\omega_b)d\omega_b$  are positive. This implies that if  $dX > 0$ ,  $d\omega_a > 0$  and  $d\omega_b < 0$ , then, unambiguously,  $dx_a > 0$ . Put differently, the data set is not rationalizable if it contains observations,  $r$  and  $s$ , such that  $X_r > X_s$ ,  $\omega_a^r > \omega_a^s$ ,  $\omega_b^r < \omega_b^s$ , and  $x_a^r < x_a^s$ .

In a fashion similar to the household problem of Section 3.4, Cherchye et al. (2013b) solve the problem with an arbitrary number  $L$  of commodities. Suppose that for each observation  $r$ , the analyst has observed each agent’s endowment of commodities  $\omega_n^r$  as well as her consumption bundle  $x_n^r$ . Suppose that the commodity price vector  $p^r \gg 0$  has also been observed, and that  $p^r \cdot x_n^r > p^r \cdot \omega_n^r$  for all  $n$  at all  $r$ .

**Definition.** The data set  $\{p^r, (x_n^r, \omega_n^r)\}$  is consistent with the NBS if there exist utility functions  $(U_n)$  such that at each  $r$ , the profile  $(x_n^r)$  solves the problem

$$\max_{(x_n)} \{ \prod_n [U_n(x_n) - v_n^r] : p^r \cdot \sum_n x_n \leq p^r \cdot \sum_n x_n^r \text{ and } U_n(x_n) \geq v_n^r \text{ for all } n \}, \tag{30}$$

where

$$v_n^r := \max_y \{U_n(y) : p^r \cdot y \leq p^r \cdot \omega_n^r\}. \tag{31}$$

In this setting, the agents bargain over the allocation of some aggregate budget, given the prices, having as outside options the maximum utility they would attain, at those same prices, given their individual endowments.

**Theorem 16 (Cherchye et al.).** Let the data set  $\{p^r, (x_n^r, \omega_n^r)\}$ , where  $p^r \cdot x_n^r > p^r \cdot \omega_n^r$  for all  $n$  at all  $r$ , be given. The following two statements are equivalent.

1. The data set is consistent with the NBS and each utility function can be chosen to be differentiable, strictly increasing and strictly concave.
2. There exist numbers  $u_n^r, v_n^r, \mu_n^r, \lambda^r > 0$  and vectors  $y_n^r \geq 0$  that solve the following system:

- (a)  $u_n^s - u_n^r - \lambda^r (u_n^r - v_n^r) p^r \cdot (x_n^s - x_n^r) \leq 0$ , with strict inequality if  $x_n^s \neq x_n^r$ , for all  $r, s$  and  $n$ ;
- (b)  $v_n^s - v_n^r - \mu_n^r p^r \cdot (y_n^s - y_n^r) \leq 0$ , with strict inequality if  $y_n^s \neq y_n^r$ , for all  $r, s$  and  $n$ ;
- (c)  $v_n^s - u_n^r - \lambda^r (u_n^r - v_n^r) p^r \cdot (y_n^s - x_n^r) \leq 0$  and  $u_n^r - v_n^s - \mu_n^s p^s \cdot (x_n^r - y_n^s) \leq 0$ , with strict inequality if  $y_n^s \neq x_n^r$ , for all  $r, s$  and  $n$ ; and
- (d)  $u_n^r > v_n^r$ , for all  $r$  and  $n$ .

To see the insight of this result, compare it with the one-commodity case. The first-order condition of the Nash bargaining problem (30) is that

$$\prod_{m \neq n} [U_m(x_m^r) - v_m^r] \cdot \nabla U_n(x_n^r) = v^r \cdot p^r,$$

for all  $r$ , where  $v^r > 0$  is the multiplier of the constraint  $p^r \cdot \sum_n (x_n^r - x_n) \geq 0$ . If one defines  $\lambda^r = v^r / \prod_{m \neq n} [U_m(x_m^r) - v_m^r]$ , it follows that  $\nabla U_n(x_n^r) = \lambda^r [U_n(x_n^r) - v_n^r] p^r$ , which explains the Afriat inequalities around each  $x_n^r$ . To understand the Afriat inequalities around each  $y_n^r$ , it suffices to note that the first-order conditions of the outside problem (31) imply that  $\nabla U_n(y_n^r) = \mu_n^r p^r$  for some multiplier  $\mu_n^r > 0$ .

#### 4.2. Imperfect competition

One way to develop testable implications in games is by using the particular structure of the game and not just the equilibrium hypothesis. This approach was successfully followed by Carvajal et al. (2013) in the case of the game of Cournot competition.

##### 4.2.1. Profit maximization by a monopolist

To understand this result more easily, consider first the case of a single-product monopolist. Suppose the data set consists of  $R$  observations of the price of its product, its output level, and the total cost of production,  $\{p^r, q^r, c^r\}$ . All these numbers are assumed to be positive.

Imagine that the analyst hypothesizes that there were unobserved demand shocks across the observations but assumes that the monopolist's cost function remains invariant. Assuming that the cost function is strictly increasing, the profit maximization hypothesis is as follows.

**Definition.** The data set  $\{p^r, q^r, c^r\}$  is consistent with monopolistic profit maximization if there exist inverse demand functions  $\{D^r\}$  and a cost function  $C$  such that  $D^r(q^r) = p^r$ ,  $C(q^r) = c^r$ , and  $q^r$  maximizes  $D^r(q) \cdot q - C(q)$  over  $q \leq 0$ , at all  $r$ .

**Theorem 17 (Carvajal et al.).** Let the data set  $\{p^r, q^r, c^r\}$  such that  $(c^r - c^s)(q^r - q^s) > 0$  at all  $r, s$  be given. The following two statements are equivalent:

1. The data set is consistent with monopolistic profit maximization, with downward sloping inverse demand functions and an upward sloping cost function with  $C(0) = 0$ .
2. For all  $r, s$ , if  $q^s < q^r$  then  $p^r \cdot q^s - c^s < p^r \cdot q^r - c^r$ .

Moreover, if the second statement is true, the functions in the first statement can be constructed to be differentiable and, for any collection of strictly positive numbers  $\{\delta^r\}$  with  $\delta^r < p^r$  at all  $r$ , the cost function can be constructed such that  $C'(q^r) = \delta^r$  at all  $r$

The inequality in the second statement is a weakening of Varian's weak axiom of profit maximization,  $WAPM$ , which we already encountered in Section 2.2. Under the notation of this section,  $WAPM$  is the

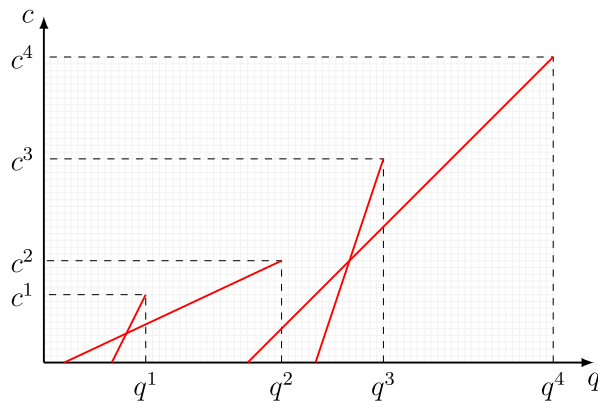


Fig. 9. Four observations such that for all  $r, s$ , if  $q^s < q^r$  then  $p^r \cdot q^s - c^s < p^r \cdot q^r - c^r$ . The segment that goes through each point  $(q^r, c^r)$  has slope  $p^r$ .

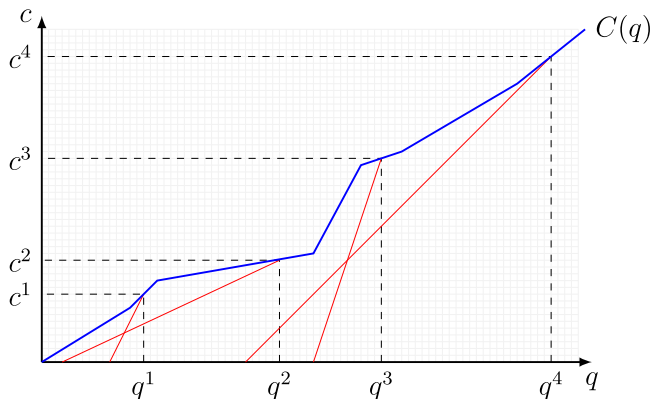


Fig. 10. Cost function  $C$ , with  $C(q^r) = c^r$  and  $C'(q^r) = \delta^r < p^r$  at all  $r$ .

requirement that  $p^r \cdot q^s - c^s \leq p^r \cdot q^r - c^r$ . This inequality must hold if a competitive firm is maximizing profits: the right-hand side of the inequality is the firm's observed profits at observation  $r$ , while the left-hand side is the counterfactual computation of the profits it would have made if, at that same observation, it decided to produce the output observed at  $s$ . The competitive firm would take the price  $p^r$  as given even if it changed its output.

For a monopolist, though,  $WAPM$  is too strong. If  $q^s > q^r$ , then  $D^r(q^s) < D^r(q^r) = p^r$ , so the left-hand side is no longer a valid imputation of counterfactual profits. On the other hand,  $q^s < q^r$  would imply  $D^r(q^s) > D^r(q^r) = p^r$ , so the counterfactual profits, even if one cannot observe them, would be  $D^r(q^s) \cdot q^s - c^s > p^r \cdot q^s - c^s$ . If the data showed  $p^r \cdot q^s - c^s \geq p^r \cdot q^r - c^r$  with  $q^s < q^r$ , the firm could not be maximizing profits (with downward sloping demand functions).

That the simple condition of the second statement suffices for the third one is remarkable. The construction of the cost function in Carvajal et al. (2013) is illustrated in Figs. 9 and 10. The first of these figures displays four observations: the quantities and costs are explicit, and the prices are used for the slopes of the four lines. The condition of the second statement in the theorem determines that any pair  $(q^s, c^s)$  is strictly above all the lines that go through any  $(q^r, c^r)$  with  $q^r > q^s$ .<sup>31</sup> Under this condition, the cost function can be constructed as in Fig. 10. In addition to the position of the lines given by the data, all that is needed is that the local slopes  $\delta^r$  be below the corresponding  $p^r$ .

Constructing the inverse demand functions is easy once the cost function has been constructed. All one needs is that each of these functions be sufficiently flat for lower production levels, sufficiently steep

<sup>31</sup> The slope of each of these latter lines is  $p^r$ .

for higher production levels, and such that the first-order conditions are satisfied at the observed output levels,<sup>32</sup> namely that

$$-\frac{dD^r}{dq}(q^r) = \frac{p^r - \delta^r}{q^r}. \tag{32}$$

4.2.2. Cournot competition

Now, consider the case of Cournot competition between  $N$  firms, and suppose that the data set includes the output and production cost of each firm at each observation.

**Definition.** The data set  $\{p_r, (q_n^r, c_n^r)\}$  is consistent with Cournot competition if there exist inverse demand functions  $\{D^r\}$  and cost functions  $(C_n)$  such that  $D^r(\sum_n q_n^r) = p^r$ ,  $C_n(q_n^r) = c_n^r$ , and each  $q_n^r$  maximizes

$$D^r(q + \sum_{m \neq n} q_m^r) \cdot q - C_n(q)$$

over  $q \geq 0$ , for all  $n$  at all  $r$ .

Carvajal et al. (2013) gives the following characterization.

**Theorem 18 (Carvajal et al.).** Let the data set  $\{p_r, (q_n^r), (c_n^r)\}$  such that  $(c_n^r - c_n^s)(q_n^r - q_n^s) > 0$  for all  $n$  at all  $r, s$  be given. The following two statements are equivalent:

1. The data set is consistent with Cournot competition, with downward sloping inverse demand functions and upward sloping cost functions such that  $C_n(0) = 0$  for all  $n$ .
2. For all  $n$  and all  $r, s$ , if  $q_n^s < q_n^r$  then  $p^r \cdot q_n^s - c_n^s < p^r \cdot q_n^r - c_n^r$ .

Moreover, if the second statement is true, the functions in the first statement can be constructed to be differentiable and, for a collection of profiles of strictly positive numbers  $\{\delta_n^r\}$ , the cost functions can be constructed with  $C_n^r(q^r) = \delta_n^r$  so long as

$$\frac{p^r - \delta_1^r}{q_1^r} = \frac{p^r - \delta_2^r}{q_2^r} = \dots = \frac{p^r - \delta_N^r}{q_N^r} > 0 \tag{33}$$

at all  $r$ .

That the result for a Cournot oligopoly resembles the theorem for the monopolist is not a surprise: Cournot oligopolists behave as monopolists over their “residual” demand functions. What is important, and rather subtle, is that all the cost functions must be constructed together. Recall Eq. (32), which amounts to the first-order condition of the monopolist’s profit maximization problem. A condition like this must hold for each of the oligopolists, but, importantly, the left-hand side of Eq. (32) must be the same for all the firms, as the demand function applies to all the market and not firm by firm. More explicitly, for each  $n$  and  $r$ , the construction must guarantee that

$$-\frac{\partial D^r}{\partial q_n}(q_n^r + \sum_{m \neq n} q_m^r) = \frac{p^r - \delta_n^r}{q_n^r}.$$

Since the quantities and prices are observed, once  $\delta_1^r$  is chosen, all other  $\delta_n^r$  are immediately determined. This is precisely what Eq. (33) does.

At a conceptual level, this observation is important. Consider what happens in the second statement of Theorem 3. In that analysis, the revealed preference arguments apply to agent  $n$  only across observations for which  $x_{-n}^s = x_{-n}^r$ . At a purely operational level, the more players a game has, the less likely this equality is to be observed. More

<sup>32</sup> For example, one can let  $D^r(q) = p^r + \gamma^r(q)(q^r - q)$ , with

$$\gamma^r(q) = \begin{cases} \alpha, & \text{if } q \leq q^r - \epsilon; \\ \frac{p^r - \delta^r}{q^r}, & \text{if } q^r - \epsilon < q < q^r + \epsilon; \\ \beta, & \text{if } q > q^r + \epsilon. \end{cases}$$

This function satisfies that  $dD^r/dq(q^r) \cdot q^r - D^r(q^r) = \delta^r = C^r(q^r)$ , while one can take  $\alpha, \epsilon > 0$  small enough, and  $\beta > 0$  big enough.

players make the test “weaker” and the equilibrium hypothesis “harder to reject”.<sup>33</sup>

In contrast, introducing more firms in a Cournot oligopoly makes Eq. (33) more “demanding”, and the Nash-Cournot equilibrium hypothesis “easier to reject”. This is not a consequence of the equilibrium concept or the particular assumptions imposed by the analyst but of the game’s structure. In a Cournot oligopoly, the first-order conditions of all the firms must be valid for a slope of the inverse demand function that is common to all of them.

This insight becomes more salient in the context where the production costs of the different firms have not been observed, so the data set reduces to only prices and quantities, as Carvajal et al. (2013) prove.

**Definition.** The data set  $\{p_r, (q_n^r)\}$  is consistent with Cournot competition if there exist inverse demand functions  $\{D^r\}$  and cost functions  $(C_n)$  such that  $D^r(\sum_n q_n^r) = p^r$  and each  $q_n^r$  maximizes

$$D^r(q + \sum_{m \neq n} q_m^r) \cdot q - C_n(q)$$

over  $q \geq 0$ , for all  $n$  at all  $r$ .

**Theorem 19 (Carvajal et al.).** Let the data set  $\{p_r, (q_n^r)\}$  be given. The following two statements are equivalent:

1. The data set is consistent with Cournot competition, with downward sloping inverse demand functions and upward sloping, convex cost functions such that with  $C_n(0) = 0$  for all  $n$ .
2. There exists a collection of profiles of strictly positive numbers  $\{\delta_n^r\}$  that (a) satisfy Eq. (33); and (b) such that  $(\delta_n^s - \delta_n^r)(q_n^s - q_n^r) \geq 0$  for all  $r, s$  and all  $n$ .

Even the weaker version of WAPM becomes moot in this setting since only revenues are observed. Instead, the convexity of the cost functions becomes the requirement that the marginal costs be co-monotone with the observed quantities, firm by firm.

It is also immediate that for the case of a monopolistic market, with  $N = 1$ , any data set is rationalizable: one simply needs to pick a sequence of marginal costs that are sufficiently low and co-monotone with the observed quantities. Notably, there are non-rationalizable data sets when  $N \geq 2$ , as per the following example from Carvajal et al. (2013). Suppose the data includes two firms and two observations for which  $q_n^r > q_n^s$  while  $q_m^r < q_m^s$ . Whatever  $p^r$  and  $p^s$  are, if one could explain these observations as Nash-Cournot competition, Eq. (33) would require

$$\frac{p^r - \delta_n^r}{q_n^r} = \frac{p^r - \delta_m^r}{q_m^r},$$

implying that  $\delta_n^r < \delta_m^r$ , while

$$\frac{p_s - \delta_n^s}{q_n^s} = \frac{p_s - \delta_m^s}{q_m^s},$$

which implies  $\delta_n^s > \delta_m^s$ . Under convex cost functions, these conditions would require that  $\delta_n^r < \delta_m^r \leq \delta_m^s < \delta_n^s \leq \delta_n^r$ .

Note that the two conditions in the second statement of Theorem 19 are linear and computationally simple. As in the literature surveyed in Section 2, quantifier elimination techniques can be used to argue that there exists a quantifier-free condition that is also equivalent to the first statement in the theorem, and the example we just gave implies that such condition is non-tautological. Still, Carvajal et al. (2013) provides a direct search algorithm that allows for the solution of the linear system in at most  $R$  iterations and uses this algorithm to “set-identify” the marginal cost of all the producers that are consistent with the observed data under the Cournot equilibrium hypothesis.

<sup>33</sup> These last statements are informal and not based on a precise concept of “hardness”. Of course, changing the number of players means a completely new game.

Carvajal et al. (2013) also considers other forms of the equilibrium hypothesis and applies their results to empirical data from a well-known non-competitive setting, the international oil markets, to study the power of the test. Carvajal et al. (2014) extends the analysis to the case of multi-product technologies,<sup>34,35</sup> while Cherchye et al. (2013a) complements the single-product analysis of Carvajal et al. (2013) in the following way.

Suppose that the analyst assumes that the aggregate inverse demand function is of the form  $D(q; \alpha)$ , while firm  $n$ 's cost function is of the form  $C_n(q, \beta)$ , where  $\alpha = (\alpha_1, \dots, \alpha_K)$  and  $\beta = (\beta_1, \dots, \beta_L)$  are vectors of parameters.

The analyst has estimated the following reduced-form functions of the industry: as a function of the parameters, the equilibrium price is  $P(\alpha, \beta)$ , while the quantity of firm  $n$  is  $Q_n(\alpha, \beta)$ .

**Definition.** The collection of functions  $\{P, \{Q_n\}\}$  is consistent with Cournot competition if there exist an inverse demand function  $D$  and cost functions  $\{C_n\}$  such that, for all possible  $(\alpha, \beta)$ : (a)  $D(\sum_n Q_n(\alpha, \beta), \alpha) = P(\alpha, \beta)$ ; and (b) for all  $n$ ,  $Q_n(\alpha, \beta)$  is the only solution to the program

$$\max_q \{D(q + \sum_{m \neq n} Q_m(\alpha, \beta), \alpha) \cdot q - C_n(q, \beta)\}.$$

In this definition, the first condition is the ‘‘observational’’ requirement that the reduced form and the structural form of the inverse demand give consistent prices, while the second condition imposes the equilibrium hypothesis.<sup>36</sup> That these conditions generate testable implications is immediate. Consider, for example, condition (a), and note that it implies that for each dimension,  $\ell$ , of the second vector of parameters,

$$\frac{\partial P}{\partial \beta_\ell}(\alpha, \beta) = \frac{\partial D}{\partial q}(\sum_n Q_n(\alpha, \beta), \alpha) \cdot \sum_n \frac{\partial Q_n}{\partial \beta_\ell}(\alpha, \beta).$$

Since the first term in the right-hand side of this equation is independent of  $\ell$ , it follows that

$$\frac{\partial P}{\partial \beta_1}(\alpha, \beta) \cdot \sum_n \frac{\partial Q_n}{\partial \beta_\ell}(\alpha, \beta) = \frac{\partial P}{\partial \beta_\ell}(\alpha, \beta) \cdot \sum_n \frac{\partial Q_n}{\partial \beta_1}(\alpha, \beta)$$

for all  $\ell$ . This condition is directly verifiable on the observed (or estimated) reduced-form functions.

Notably, under some minor assumptions, Cherchye et al. (2013a) develops the complete set of testable implications in the sense they are necessary and sufficient for the hypothesis. Note that, unlike Carvajal et al. (2013, 2014), this analysis models the firm cost functions as dependent on observable ‘‘shifters’’ but does not restrict the shape of these functions.

## 5. Other results in game theory

### 5.1. Strategic complementarity and substitution

Consider again the context of Carvajal (2003), which was introduced in Section 1.2, but suppose for simplicity that only upper bounds are observed for the actions of the players. That is, each agent chooses

<sup>34</sup> A caveat is important: in the multi-product case, the hypothesis that the inverse demand function is downward sloping is replaced by the requirement that its Jacobian be negative semi-definite. Of course, this fact introduces a non-linearity into the system.

<sup>35</sup> For the case of price competition, see Yasui (2023).

<sup>36</sup> To be precise, Cherchye et al. (2013a) require that

$$\frac{\partial D}{\partial q}(\sum_m Q_m(\alpha, \beta), \alpha) \cdot Q_n(\alpha, \beta) + D(\sum_n Q_n(\alpha, \beta), \alpha) = \frac{\partial C_n}{\partial q}(Q_n(\alpha, \beta), \beta)$$

and

$$2 \frac{\partial D}{\partial q}(\sum_m Q_m(\alpha, \beta), \alpha) + \frac{\partial^2 D}{\partial q^2}(\sum_m Q_m(\alpha, \beta), \alpha) \cdot Q_n(\alpha, \beta) = \frac{\partial^2 C_n}{\partial q^2}(Q_n(\alpha, \beta), \beta)$$

namely, the first- and second-order conditions of the latter program.

an action from  $[0, 1]$ , but in addition to her choice  $x_n^r$ , the analyst observes an exogenous upper bound  $\beta_n^r$  that constrained the agent's choice at observation  $r$ .

**Definition.** The data set  $\{(x_n^r, \beta_n^r)\}$ , with  $x_n^r \leq \beta_n^r$  for all  $n$  and  $r$ , is consistent with Nash equilibrium if there exist functions  $\{U_n\}$  defined on  $[0, 1]^N$  such that

$$\forall x \in [0, \beta_n^r], U_n(x, x_{-n}^r) \leq U_n(x_n^r, x_{-n}^r)$$

for all  $n$  and  $r$ .

It follows from Theorem 3 that the data set is consistent with Nash equilibrium and the utility functions can be chosen to be continuous and strictly concave in the respective player's own action if, and only if, the system consisting of conditions (2a) and (2b) in that theorem has a solution.

Consider first the case where there are only two players,  $n = a, b$  and their payoff functions are differentiable. For player  $n$ , the actions of her opponent,  $m$ , are complementary to her own if whenever  $m$  increases the value of her choice, the marginal utility that  $n$  derives from her own action increases. If, under the same premise,  $n$ 's marginal utility decreases, then the actions of  $m$  are substitutes to those of  $n$ , from the point of view of  $n$ . Formally,  $U_n$  is said to exhibit strategic complementarity over the interval  $S \subseteq [0, 1]$  if for all  $x_n, \partial U_n / \partial x_n(x_n, x_m)$  is non-decreasing in  $x_m \in S$ ; if, under the same assumptions,  $\partial U_n / \partial x_n(x_n, x_m)$  is non-increasing in  $x_m \in S$ ,  $U_n$  is said to exhibit strategic substitutability.

Suppose the analyst hypothesizes that for each  $n$  the actions of her opponent are complementary on  $[0, \alpha_m]$  and substitute on  $[\alpha_m, 1]$ . These two new hypotheses can be tested, and all the restrictions of Nash equilibrium in this new context are as in the following result:

**Theorem 20 (Carvajal).** Let the data set  $\{(x_n^r, \beta_n^r)\}$  and the constants  $(\alpha_n)$  be given. The following two statements are equivalent:

1. The data set is consistent with Nash equilibrium and the utility functions  $U_n$  can be chosen to be continuous and strictly concave in the players' own actions and to display strategic complementarity on  $[0, \alpha_m]$  and strategic substitutability on  $[\alpha_m, 1]$ .
2. For all  $r, s$  and all  $n$ , denoting  $m \neq n$ ,

- (a)  $x_m^r = x_m^s, x_n^r \leq \beta_n^s$ , and  $x_n^s \leq \beta_n^r$  together imply  $x_n^r = x_n^s$ ;
- (b)  $x_m^r = x_m^s, 0 < x_n^r$ , and  $x_n^r \geq \beta_n^s$  together imply  $x_n^s = \beta_n^s$ ;
- (c)  $x_m^s \leq x_m^r \leq \alpha_m$  implies  $\min\{\beta_n^r, x_n^s\} \leq x_n^r \leq \beta_n^r$ ; and
- (d)  $x_m^s \geq x_m^r \geq \alpha_m$  implies  $\min\{\beta_n^r, x_n^s\} \leq x_n^r \leq \beta_n^r$ .

The first two conditions in the second statement are already present in Theorem 3. Strategic complementarity and substitutability imply the third and fourth conditions, which are new. There two conditions are expressions of the monotonicity that the two new hypotheses impose on the best responses of each player: if the observed upper bounds allow it, across two observations the optimal action of a player must increase when either: (2c) the observations are in the domain of strategic complementarity and the opponent has increased her action; or (2d) the observations are in the domain of substitutability and the opponent has decreased her action.

Lazzati et al. (2021) study a similar problem in a much richer setting. Allowing for randomness in preferences and in equilibrium selection, they assume that the analyst observes the actions chosen by the players, contingent on the observed values of some exogenous, payoff-relevant co-variables. They study how the probability distributions of endogenous variables responds to changes in the co-variables under the assumptions that the agents play Nash equilibria of the game, given the co-variables, and that their preferences are such that their best-responses are monotone in both the co-variables and the actions of the other players.

A simplified, deterministic version of the analysis in Lazzati et al. (2021) is as follows. Player  $n$  has a parameter  $\beta_n$  of her utility function that is observed by the analyst.<sup>37</sup> For all players, the set of actions and the set of possible values of their parameters are finite subsets of  $\mathbb{R}$ . The utility functions are of the form  $U_n(x_n, x_{-n}, \beta_n)$ .

Assume that the data set is comprehensive in the sense that for each possible profile  $\beta = (\beta_n)$  of parameters, a unique profile of actions is observed to be played, and denote such profile by  $f(\beta)$ . The action played by player  $n$  will be denoted by  $f_n(\beta)$  and, when needed,  $f(\beta)$  will be written as  $(f_n(\beta), f_{-n}(\beta))$ . Similarly, any profile of actions can be written as  $x$  or as  $(x_n, x_{-n})$ .

**Definition.** Function  $f$  is consistent with Nash equilibrium if there exist utility functions  $(U_n)$  such that: (a) given  $(x_{-n}, \beta_n)$ ,  $U_n(\cdot, x_{-n}, \beta_n)$  orders  $X_n$  strictly; and (b) for each profile  $\beta$ ,  $f(\beta)$  is a Nash equilibrium of the game played over  $X$  when the payoff functions are  $U_n(\cdot, \beta_n)$ .

Lazzati et al. (2021) focuses on strategic complementarity between the action of each players and the joint changes in her parameter and the actions of the other players. In the context of Carvajal (2003), we would have defined strategic complementarity for function  $U_n$  as the requirement that for any actions  $x_n$  and  $y_n$  of player  $n$ ,  $x > y$ , the difference

$$U_n(x_n, x_{-n}, \beta_n) - U_n(y_n, x_{-n}, \beta_n)$$

be non-decreasing in  $(x_{-n}, \beta_n)$ . This definition is cardinal and can be weakened to the ordinal requirement that for any pairs  $(x, \beta_n)$  and  $(y, \gamma_n)$ ,  $(x, \beta_n) > (y, \gamma_n)$ ,

$$U_n(x_n, y_{-n}, \gamma_n) - U_n(y_n, \gamma_n) > 0 \Rightarrow U_n(x, \beta_n) - U_n(y_n, x_{-n}, \beta_n) > 0.$$

Following existing literature, Lazzati et al. (2021) refers to this condition as the function  $U_n$  having single-crossing differences in  $(x_n; x_{-n}, \beta_n)$ .<sup>38</sup>

The results of Carvajal (2003) rely on the monotonicity of the players' best responses under strategic complementarity and substitutability. Lazzati et al. (2021) use the fact that the weaker condition of single-crossing differences also suffices to imply that each player's best-response function is non-decreasing in the other players' actions and her own parameter. Formally,

**Theorem 21 (Lazzati et al.).** Let function  $f$  be given. The following three statements are equivalent.

1. The function is consistent with Nash equilibrium and the utility functions can be chosen to have single-crossing differences in  $(x_n; x_{-n}, \beta_n)$ .
2. For each pair of parameter profiles  $\beta$  and  $\gamma$ , and for each  $n$  such that  $\beta_n > \gamma_n$ ,
 
$$f_{-n}(\beta) > f_{-n}(\gamma) \Rightarrow f_n(\beta) > f_n(\gamma).$$
3. The function is consistent with Nash equilibrium and the utility functions can be chosen to display strategic complementarity.

The condition of the second statement is called revealed monotonicity.

### 5.2. Matching

Another setting where the structure of the game imposes testable implications is the one-to-one matching problem. Let  $M$  and  $W$  be, respectively, the sets of "men" and "women", and assume that they contain the same number of agents. Suppose that an analyst observes a collection of matchings  $\{\mu^r\}$  such that all agents are matched in all

matchings:  $\mu^r(m) \in W$  for every man  $m$  and  $\mu^r(w) \in M$  for every woman  $w$ , at all  $r$ .<sup>39</sup> The analyst wants to find (strict) preferences for each man and each woman, each defined over the set of people of the other gender and being unmatched, such that every observed matching is stable under those preferences. More formally:

**Definition.** The collection  $\{\mu^r\}$  of matchings is consistent with stability in matching if there exist, for each  $m$ , a strict order  $P(m)$  over  $W \cup \{\emptyset\}$ , and for each  $w$ , one  $Q(w)$  over  $M \cup \{\emptyset\}$  such that for all  $r$ : (a) the matching is individually rational: for each  $m$ ,  $\mu^r(m)P(m)\emptyset$ , and for each  $w$ ,  $\mu^r(w)Q(w)\emptyset$ ; and (b) there is no blocking pair: for each pair  $(m, w)$ ,  $wP(m)\mu^r(m)$  if, and only if,  $\mu^r(w)Q(w)m$ . If such a preference profile exists, we denote it as  $(P, Q)$ .

In order to understand the insight of the results in Echenique (2008), it is useful to note first that no structure is imposed on a pair of matches. To see why, suppose that the analyst has to determine whether the distinct matchings  $\mu$  and  $\nu$  are both stable under some preferences. Whatever the two matchings are, one can always define the preferences as follows. For each  $m$ , let  $P(m)$  be such that  $\mu(m)$  be the most preferred element of  $W$  and let  $\emptyset$  be the least preferred option; the rest of the order is immaterial. Similarly, define  $Q(w)$  by letting  $\nu(w)$  be the most preferred element of  $M$  and  $\emptyset$  the least preferred option. That both matchings are individually rational under these preferences is immediate. Now, fix any  $(m, w)$  such that  $wP(m)\nu(m)$ . By construction  $w \neq \nu(m)$  so  $\nu(w) \neq m$  and  $\nu(w)Q(w)m$ . This proves that  $\nu$  is stable, and a similar argument does the same for  $\mu$ . In this construction, the preferences of men and women were designed with maximal disagreement: every man got their most preferred match under  $\mu$ , every woman did it under  $\nu$ .

Now, suppose that there are multiple matchings  $\{\mu^r\}$ , but imagine no individual is matched to the same person in two of the observed matchings. This means that that for every agent  $a$ , man or woman,  $\mu^r(a) \neq \mu^s(a)$  whenever  $r \neq s$ .<sup>40</sup> Echenique (2008) argues that data sets with this property are always consistent with stability in matching. The argument follows the same idea as before, by constructing the orderings of the two genders with maximal conflict. For man  $m$ ,

$$\mu^1(m)P(m)\mu^2(m)P(m) \dots P(m)\mu^R(m)P(m)\emptyset;$$

women who were not matched to  $m$  are arbitrarily ordered below  $\emptyset$ . For woman  $w$ ,

$$\mu^R(w)Q(w)\mu^{R-1}(w)Q(w) \dots Q(w)\mu^1(w)Q(w)\emptyset;$$

while men not matched to  $w$  are arbitrarily ordered below  $\emptyset$ . Evidently, each matching  $\mu^r$  is individually rational. To see that there is no blocking pair, let  $(m, w)$  be such that  $wP(m)\mu^r(m)$ . By construction,  $w = \mu^s(m)$  for some  $s < r$ , which implies that  $\mu^r(w)Q(w)\mu^s(w) = m$ .

It follows that inconsistency with stability in matching requires at least three matchings and that at least one pair of agents be matched together in at least two matchings. For each pair  $r, s$ , define the following binary relations on the set of men. First, say that  $m \mapsto_{r,s} n$  if  $m$ 's match under  $\mu^r$  is  $n$ 's match under  $\mu^s$ , namely if  $\mu^r(m) = \mu^s(n)$ . If  $m \mapsto_{r,s} o \mapsto_{r,s} p \mapsto_{r,s} \dots \mapsto_{r,s} q \mapsto_{r,s} n$  is possible, further say that  $m \rightarrow_{r,s} n$ ; note that from  $\mu^r$  to  $\mu^s$  there is a path of woman re-assignments that trickles from man  $m$  to man  $n$  when this is the case. Finally, say that  $m \leftrightarrow_{r,s} n$  if  $m \rightarrow_{r,s} n$  and  $n \rightarrow_{r,s} m$ . Importantly, relation  $\leftrightarrow_{r,s}$  is an equivalence, so its equivalence classes partition set  $M$ ; denote the resulting partition by  $\mathcal{P}^{r,s}$ , and let  $\mathcal{P}^{r,s}$  be the subset of non-singleton classes in that partition. Intuitively, each class  $E \in \mathcal{P}^{r,s}$  is defined by a cycle of woman "trades" that reassign the matches of the men in the class under  $\mu^r$  as their matches under  $\mu^s$ .

<sup>37</sup> These exogenous parameters are what Lazzati et al. (2021) calls co-variates.

<sup>38</sup> As opposed to having increasing differences, which the stronger condition introduced before.

<sup>39</sup> As Echenique (2008) argues, this assumption implies no loss of generality for the purpose of this exercise.

<sup>40</sup> Of course, this requires that there be at most as many mathings as there are individuals of each gender.

The crucial structure for the restrictions is that stability in matching implies a particular form of coincidence of interests within a gender and opposition of interests across genders. To keep notation light, we will write that  $\mu P(E)\nu$ , where  $\mu$  and  $\nu$  are matchings and  $E \subseteq M$ , when  $\mu(m)P(m)\nu(m)$  for all  $m \in E$ .

**Lemma (Echenique).** Suppose that two matchings  $\mu^r$  and  $\mu^s$  are stable under the preference profile  $(P, Q)$  and let  $E \in \mathcal{P}^{r,s}$ . Then, either  $\mu^r P(E)\mu^s$  or  $\mu^s P(E)\mu^r$ .

The following example from Echenique (2008) shows that consistency with stability in matching is refutable. Suppose that there are four men,  $M = \{a, b, c, d\}$  and data set is  $\{\mu^1, \mu^2, \mu^3\}$  such that:<sup>41</sup>

$$\begin{aligned} \bar{\mathcal{P}}^{1,2} &= \{\{a\}, \{b, c, d\}\}, \bar{\mathcal{P}}^{2,3} = \{\{b\}, \{a, c, d\}\}, \text{ and} \\ \bar{\mathcal{P}}^{1,3} &= \{\{c\}, \{a, b, d\}\}. \end{aligned} \tag{34}$$

The lemma implies that either  $\mu^2 P(\{b, c, d\})\mu^1$  or  $\mu^1 P(\{b, c, d\})\mu^2$ . Say the former is true. Since  $\mu^2(b) = \mu^3(b)$ , it immediately follows that  $\mu^3(b)P(b)\mu^1(b)$  and hence that  $\mu^3 P(\{a, b, c\})\mu^1$ , again by the lemma. Similarly,  $\mu^1(c) = \mu^3(c)$  implies that  $\mu^2(c)P(c)\mu^3(c)$  and therefore that  $\mu^2 P(\{a, c, d\})\mu^3$ . But then,

$$\mu^2(a)P(a)\mu^3(a)P(a)\mu^1(a) = \mu^2(a),$$

which is impossible since  $P(a)$  is strict. A similar argument gives that  $\mu^1 P(\{b, c, d\})\mu^2$  implies the impossibility

$$\mu^1(a)P(a)\mu^3(a)P(a)\mu^2(a) = \mu^1(a).$$

A data set that implies the partitions in Eq. (34) cannot be explained by any pair  $(P, Q)$  of strict preferences.

For the sake of simplicity in presentation, we maintain the assumption that the data set is just  $\{\mu^1, \mu^2, \mu^3\}$  such that for each pair  $r, s$  there exists  $m$  such that  $\mu^r(m) = \mu^s(m)$ . This setting is much less general than the one in Echenique (2008) but allows us to express a testable implication of the hypothesis that the observed matchings are consistent with stability. Define three ancillary binary relations on the set  $\mathcal{P}^{1,2} \cup \mathcal{P}^{2,3} \cup \mathcal{P}^{1,3}$ . First, say that  $E \sim F$  if either

$$E \in \mathcal{P}^{1,2}, F \in \mathcal{P}^{1,3}, \text{ and } \exists m \in E \cap F : \mu^2(m) = \mu^3(m)$$

or

$$E \in \mathcal{P}^{1,3}, F \in \mathcal{P}^{2,3}, \text{ and } \exists m \in E \cap F : \mu^1(m) = \mu^2(m).$$

Alternatively, say that  $E \succsim F$  if either

$$E \in \mathcal{P}^{1,2}, F \in \mathcal{P}^{2,3}, \text{ and } \exists m \in E \cap F : \mu^1(m) = \mu^3(m)$$

or

$$E \in \mathcal{P}^{2,3}, F \in \mathcal{P}^{1,2}, \text{ and } \exists m \in E \cap F : \mu^1(m) = \mu^3(m)$$

Finally, say that  $E \asymp F$  if either  $E \sim F$  or  $E \succsim F$ .

**Theorem 22 (Echenique).** Let the data set  $\{\mu^1, \mu^2, \mu^3\}$  be given. If it is consistent with stability in matching and  $E \asymp F \asymp G \dots \asymp H \asymp E$  is possible, then the sign  $\asymp$  cannot mean  $\sim$  an odd number of times.

To see that the example in Eq. (34) above is inconsistent with stability in matching, note that

$$\mathcal{P}^{1,2} = \{\{b, c, d\}\}, \mathcal{P}^{2,3} = \{\{a, c, d\}\}, \text{ and } \mathcal{P}^{1,3} = \{\{a, b, d\}\},$$

while

$$\mu^1(a) = \mu^2(a), \mu^2(b) = \mu^3(b) \text{ and } \mu^1(c) = \mu^3(c)$$

This implies that  $\{a, b, d\} \sim \{a, c, d\} \sim \{b, c, d\} \sim \{a, b, d\}$ , violating the necessary condition of the theorem. Importantly, Echenique (2008) further obtains a strengthening of this condition that is both necessary and sufficient for the hypothesis of consistency with stability in matching.

### 5.3. Aggregate matching

The aggregate matching framework models matching as between types of individuals, where individuals with the same observable characteristics form a type that has the same preferences over types of spouses.

Let  $M$  and  $W$  be, respectively, the sets of types of men and women, and let  $m$  and  $w$  denote their generic elements. An aggregate matching is a matrix that contains information on the number of matches between each type of agent of one side (men) and each type of agent of the other (women). That is, the analyst observes an  $|M| \times |W|$  integer matrix  $X = (x_{m,w})$ , where  $x_{m,w}$  is the number of men of type  $m$  matched to women of type  $w$ .<sup>42</sup>

**Definition.** Aggregate matching  $X$  is consistent with stability if there exist profiles of strict orders  $P$  and  $Q$  over the types of the other gender such that for no pair of types  $(m, w)$  does there exist another pair  $(n, y)$  such that  $mQ(w)n$ ,  $wP(m)y$ ,  $x_{m,y} > 0$ , and  $x_{n,w} > 0$ .

Given their (strict) preferences, a pair of types  $(m, w)$  would block matching  $X$  if the pair  $(n, y)$  such that  $mQ(w)n$ ,  $wP(m)y$ ,  $x_{m,y} > 0$ , and  $x_{n,w} > 0$  exists. In this case, at least one man of type  $m$  is matched to a woman of type  $y$  but would strictly prefer to match to one of type  $w$ , while a woman of type  $w$  would prefer a type- $m$  man to the type- $n$  one she is matched with. Matching  $X$  would not be stable under those profiles of preferences.

Echenique et al. (2013) develops the testable implications of consistency with stability for aggregate matchings. The results rely on graph-theoretical language, so they are presented in full detail here. Instead, we illustrate their insight with an example.

Suppose that the following information has been observed by the analyst. There are three types of men,  $M = \{a, b, c\}$ , and three types of women,  $W = \{1, 2, 3\}$ ; there are three men of type  $b$ , three women of type 2, and two people of each of the other types of each gender. Seven couples form—one of each of the following pairs:  $(a, 1)$ ,  $(a, 2)$ ,  $(b, 1)$ ,  $(b, 2)$ ,  $(b, 3)$ ,  $(c, 2)$ , and  $(c, 3)$ . The observed matching, therefore, is

$$X = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{35}$$

Given a matching, one can construct a graph that takes all the positive entries of  $X$  as vertices, as in Fig. 11, and draws edges between any pair of vertices that are on the same row or same column. In the example, this graph is in Fig. 12. If this matching is stable under some strict preferences, all the edges can be oriented according to the preferences of the corresponding common type in the two vertices that define the edge. In Fig. 13, for example, the vertical arrow means that  $aQ(1)b$ , while the horizontal arrow means that  $1P(a)2$ . It must be possible to assign direction to each of the edges, turning them into arrows that inform us of the direction of preference.

Now, notice that if the arrows were as in Fig. 13, the pair  $(m, w) = (a, 1)$  would block the matching: there is one man of type  $a$  coupled with a woman of type 2 and a woman of type 1 coupled with a man of type  $b$ , and they would strictly prefer being coupled together. If  $X$  is to be consistent with stability, the orientation of the edges must be so that there is no pair of a vertical and a horizontal arrow pointing to the same vertex.

Say that the edge that goes from  $(a, 1)$  to  $(b, 1)$  points upwards, as in Fig. 13. Stability then requires that the edge from  $(a, 1)$  to  $(a, 2)$  point to the right, as in Fig. 14. The same stability argument now implies that the edges drawn from  $(a, 2)$  point away from it, as in Fig. 15. Fig. 16 follows, and it now implies that it is impossible to reconcile matching  $X$  with stability: the edge connecting  $(b, 3)$  and  $(c, 3)$  must point away from both of its vertices, as illustrated by Fig. 17.

<sup>42</sup> An assumption is that there are  $\sum_{m,w} x_{m,w}$  men and women, all of whom are matched.

<sup>41</sup> A singleton class  $\{m\}$  is in the partition  $\mathcal{P}^{r,s}$  when  $\mu^r(m) = \mu^s(m)$ .

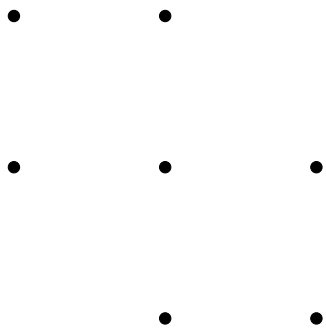


Fig. 11. The vertices induced by Eq. (35).

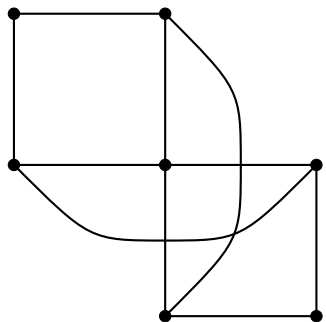


Fig. 12. The graph induced by Eq. (35).

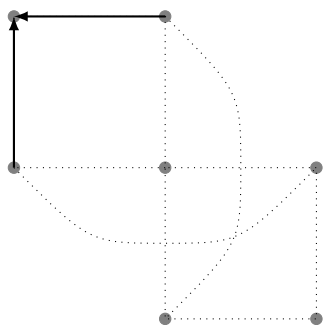


Fig. 13. A blocking pair.

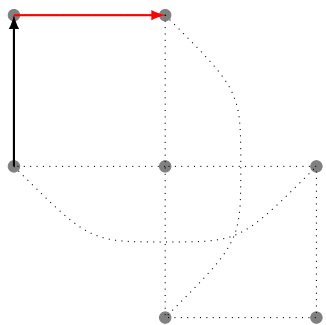


Fig. 14. Inference on  $P(a)$ .

The results in Echenique et al. (2013) extend this idea. They prove that an aggregate matching is consistent with stability if, and only if, its graph does not contain two cycles that are connected through some path, with some qualification. Demuyck and Salman (2022) offer an alternative characterization using bi-partite graph, while Carvajal and Zhang (2024) obtain a different one using only matrix-theoretical language. The latter paper also assesses the power of the test, its

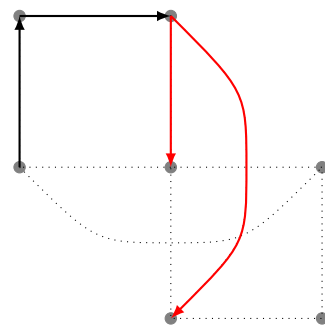


Fig. 15. Inference on  $Q(2)$ .

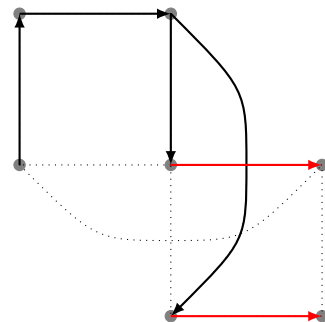


Fig. 16. Inference on  $P(b)$  and  $P(c)$ .

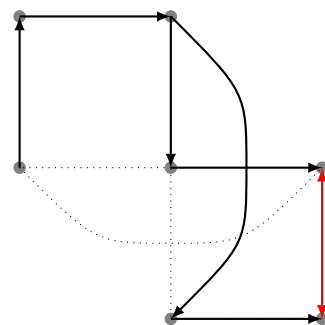


Fig. 17. Contradictory inference on  $Q(3)$ .

dependence on the size of the population of agents, and its robustness to the aggregation and disaggregation of types.

#### 5.4. Zero-sum games

Carvajal et al. (2004) covered the results from Sprumont (2000) on the testable implications of Nash equilibrium in finite, normal form games.<sup>43</sup> The observational assumption is that the analyst has very comprehensive information. The players are  $n = 1, \dots, N$ , and the set of actions of player  $n$  is  $X_n$ . The product set,  $X$ , is called the set of joint actions. Let  $\mathcal{X}$  denote the set of all non-empty subsets of  $X$  which are Cartesian products; a typical element of  $B$  of  $\mathcal{X}$  is written as  $B = \prod_n B_n$ . The analyst observes a choice function  $f$  that assigns to every possible subset  $B \in \mathcal{X}$  a non-empty subset  $f(B) \subseteq B$ . The hypothesis is that  $f(B)$  contains all the Nash equilibria of the game when the feasible sets of the players are restricted to be  $B_n$ , so that the game is played over  $B$ .

<sup>43</sup> For the analysis of mixed strategies, see Demuyck and Lauwers (2009) and Rehbeck (2018).

**Definition.** The observed function  $f$  captures all the Nash equilibria of some game if there exist complete pre-orders  $(P_n)$  over  $X$ , one for each player, such that for every  $B \in \mathcal{X}$ ,  $f(B)$  coincides with the set of Nash equilibria of the game defined over the feasible set  $B$  by those preferences.

**Sprumont (2000)** shows that two conditions characterize the hypothesis that a function captures all the Nash equilibrium of some game in this setting. The first condition says that when the set of actions shrinks, the equilibria that remain feasible must remain equilibria:  $f$  is said to satisfy *consistency under contraction* if for all  $B, C \in \mathcal{X}$  with  $C \subseteq B$ : (a)  $f(B) \cap C \subseteq f(C)$ ; and (b) if  $B_n$  is singleton for  $N - 1$  players and  $f(B) \cap C \neq \emptyset$ , then  $f(C) \subseteq f(B)$ . The second condition, called *consistency under expansion* says that a profile of actions that is an equilibrium in two games must remain equilibrium when the two games are “merged”: for any  $B, C \in \mathcal{X}$ ,  $f(B) \cap f(C) \subseteq f(\times_n(B_n \cup C_n))$ .

**Lee (2012)** refines this result to the case of two-person, zero-sum games. Let the two players be  $n = a, b$  and denote their preferences by  $P$  and  $Q$ , respectively.

**Definition.** The observed function  $f$  captures all the Nash equilibria of some two-person, zero-sum game if it captures all the Nash equilibria of some game and the two pre-orders  $(P, Q)$  satisfy  $xPy \Leftrightarrow yQx$ .

It is well known that in any zero-sum such game if  $x = (x_a, x_b)$  and  $y = (y_a, y_b)$  are both Nash equilibria, then so too are  $(x_a, y_b)$  and  $(y_a, x_b)$ . **Lee (2012)** proves that this extra structure is all that is needed to test Nash rationalizability in zero-sum games, in addition to the two conditions developed by **Sprumont (2000)**.

**Theorem 23 (Lee).** Let function  $f$  be given in a setting with two players,  $n = a, b$ . The following two statements are equivalent:

1. The function captures all the Nash equilibria of some zero-sum game.
2. The function satisfies consistency under contraction and under expansion and, for all  $B \in \mathcal{X}$ , if  $x = (x_a, x_b)$  and  $y = (y_a, y_b)$  are both in  $f(B)$ , then so too are  $(x_a, y_b)$  and  $(y_a, x_b)$ .

**Lee (2012)** calls the extra condition *interchangeability*.

### 5.5. Other literature and applications

This section of the paper covered important results in applying revealed preference analysis in game theory. By necessity, some work could not be covered in detail. First among this literature are the results on equilibrium in sequential games that followed the work of **Sprumont (2000, 2001)** and **Ray and Zhou (2001)**. Their extension to subgame-perfect Nash equilibrium by **Ray and Snyder (2013)** had already been covered in **Carvajal et al. (2004)**, and the reader interested in this literature should also study the results on “backward-induction rationalizability” by **Bossert and Sprumont (2013)** and **Rehbeck (2014)**.

As mentioned before, the analysis of mixed strategies can be found in **Demuyne and Lauwers (2009)** and **Rehbeck (2018)**. Notably, **Chambers et al. (2017)** argue that the revealed preference formulation of Nash equilibrium in game theory has universal implications, while **Galambos (2019)** shows that the problem of verifying if the observed choices of a group of players in a normal form game is consistent with Nash equilibrium is NP-complete, which poses a challenge for the formulation of a version of  $SARP$  that applies in this context.

In terms of applications, the surveyed literature has also been fruitful. **Banzhaf et al. (2019)**, for example, extends the tests of **Carvajal et al. (2013)** to the “tragedy of the commons” context of the exploitation of common-pool resources,<sup>44</sup> while **Matsukawa (2019)** uses the same tests to argue that collusive behavior among the participants of Japan’s electricity markets cannot be ruled out. Similarly, **Chassang**

**et al. (2022)** uses the tests to determine the existence of potentially non-competitive bidding in procurement auctions in Japan.

Methodologically, **Magnolfi et al. (2022)** extend the non-parametric tests of the Cournot model proposed by **Carvajal et al. (2013)** and **Cherchye et al. (2013a)** to provide an econometric framework for testing models of firm conduct in potentially non-competitive settings.

### CRedit authorship contribution statement

**Andrés Carvajal:** Writing – original draft, Investigation, Formal analysis.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Data availability

No data was used for the research described in the article.

### Appendix

**Proof of Claim 1.** From Eqs. (10) and (11),  $100X_1^r \geq 100X_1^s + (X_2^s - X_2^r) \geq 100X_1^s + \frac{1}{100}(X_1^r - X_1^s)$ , so  $X_1^r \geq X_1^s$ . That  $X_1^s \geq -2$  follows from Eqs. (18) and (20). The other inequalities are proven similarly.  $\square$

**Proof of Claim 2.** From Eq. (13), notice that  $x_{1,1}^r = \frac{1}{100}(901 + \pi^r - x_{1,2}^r)$ , so

$$\begin{aligned} p^s \cdot x_1^r &= \frac{1}{100}(901 + \pi^r - x_{1,2}^r) + 100x_{1,2}^r \\ &= 9.01 + \frac{1}{100}\pi^r + (100 - \frac{1}{100})x_{1,2}^r \\ &= 9.01 + \frac{1}{100}(100X_1^r + X_2^r) + (100 - \frac{1}{100})(2 + X_2^r - x_{2,2}^r) \\ &\leq 208.99 + X_1^r + 100X_2^r \\ &\leq 208.99 + X_1^s + 100X_2^s \\ &< 901 + X_1^s + 100X_2^s \\ &= p^s \cdot \omega_1^s + \pi^s \end{aligned}$$

where the third equality follows from Eq. (17), the first inequality from Eq. (19) and the second one from Eq. (11). The other inequality is proven similarly.  $\square$

**Proof of Claim 3.** By Eq. (21), either  $X_1^r \leq 0$  or  $X_2^r < 0$ . Consider first the case  $X_1^r \leq 0$ , which implies, by Claim 1, that  $X_1^s \leq 0$ . Then, by Eqs. (18), and (20) and the fact that  $X_1^s \leq 0$ , we have that  $x_{1,1}^s = 2 + X_1^s - x_{2,1}^s \leq 2$ . On the other hand, by Eq. (17),  $x_{1,1}^r = 10 + X_1^r - x_{2,1}^r \geq 8 - x_{2,1}^r \geq 6.99$ , where the first inequality follows from Claim 1, and the second inequality from Eqs. (15) and (19), which imply that  $x_{2,1}^r = \frac{1}{100}(101 - x_{2,2}^r) \leq 1.01$ .

Now, suppose that  $X_2^r < 0$ . Then, by Eqs. (17) and (19),  $x_{1,2}^r = 2 + X_2^r - x_{2,2}^r < 2 - x_{2,2}^r \leq 2$ , whereas  $x_{1,2}^s = 10 + X_2^s - x_{2,2}^s \geq 8 - x_{2,2}^s \geq 6.99$ , which follows from Eq. (18), Claim 1, and Eqs. (16) and (20).<sup>45</sup>  $\square$

<sup>44</sup> See also **Koch and Nax (2022)**.

<sup>45</sup> The last two conditions imply that  $x_{2,2}^s \leq 1.01$ .

## References

- Afriat, S.N., 1967. The construction of utility functions from expenditure data. *Internat. Econom. Rev.* 8 (1), 67–77.
- Agatsuma, Y., 2016. Testable implications of the core in TU market games. *J. Math. Econom.* 64, 23–29.
- Amarante, V., Rossel, C., Scalse, F., 2023. Housework and earnings: intrahousehold evidence from Latin America. *J. Fam. Stud.* 1–21.
- Bachmann, R., 2006. Testable implications of Pareto efficiency and individual rationality. *Econom. Theory* 29 (3), 489–504.
- Balasko, Y., 2004. The equilibrium manifold keeps the memory of individual demand functions. *Econom. Theory* 24 (3), 493–501.
- Banzhaf, H.S., Liu, Y., Smith, M., Asche, F., 2019. Non-Parametric Tests of the Tragedy of the Commons. Technical Report, National Bureau of Economic Research.
- Blundell, R., Pashardes, P., Weber, G., 1993. What do we learn about consumer demand patterns from micro data? *Am. Econ. Rev.* 570–597.
- Bossert, W., Sprumont, Y., 2013. Every choice function is backwards-induction rationalizable. *Econometrica* 81 (6), 2521–2534.
- Brown, D.J., Kannan, R., 2008. Two algorithms for solving the Walrasian equilibrium inequalities. In: Brown, D.J., Kübler, F. (Eds.), *Computational Aspects of General Equilibrium Theory: Refutable Theories of Value*. In: *Lecture Notes in Economics and Mathematical Systems*, vol. 604. Springer, pp. 67–77.
- Brown, D.J., Matzkin, R.L., 1996. Testable restrictions on the equilibrium manifold. *Econometrica* 64 (6), 1249–1262.
- Browning, M., Chiappori, P.-A., 1998. Efficient intra-household allocations: A general characterization and empirical tests. *Econometrica* 1241–1278.
- Carvajal, A.M., 2003. On Individually Rational Choice and Equilibrium. Brown University.
- Carvajal, A., 2010. The testable implications of competitive equilibrium in economies with externalities. *Econ. Theory* 45 (1–2), 349–378.
- Carvajal, A., Deb, R., Fenske, J., Quah, J.K.-H., 2013. Revealed preference tests of the Cournot model. *Econometrica* 81 (6), 2351–2379.
- Carvajal, A., Deb, R., Fenske, J., Quah, J.K.-H., 2014. A nonparametric analysis of multi-product oligopolies. *Econom. Theory* 57 (2), 253–277.
- Carvajal, A., González, N., 2014. On refutability of the Nash bargaining solution. *J. Math. Econom.* 50, 177–186.
- Carvajal, A., Ray, I., Snyder, S., 2004. Equilibrium behavior in markets and games: testable restrictions and identification. *J. Math. Econom.* 40 (1–2), 1–40.
- Carvajal, A., Riasco, A., 2005. Identification of preferences from market data. *BE J. Theor. Econ.* 5 (1), 0000102202153459631195.
- Carvajal, A., Riasco, A.J., 2008. Identification of individual demands from market data under uncertainty. *BE J. Theor. Econ.* 8 (1), 0000102202193517041420.
- Carvajal, A., Song, X., 2018. Testing Pareto efficiency and competitive equilibrium in economies with public goods. *J. Math. Econom.* 75, 19–30.
- Carvajal, A., Zhang, T.D., 2024. On the revealed preference test for aggregate matching: equivalent formulations, properties, and an application.
- Chambers, C.P., Echenique, F., 2014. On the consistency of data with bargaining theories. *Theor. Econ.* 9 (1), 137–162.
- Chambers, C.P., Echenique, F., 2016. *Revealed Preference Theory*. Vol. 56, Cambridge University Press.
- Chambers, C.P., Echenique, F., 2021. Empirical welfare economics. *arXiv preprint arXiv:2108.03277*.
- Chambers, C.P., Echenique, F., Shmaya, E., 2017. General revealed preference theory. *Theor. Econ.* 12 (2), 493–511.
- Chassang, S., Kawai, K., Nakabayashi, J., Ortner, J., 2022. Robust screens for noncompetitive bidding in procurement auctions. *Econometrica* 90 (1), 315–346.
- Cherchye, L., De Rock, B., Vermeulen, F., 2007. The collective model of household consumption: a nonparametric characterization. *Econometrica* 75 (2), 553–574.
- Cherchye, L., Demuynck, T., De Rock, B., 2011. Testable implications of general equilibrium models: an integer programming approach. *J. Math. Econom.* 47 (4–5), 564–575.
- Cherchye, L., Demuynck, T., De Rock, B., 2013a. The empirical content of Cournot competition. *J. Econom. Theory* 148 (4), 1552–1581.
- Cherchye, L., Demuynck, T., De Rock, B., 2013b. Nash-bargained consumption decisions: a revealed preference analysis. *Econ. J.* 123 (567), 195–235.
- Chiappori, P.-A., 1988. Rational household labor supply. *Econometrica* 63–90.
- Chiappori, P.-A., 1992. Collective labor supply and welfare. *J. Political Econ.* 100 (3), 437–467.
- Chiappori, P.-A., Donni, O., Komunjer, I., 2012. Learning from a piece of pie. *Rev. Econ. Stud.* 79 (1), 162–195.
- Chiappori, P.-A., Ekeland, I., Kübler, F., Polemarchakis, H.M., 2004. Testable implications of general equilibrium theory: a differentiable approach. *J. Math. Econom.* 40 (1–2), 105–119.
- Chiappori, P.-A., Rochet, J.-C., 1987. Revealed preferences and differentiable demand. *Econometrica* 55 (3), 687–691.
- De Clippel, G., Rozen, K., 2021. Bounded rationality and limited data sets. *Theor. Econ.* 16 (2), 359–380.
- de Clippel, G., Rozen, K., 2022. Empirical content of classic assignment methods: Jungle and market economy. *Econom. Theory* 1–13.
- Deb, R., 2009. A testable model of consumption with externalities. *J. Econom. Theory* 144 (4), 1804–1816.
- Deb, R., Gazzale, R.S., Kotchen, M.J., 2014. Testing motives for charitable giving: A revealed-preference methodology with experimental evidence. *J. Public Econ.* 120, 181–192.
- Demuynck, T., Lauwers, L., 2009. Nash rationalization of collective choice over lotteries. *Math. Social Sci.* 57 (1), 1–15.
- Demuynck, T., Salman, U., 2022. On the revealed preference analysis of stable aggregate matchings. *Theor. Econ.* 17 (4), 1651–1682.
- Donni, O., 2016. Collective models of the household. In: *The New Palgrave Dictionary of Economics*. Palgrave Macmillan UK, London, pp. 1–4.
- Dziewulski, P., Lanier, J., Quah, J.K.-H., 2024. Revealed preference and revealed preference cycles: a survey.
- Echenique, F., 2008. What matchings can be stable? The testable implications of matching theory. *Math. Oper. Res.* 33 (3), 757–768.
- Echenique, F., Lee, S., Shum, M., Yenmez, M.B., 2013. The revealed preference theory of stable and extremal stable matchings. *Econometrica* 81 (1), 153–171.
- Forges, F., Minelli, E., 2009. Afriat's theorem for general budget sets. *J. Econom. Theory* 144 (1), 135–145.
- Galambos, A., 2019. Descriptive complexity and revealed preference theory. *Math. Social Sci.* 101, 54–64.
- Koch, C.M., Nax, H.H., 2022. Groundwater usage and strategic complements, part II: revealed preferences. *Games* 13 (5), 68.
- Kübler, F., Chiappori, P.-A., Ekeland, I., Polemarchakis, H.M., 2002. The identification of preferences from equilibrium prices under uncertainty. *J. Econom. Theory* 102 (2), 403–420.
- Lazzati, N., Quah, J.K.-H., Shirai, K., 2021. An ordinal approach to the empirical analysis of games with monotone best responses. URL <https://people.ucsc.edu/~nlazzati/Papers/LQS.pdf>.
- Lee, S., 2012. The testable implications of zero-sum games. *J. Math. Econom.* 48 (1), 39–46.
- Magnolfi, L., Quint, D., Sullivan, C., Waldfoegel, S., 2022. *Falsifying Models of Firm Conduct*. Technical Report, Working paper.
- Matsukawa, I., 2019. Detecting collusion in retail electricity markets: results from Japan for 2005 to 2010. *Util. Policy* 57, 16–23.
- Matzkin, R.L., Richter, M.K., 1991. Testing strictly concave rationality. *J. Econom. Theory* 53 (2), 287–303.
- Monderer, D., Shapley, L.S., 1996. Potential games. *Games Econ. Behav.* 14 (1), 124–143.
- Nishimura, H., Ok, E.A., Quah, J.K.-H., 2017. A comprehensive approach to revealed preference theory. *Amer. Econ. Rev.* 107 (4), 1239–1263.
- Ray, I., Snyder, S., 2013. Observable implications of Nash and subgame-perfect behavior in extensive games. *J. Math. Econom.* 49 (6), 471–477.
- Ray, I., Zhou, L., 2001. Game theory via revealed preferences. *Games Econ. Behav.* 37 (2), 415–424.
- Rehbeck, J., 2014. Every choice correspondence is backwards-induction rationalizable. *Games Econ. Behav.* 88, 207–210.
- Rehbeck, J., 2018. Note on unique Nash equilibrium in continuous games. *Games Econ. Behav.* 110, 216–225.
- Samuelson, P.A., 1938. A note on the pure theory of consumer's behaviour. *Economica* 5 (17), 61–71.
- Samuelson, P.A., 1947. *Foundations of Economic Analysis*. Harvard University Press.
- Slade, M.E., 1994. What does an oligopoly maximize? *J. Ind. Econ.* 45–61.
- Snyder, S.K., 1999. Testable restrictions of Pareto optimal public good provision. *J. Public Econ.* 71 (1), 97–119.
- Snyder, S.K., 2000. Nonparametric testable restrictions of household behavior. *South. Econ. J.* 67 (1), 171–185.
- Sprumont, Y., 2000. On the testable implications of collective choice theories. *J. Econom. Theory* 93 (2), 205–232.
- Sprumont, Y., 2001. Paretian quasi-orders: the regular two-agent case. *J. Econ. Theory* 101 (2), 437–456.
- Toda, A.A., Walsh, K.J., 2024. Recent advances on uniqueness of competitive equilibrium. *arXiv:2402.00998*.
- Varian, H.R., 1982. The nonparametric approach to demand analysis. *Econometrica* 50 (4), 945–973.
- Varian, H.R., 1984. The nonparametric approach to production analysis. *Econometrica* 52 (3), 579–597.
- Vermeulen, F., 2002. Collective household models: principles and main results. *J. Econ. Surv.* 16 (4), 533–564.
- Vu, T.M., 2023. Temporary migrants and gender housework division among left-behind household members. *Rev. Dev. Econ.* 27 (3), 1834–1854.
- Yasui, Y., 2023. Revealed preference tests for price competition in multi-product differentiated markets. *Econom. Theory* 1–30.
- Zhao, C., Qu, X., 2024. Place-based policies, rural employment, and intra-household resources allocation: Evidence from China's economic zones. *J. Dev. Econ.* 167, 103210.
- Zhou, L., 2006. The structure of the Nash equilibrium sets of standard 2-player games. In: *Rationality and Equilibrium: A Symposium in Honor of Marcel K. Richter*. Springer, pp. 57–66.