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Implementing Lindahl allocations in a warm-glow economy☆

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ABSTRACT

We provide a mechanism, the Nash equilibria of which coincide with the Lindahl allocations with two personalized prices in an economy that may display warm-glow preferences. The traditional techniques in implementing Lindahl allocations crucially depend on one Lindahl price for each consumer. The novel contribution of our mechanism is that it offers a construction of two personalized prices for each consumer, their individual contributions and the corresponding tax rules that align consumers' incentives to restore efficiency. Our construction has potential applications to efficient provision of public goods in networks, which would require multiple Lindahl prices.

1. Introduction

The warm-glow model, based on the earlier work by Becker (1974) and Cornes and Sandler (1984), was proposed by Andreoni (1989, 1990) as an alternative to the classical public goods model to explain consumers' public goods contribution behavior. In the classical model, consumer i's utility function is $\upsilon^i(x^i, Y)$, where x^i is her private consumption and Y denotes the aggregate provision of the public goods. Therefore, consumers are assumed to care about the aggregate level of public goods provision only. Bergstrom et al. (1986) demonstrated that the classical model implies that government provision will crowd out private contribution dollar for dollar, and the public goods provision is neutral to the income redistribution up to the set of contributors not changing. Andreoni (1988) showed that the classical model implies that virtually no one makes contributions in a large economy. These theoretical implications have been challenged by field and experimental evidence, as shown in Andreoni (1989). Instead, in the warm-glow model, consumer i's utility function is $\upsilon^i(x^i, y^i, Y)$, where y^i is her provision of the public goods, and the consumer derives 'warm-glow' utility from giving. From the perspective of a consumer with warm-glow preferences, her contribution is not a perfect substitute for other consumers' contribution, which implies that the crowding-out is not complete, income redistribution effect is not neutral and free-riding is less severe. The warm-glow model is considered to be more consistent with observed consumer behavior and is widely used in public economics.

Significant progress has been made in terms of the study of institutions that achieve Pareto efficiency of public goods provision without warm-glow. In the classical model, competitive markets fail the conditions for Pareto efficiency characterized by Samuelson (1954). To restore the efficiency of competitive markets, each consumer must be charged differential taxes (Lindahl prices) proportional to her benefit, as argued by Lindahl (1958) and Foley (1970). As Arrow (1970) pointed out, however, Lindahl's solution does not satisfy the incentive compatibility constraints of Hurwicz (1972), and consumers have no incentive to truthfully reveal their valuation for the public goods. Groves and Ledyard (1977) and Walker (1981) invented a mixed competitive mechanism, in which the private goods are allocated by competitive markets and the government designs a game to allow consumers (strategically reporting messages concerning their preferences) to determine the aggregate supply of public goods and the taxes imposed on each consumer. The equilibria of this mechanism are Pareto optimal. Tian (1989) designed a

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game form to allocate both private and public goods to achieve Lindahl equilibria, and the mechanism is single-valued, feasible and continuous. As argued by Tian (1989) and tested in laboratory experiments (Chen, 2008), these properties are important for the mechanism to work in practice.

However, the question of achieving Pareto efficiency of public goods provision with warm-glow by institution design has been largely unexplored until recently. Allouch (2013) extended Lindahl's solution to achieve Pareto efficient provision of public goods in a warm-glow economy. In his *warm-glow equilibrium*, each consumer has two personalized Lindahl prices for each public good, instead of one as in the classical public goods model, with one Lindahl price q^i for her own contribution and the other q^{-i} for other consumers' contribution. Carvajal and Song (2022) offered an alternative solution. In their *warm-glow Lindahl equilibrium*, each 'impure' public good has two market prices: one as a private good and the other as a pure public good. Each consumer pays the same price for the private good and also has one personalized price, the sum of which is the production price for the supply of public good. Carvajal and Song (2022) took advantage of one personalized price in their warm-glow Lindahl equilibrium and modified Tian's mechanism to implement Pareto efficient provision of public goods with warm-glow.

It is an interesting and important question whether the warm-glow equilibrium with two personalized prices in Allouch (2013) is implementable. First, to implement the warm-glow equilibrium with two personalized prices requires inventing new techniques. The mechanism has to account for two new issues: the first is to decide two Lindahl prices for each consumer, instead of one Lindahl price; the second is to decide the individual contribution of public goods, instead of the aggregate level only. The traditional techniques in implementing Lindahl allocations, as used by Carvajal and Song (2022), crucially depend on there being only one Lindahl price for each consumer, and are not straightforward to deal with these issues. Second, the study of implementation of the warm-glow equilibrium with two personalized prices has potential applications in the burgeoning literature on efficient provision of public goods in networks (Elliot and Golub, 2019). Consumers' altruistic behavior could be motivated by geographic spillovers (Allouch and King, 2019) or social relationships (Scharf and Smith, 2016). Both the classical model and the warm-glow model can be incorporated in a network model: consumer i's utility function is $\upsilon^i(x^i, \sum_{j\in C^s} y^j, Y)$, where C^s is a geographic or social community that consumer i is in, and the consumer cares about the group identity.[1] Lindahl allocations will require two personalized Lindahl prices for each consumer, with one Lindahl price q^i for her community's contribution and the other q^{-i} for the contribution outside of her community. The equilibrium concept and implementation mechanism in Carvajal and Song (2022) has limited use in this general network model.

In this paper, we propose a novel mechanism to achieve warm-glow equilibria in an economy where the agents may have warm-glow. Our mechanism is novel in redefining the Lindahl prices, individual contributions and taxes levied on consumers to align the consumers' incentives in the right way. As in the Nash implementation literature of Hurwicz (1972) and Maskin (1999), consumers strategically report messages concerning their preferences, and the mechanism designer simply collects these messages and implements the outcome according to the mechanism. In equilibria, all consumers will report truthfully, and the Nash equilibria fully implement the warm-glow equilibria of the economy.

[1] In the classical model, all consumers are in one community; in the warm-glow model, each community has one consumer.

Our mechanism possesses nice properties: it is individually rational, single-valued, feasible and continuous. We allow for warm-glow effects but do not assume that they are present in the economy.[2] When no warm-glow preference is present in the economy, our mechanism fully implements the classical Lindahl allocations. Importantly, our construction can shed light on how to design a mechanism to implement Lindahl allocations on the network structure mentioned above or on the network structure in Elliot and Golub (2019), which is a subject of future research.[3]

2. Environment

2.1. The economy

Consider an economy with L private goods and K public goods, which we will index by $\ell = 1, \ldots, L$ and $k = 1, \ldots, K$, respectively. The public goods are produced using the private goods, and the aggregate production function is $f : X \subseteq \mathbb{R}^L_+ \mapsto Y \subseteq \mathbb{R}^K_+$.

There are finitely many consumers in the economy, indexed by $i = 1, \ldots, I$ with $I \geq 3$. Consumer i's endowment, ω^i, is a bundle of the private goods, and there are no endowments of the public goods. Following Andreoni (1989, 1990), we assume she derives utility from her consumption of the private goods, (potentially) from her individual provision of the public goods, and from the aggregate provision of the public goods, according to a utility function $\upsilon^i(x, y, Y)$, where $x \geq 0$ is her private consumption, $y \geq 0$ her provision of the public goods, and $Y \geq y$ the aggregate provision of the latter.[4]

Throughout the paper we will maintain the following assumptions: each function $\upsilon^i(x, y, Y)$ is continuous, strictly increasing and strictly quasi-concave in (x, Y) and non-decreasing in y, and satisfies that (i) for any (x, Y), if $\upsilon^i(x, y, Y) > \upsilon^i(x, \bar{y}, Y)$, then $\upsilon^i(x, \alpha y + (1-\alpha)\bar{y}, Y) > \upsilon^i(x, \bar{y}, Y)$ for all $\alpha \in (0, 1]$, while (ii) for any $(\bar{x}, \bar{Y}) \in \partial\mathbb{R}^{L+K}_+$, any $(x, Y) \in \mathbb{R}^{L+K}_{++}$, any $y \leq Y$ and any $\bar{y} \leq \bar{Y}$, $\upsilon^i(x, y, Y) > \upsilon^i(\bar{x}, \bar{y}, \bar{Y})$; each $\omega^i \gg 0$; and function f displays constant returns to scale.[5]

With these assumptions, our setting *allows for* warm glow effects but *does not assume* that they are present in the economy.

We will use capital letters to denote the aggregate variables of the economy, and lower case for the ones corresponding to an individual; we will use bold-face to denote profiles of individual variables and denote $Y^{-i} = \sum_{j\neq i} y^j$ for a profile $\mathbf{y} = (y^1, \ldots, y^I) \in \mathbb{R}^{KI}$.

2.2. Warm-glow equilibrium

An allocation in this economy is a tuple $(\mathbf{x}, \mathbf{y}, X, Y)$; it is feasible if

$$\textstyle\sum_i x^i + X \leq \sum_i \omega^i \quad \text{and} \quad \sum_i y^i \leq Y \leq f(X). \tag{1}$$

If private markets were to operate in the economy, a competitive equilibrium would consist of a price for the private goods, p, a price for the public goods, q, and an allocation that satisfies Eq. (1) with equalities and such that: (X, Y) solves $\max_{X,Y} \{q \cdot Y - p \cdot X : Y \leq f(X)\}$; and each (x^i, y^i) solves $\max_{x,y} \left\{\upsilon^i(x, y, y + Y^{-i}) : p \cdot x + q \cdot y \leq p \cdot \omega^i\right\}$.

[2] As shown by Carvajal and Song (2018), both Lindahl and Nash–Walras equilibria of a warm-glow economy have strong non-parametric testable implications. Therefore, the warm-glow effect can be identified.

[3] Elliot and Golub (2019) proved that the Lindahl allocations in their externality network satisfy the Maskin monotonicity, hence are implementable. However, a concrete implementation mechanism is unknown.

[4] Sometimes we will write $\upsilon^i(x, y, y+Y)$, in which case Y is to be understood as the provision of the public good that is made in addition to y.

[5] Under our assumptions, it will not be necessary to specify a distribution for the ownership of the firm.

Because of the existence of public goods, the competitive equilibrium allocation would not be efficient as demonstrated by Samuelson (1954).[6] To restore efficiency, the following Lindahl solution is given by Allouch (2013) for the present context, which requires the introduction of two personalized prices for each individual, q^i and $q^{\neg i}$.

Definition 1. A *warm-glow equilibrium* is a tuple $[P, (q^i, q^{\neg i})_{i=1}^I, Q, \bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{X}, \bar{Y}]$ such that:

(a) pair $(\bar{X}, \bar{Y})$ solves the program $\max_{X,Y}\{Q \cdot Y - P \cdot X : Y \leq f(X)\}$;
(b) each triple $(\bar{x}^i, \bar{y}^i, \bar{Y}^{\neg i})$ solves the program

$$\max_{x,y,Y}\left\{\upsilon^i(x, y, y+Y) : P \cdot x + q^i \cdot y + q^{\neg i} \cdot Y \leq P \cdot \omega^i\right\};$$

(c) Eq. (1) holds with equalities; and
(d) for each individual i, $q^i + \sum_{j \neq i} q^{\neg j} = Q$.

Remark 1. In the classical public goods model, each consumer regards her contribution and others' contribution as perfect substitute, and this entails $q^i = q^{\neg i}$. Therefore, this warm-glow equilibrium accommodates the classical Lindahl equilibrium as a special case.

3. Implementation

3.1. Mechanisms

A mechanism (or game form) fixes a message space for each individual and an outcome function defined over the product of the message spaces. Let M^i denote the message space of individual i, with generic element m^i. Let $M = \prod_i M^i$ and denote the outcome function $\varphi : M \to \mathbb{R}^{(K+L)I}$ as

$$\varphi(\mathbf{m}) = \left(x^i(\mathbf{m}), y^i(\mathbf{m})\right)_{i=1}^I$$

with

$$\varphi^i(\mathbf{m}) = \left(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})\right).$$

The outcome function determines, for each profile of messages, an allocation for the economy: the individual variables are given by the function, usage of the private goods by the firm is $X(\mathbf{m}) = \sum_i[\omega^i - x^i(\mathbf{m})]$, its output is bundle $Y(\mathbf{m}) = f(X(\mathbf{m}))$, and the mechanism is feasible if $Y(\mathbf{m}) \geq \sum_i y^i(\mathbf{m})$, for all profiles of messages.

3.2. Nash implementation

The mechanism defines a game, with strategy spaces given by the message spaces and payoff functions $\pi^i(\mathbf{m}) = \upsilon^i(\varphi^i(\mathbf{m}))$.

The set of allocations that are attained at the Nash equilibria of the game defined by the mechanism, given the economy, is the set *implemented* by the mechanism (in Nash equilibrium).

In what follows, we will further denote $Y^{\neg i}(\mathbf{m}) = \sum_{j \neq i} y^j(\mathbf{m})$ and $\mathbf{m}^{\neg i} = (m^j)_{j \neq i}$.

3.3. The mechanism

Building on the ideas of Walker (1981) and Tian (1989), we invent a new mechanism to implement the warm-glow equilibria. The mechanism designer does not know consumers' preferences. For the simplicity of exposition, we assume the mechanism designer knows consumers' endowments. The message space for all individuals is now

$$M^i = \mathbb{R}^L_{++} \times \mathbb{R}^K_{++} \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^K_{++} \times \mathbb{R}^L_{+} \times \mathbb{R}_{++},$$

with typical element $m^i = (P^i, Q^i, a^i, b^i, c^i, d^i, e^i, n^i)$. Intuitively, c^i denotes a *claim* about agent i's public goods contribution. Vector e^i is agent i's demand for private goods, while vectors P^i and Q^i can be understood as the agent's recommendation of the prices of the private goods and the prices paid to the firm for the public goods; their entries are denoted e^i_ℓ, P^i_ℓ and Q^i_k, respectively. Number n^i, finally, will be used in a "shouting" game.

The definition of the outcome function, which makes the mechanism single-valued, feasible and continuous, is as follows: given any profile $\mathbf{m}$ of messages,

1. The market prices of the private goods and the producer prices of the public goods are constructed as follows: first, let

$$s^i(\mathbf{m}) = \sum_{j,j' \neq i} \|(P^j, Q^j) - (P^{j'}, Q^{j'})\|$$

and $s(\mathbf{m}) = \sum_i s^i(\mathbf{m})$; then, if $s(\mathbf{m}) = 0$, let $P(\mathbf{m}) = P^1$ and $Q(\mathbf{m}) = Q^1$; otherwise,

$$[P(\mathbf{m}), Q(\mathbf{m})] = \frac{1}{s(\mathbf{m})} \sum_i s^i(\mathbf{m})(P^i, Q^i).$$

2. For each individual, there are two personalized prices for each public good k:

$$q^i_k(\mathbf{m}) = \frac{1}{I} Q_k(\mathbf{m}) - \sum_{j \neq i} a^j_k \tag{2}$$

and

$$q^{\neg i}_k(\mathbf{m}) = \frac{1}{I} Q_k(\mathbf{m}) - \sum_{j \neq i} a^j_k + \frac{1}{I-1} \sum_{j \neq i} b^j_k. \tag{3}$$

3. The set of bundles of public goods that can be produced and is affordable by all individuals, as a result of the prices, is the set of $Y \in R^K_+$ for which

   (a) there exists a profile $\mathbf{y}$ such that $Y = \sum_i y^i$,

$$P(\mathbf{m}) \cdot \omega^i - q^i(\mathbf{m}^{\neg i}) \cdot y^i - q^{\neg i}(\mathbf{m}) \cdot Y^{\neg i} \geq 0,$$

   and

$$\sum_i \left[q^i(\mathbf{m}) + \sum_{j \neq i} q^{\neg j}(\mathbf{m})\right] \cdot y^i \geq Q(\mathbf{m}) \cdot Y$$

   for all i; and
   (b) $f(\sum_i \omega^i) \geq Y$.

   Denoting this set by $\mathcal{B}(\mathbf{m})$, the actual supply of public goods is

$$Y(\mathbf{m}) = \operatorname*{argmin}_Y \left\{\| \textstyle\sum_i c^i - Y \| : Y \in \mathcal{B}(m)\right\},$$

   and its allocation is, for each good k,

$$y^i_k(\mathbf{m}) = \frac{d^i_k}{\sum_j d^j_k} Y_k(\mathbf{m}).$$

4. There is a *punishment tax* levied on individual i at a rate

$$\tau^i(\mathbf{m}) = 1 - \exp\{-\|b^i - \textstyle\sum_j a^j\|\},$$

   so the total tax paid by her is the sum of

$$q^i(\mathbf{m}) \cdot y^i(\mathbf{m}) + q^{\neg i}(\mathbf{m}) \cdot [Y(\mathbf{m}) - y^i(\mathbf{m})]$$

[6] Actually it can be shown that the competitive allocation is generically constrained suboptimal (Geanakoplos and Polemarchakis, 2008).

and

$$\tau^i(\mathbf{m})\cdot\left[P(\mathbf{m})\cdot\omega^i - q^i(\mathbf{m})\cdot y^i(\mathbf{m}) - q^{-i}(\mathbf{m})\cdot[Y(\mathbf{m}) - y^i(\mathbf{m})]\right].$$

We denote this sum $T^i(\mathbf{m})$.

5. The set of bundles of private goods that can be produced and is affordable by individual i is $\mathcal{B}^i(\mathbf{m})$, defined as

$$\{x^i \in R^L_+ \mid P(\mathbf{m})\cdot x^i \le P(\mathbf{m})\cdot\omega^i - T^i(\mathbf{m}) \text{ and } f(\textstyle\sum_j \omega^j - x^i) \ge Y(\mathbf{m})\},$$

and the bundle of private goods for individual i closest to her claim is

$$\hat{x}^i(\mathbf{m}) = \operatorname*{argmin}_x \left\{\|x - e^i\| : x \in \mathcal{B}^i(\mathbf{m})\right\}. \tag{4}$$

In order to determine the actual allocation of the private good define a "shrinking factor"

$$N(\mathbf{m}) = \left\{N \in R_{++} : f\left(\textstyle\sum_i \omega^i - \sum_i \frac{n^i}{N}\hat{x}^i(\mathbf{m})\right) \ge Y(\mathbf{m}) \text{ and } N \ge n^i \text{ for all } i\right\}, \tag{5}$$

and then let

$$x^i(\mathbf{m}) = \frac{n^i}{N(\mathbf{m})}\hat{x}^i(\mathbf{m}). \tag{6}$$

Importantly, note that the personalized prices of individual i do *not* depend on her message, so we can denote them as $q^i(\mathbf{m}^{-i})$ and $q^{-i}(\mathbf{m}^{-i})$. Similarly, the punishment tax paid by i depends on $\mathbf{a} = (a^1, \dots, a^I)$ and b^i, but not on other arguments, so we denote it as $\tau^i(\mathbf{a}, b^i)$. By construction, $0 \le \tau^i(\mathbf{a}, b^i) < 1$, and the punishment tax is null when, and only when, $b^i = \sum_j a^j$.

Remark 2. In the above construction, consumers' endowment is assumed to be known to the mechanism designer. This is just for simplicity of the argument. Instead, we can ask consumers to report their endowment as in Tian (1989), and consumers will truthfully report their endowment in equilibria. This does not bring more insight.

The following lemmata will be useful in the proof of the implementation theorem. The first one regards properties of the mechanism, the second is on properties of the Nash equilibria of the game it induces.

Lemma 1. *The mechanism gives non-negative consumption to all individuals, is feasible and satisfies:*

(a) If $\upsilon^i(\hat{x}^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})) > \upsilon^i(x, y, Y)$, then individual i can choose a message $\hat{m}$ such that $\pi^i(\hat{m}, \mathbf{m}^{-i}) > \upsilon^i(x, y, Y)$.

(b) Suppose that $x^i(\mathbf{m}) \in R^L_{++}$ for all individuals and $Y(\mathbf{m}) \in R^L_{++}$. If for some i there exists (x, y, Y) with $Y = \sum_{j\neq i} y^j$ such that $p(\mathbf{m})\cdot x \le p(\mathbf{m})\cdot\omega^i - T^i(\mathbf{m})$,

$$[q^i(\mathbf{m}) + \textstyle\sum_{j\neq i} q^{-j}(\mathbf{m})]\cdot y + \sum_{j\neq i}[q^j(\mathbf{m}) + \sum_{j'\neq j} q^{-j'}(\mathbf{m})]\cdot y^j \ge Q(\mathbf{m})\cdot(y + Y),$$

and

$$\upsilon^i(x, y, y + Y) > \upsilon^i(x^i(\mathbf{m}), y^i(\mathbf{m}), y^i(\mathbf{m}) + Y^{-i}(\mathbf{m})),$$

then there exists a message that i can send, $\hat{m}$, such that $\pi^i(\hat{m}, \mathbf{m}^{-i}) > \pi^i(\mathbf{m})$.

Proof. Non-negativity of individual consumption and feasibility of the individual bundles follow from the properties of the punishment function τ^i and the construction of $Y(\mathbf{m})$ and $x^i(\mathbf{m})$.

For property (a), since $N(\hat{n}, \mathbf{m}^{-i}) \ge \hat{n}$, individual i can announce a large enough $\hat{n}$ to make $x^i(\hat{m}, \mathbf{m}^{-i}) \approx \hat{x}^i(\mathbf{m})$, which, suffices for the result, by continuity of υ^i. For property (b), define, for $\lambda \in (0, 1)$, the convex combination

$$(x_\lambda, y_\lambda, Y_\lambda) = \lambda(x, y, Y) + (1-\lambda)[x^i(\mathbf{m}), y^i(\mathbf{m}), Y^{-i}(\mathbf{m})].$$

Note that

$$P(\mathbf{m})\cdot x_\lambda \le P(\mathbf{m})\cdot\omega^i - T^i(\mathbf{m}),$$

$$[q^i(\mathbf{m}) + \textstyle\sum_{j\neq i} q^{-j}(\mathbf{m})]\cdot y_\lambda + \sum_{j\neq i}[q^j(\mathbf{m}) + \sum_{j'\neq j} q^{-j'}(\mathbf{m})]\cdot y^j_\lambda \ge Q(\mathbf{m})\cdot(y_\lambda + Y_\lambda),$$

and

$$\upsilon^i(x_\lambda, y_\lambda, y_\lambda + Y_\lambda) > \upsilon^i(x^i(\mathbf{m}), y^i(\mathbf{m}), y^i(\mathbf{m}) + Y^{-i}(\mathbf{m})).$$

Define the message $\hat{m}$ as: $\hat{P} = P^i$, $\hat{Q} = Q^i$, $\hat{a} = a^i$, $\hat{b} = b^i$, $\hat{c} = Y_\lambda + y_\lambda - \sum_{j\neq i} c^j$, $\hat{e} = x_\lambda$, and $\hat{d}$ such that

$$\frac{\hat{d}_k}{\hat{d}_k + \sum_{j\neq i} d^j_k}(Y_{\lambda,k} + y_{\lambda,k}) = y_{\lambda,k}$$

for each public good. If λ is small enough, by continuity $\hat{x}^i(\hat{m}, \mathbf{m}^{-i}) = x_\lambda$, and $Y(\hat{m}, \mathbf{m}^{-i}) = Y_\lambda + y_\lambda$, which further implies that $y^i(\hat{m}, \mathbf{m}^{-i}) = y_\lambda$.

To prove that message $\hat{m}$ is a beneficial deviation for agent i, choose $\hat{n}$ large enough such that

$$\pi^i(\hat{m}, \mathbf{m}^{-i}) = \upsilon^i(x_\lambda, y_\lambda, y_\lambda + Y_\lambda) > \upsilon^i(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})) = \pi^i(\mathbf{m}),$$

by strict quasi-concavity of υ^i.[7] □

Lemma 2. *Let $\bar{\mathbf{m}}$ be a Nash equilibrium of the game induced by the mechanism.*

(a) $x^i(\bar{\mathbf{m}}) \gg 0$ for all i, and $Y(\bar{\mathbf{m}}) \gg 0$;

(b) $q^i(\bar{\mathbf{m}}) \gg 0$ and $q^{-i}(\bar{\mathbf{m}}) \gg 0$;

(c) $\bar{b}^i = \sum_{j=1}^I \bar{a}^j$, $q(\bar{\mathbf{m}}) = q^i(\bar{\mathbf{m}}) + \sum_{j\neq i} q^{-j}(\bar{\mathbf{m}})$ and $\tau^i(\bar{\mathbf{m}}) = 0$.

(d) $P(\bar{\mathbf{m}})\cdot x^i(\bar{\mathbf{m}}) + q^i(\bar{\mathbf{m}})\cdot y^i(\bar{\mathbf{m}}) + q^{-i}(\bar{\mathbf{m}})\cdot Y^{-i}(\bar{\mathbf{m}}) = P(\bar{\mathbf{m}})\cdot\omega^i$; and

(e) $N(\bar{\mathbf{m}}) = n^i$ and thus $x^i(\bar{\mathbf{m}}) = \hat{x}^i(\bar{\mathbf{m}})$.

Proof. For property (a), suppose, by way of contradiction, that either $x^i(\bar{\mathbf{m}}) = 0$ for some i or $Y(\bar{\mathbf{m}}) = 0$. Consider the following message $\hat{m} = (\hat{P}, \hat{Q}, \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, \hat{n})$ that individual i can play[8]: $\hat{P} = \bar{P}$, $\hat{Q} = \bar{Q}$, $\hat{a}$ and $\hat{b}$ are such that

$$q^j(\hat{m}, \bar{\mathbf{m}}^{-i}) + \sum_{j'\neq j} q^{-j'}(\hat{m}, \bar{\mathbf{m}}^{-i}) \ge Q(\bar{\mathbf{m}}) + \varepsilon,$$

where $\varepsilon > 0$, for all j. This guarantees that

$$\exists\, Y > 0 \text{ such that } Y \in \mathcal{B}(\hat{m}, \bar{\mathbf{m}}^{-i}),$$

since $\omega^i > 0$. Now, define $\hat{c} = \varepsilon - \sum_{j\neq i}\bar{c}^j$, $\hat{d} = \varepsilon$ and $\hat{e} = \varepsilon$. If ε is small enough, then

$$Y(\hat{m}, \bar{\mathbf{m}}^{-i}) = \hat{c} + \sum_{j\neq i}\bar{c}^j = \varepsilon > 0$$

(and $y^i(\hat{m}, \bar{\mathbf{m}}^{-i}) = \varepsilon^2/(\varepsilon + \sum_{j\neq i}\bar{d}^j) > 0$). Moreover,

$$P(\hat{m}, \bar{\mathbf{m}}^{-i})\cdot x^i(\hat{m}, \bar{\mathbf{m}}^{-i}) < [1 - \tau^i(\hat{a}, \bar{\mathbf{a}}^{-i}, \hat{b})]\cdot [P(\hat{m}, \bar{\mathbf{m}}^{-i})\cdot\omega^i - q^i(\bar{\mathbf{m}}^{-i})\cdot y^i(\hat{m}, \bar{\mathbf{m}}^{-i}) - q^{-i}(\bar{\mathbf{m}}^{-i})\cdot Y^{-i}(\hat{m}, \bar{\mathbf{m}}^{-i})].$$

[7] The argument is a little more subtle than it seems. If $(x, y + Y) \neq [x^i(\mathbf{m}), Y(\mathbf{m})]$, strict quasi-concavity of υ^i in its first and third arguments yields the inequality. Else, the result follows from the (weaker) quasi-concavity property imposed on the second argument of υ^i.

[8] If $x^i(\bar{\mathbf{m}}) > 0$ for all i, then let $i = 1$.

And

$$f\Big(\sum_{j=1}^{I}\omega^j - x^i(\hat{m}, \bar{\mathbf{m}}^{-i})\Big) \geq Y(\hat{m}, \bar{\mathbf{m}}^{-i}).$$

By the interiority assumption, $\pi^i(\hat{m}, \bar{\mathbf{m}}^{-i}) > \pi^i(\bar{\mathbf{m}})$, if individual i chooses a large $\hat{n}$. So $\bar{\mathbf{m}}$ cannot be a Nash equilibrium.

By property (b) in Lemma 1 and property (a) here, note that if $\bar{\mathbf{m}}$ is a Nash equilibrium, then each $(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), Y^{-i}(\bar{\mathbf{m}}))$ must solve the problem

$$\max_{x,y,Y}\Big\{\upsilon^i(x, y, y+Y) : P(\bar{\mathbf{m}})\cdot x \leq [1-\tau^i(\bar{\mathbf{a}}, \bar{b}^i)]\cdot[P(\bar{\mathbf{m}})\cdot\omega^i - q^i(\bar{\mathbf{m}}^{-i})\cdot y - q^{-i}(\bar{\mathbf{m}}^{-i})\cdot Y]\Big\}.$$

Since function υ^i is strictly increasing in all arguments, the latter implies property (b). For property (c), suppose otherwise: $\bar{b}^i \neq \sum_{j=1}^{I}\bar{a}^j$ for some individual, at some equilibrium. By property (a), it follows that

$$P(\bar{\mathbf{m}})\cdot\omega^i - q^i(\bar{\mathbf{m}}^{-i})\cdot y^i(\bar{\mathbf{m}}) - q^{-i}(\bar{\mathbf{m}}^{-i})\cdot\sum_{j\neq i} y^j(\bar{\mathbf{m}}) > 0$$

and $f\big(\sum_i \omega^i\big) > \sum_i y^i(\bar{\mathbf{m}})$. A message $\hat{m}$ where $\hat{b}^i$ and $\hat{a}^i$ are chosen to reduce the difference $|b^i - \sum_j a^j|$ and to keep

$$\sum_i\Big[q^i(\hat{\mathbf{m}}^{-i}) + \textstyle\sum_{j\neq i} q^{-j}(\hat{\mathbf{m}}^{-j})\Big]\cdot y^i(\bar{\mathbf{m}}) \geq Q(\bar{\mathbf{m}})\cdot\sum_i y^i(\bar{\mathbf{m}})$$

will reduce $\tau^i(\hat{m}, \bar{\mathbf{m}}^{-i})$ and increase $x^i(\hat{m}, \bar{\mathbf{m}}^{-i})$, while leaving $y^i(\hat{m}, \bar{\mathbf{m}}^{-i})$ and $Y^{-i}(\hat{m}, \bar{\mathbf{m}}^{-i})$ unchanged. Since υ^i is strictly increasing in x, such deviation would increase $\pi^i(\hat{m}, \bar{\mathbf{m}}^{-i})$, which is impossible as $\bar{\mathbf{m}}$ is an equilibrium.

For property (d), suppose, by way of contradiction, that for some individual i

$$P(\bar{\mathbf{m}})\cdot x^i(\bar{\mathbf{m}}) + q_i(\bar{\mathbf{m}})\cdot y^i(\bar{\mathbf{m}}) + q_{-i}(\bar{\mathbf{m}})\cdot Y^{-i}(\bar{\mathbf{m}}) < P(\bar{\mathbf{m}})\cdot\omega^i.$$

Then there is (x^i, y^i, Y^{-i}) such that

$$P(\bar{\mathbf{m}})\cdot x^i + q_i(\bar{\mathbf{m}})\cdot y^i + q_{-i}(\bar{\mathbf{m}})\cdot Y^{-i} \leq P(\bar{\mathbf{m}})\cdot\omega^i.$$

and

$$\sum_i^I y^i\cdot[q_i(\bar{\mathbf{m}}) + \sum_{j\neq i} q_{-i}(\bar{\mathbf{m}})] \geq Q(\bar{\mathbf{m}})\cdot\sum_i^I y^i.$$

By monotonicity of preferences,

$$\upsilon^i(x^i, y^i, y^i + Y^{-i}) > \upsilon^i(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}) + Y^{-i}(\bar{\mathbf{m}})),$$

From property (a), $x^i(\bar{\mathbf{m}}) \gg 0$ for all i and $Y(\bar{\mathbf{m}}) \gg 0$. Thus, by property (b) in Lemma 1, there exists a message that i can send, $\hat{m}$, such that $\pi^i(\hat{m}, \bar{\mathbf{m}}^{-i}) > \pi^i(\bar{\mathbf{m}})$. This contradicts $\bar{\mathbf{m}}$ being a Nash equilibrium.

Finally, for property (e), suppose $N(\bar{\mathbf{m}}) > n^i$ for some individual i. Then

$$x^i(\bar{\mathbf{m}}) = \frac{n^i}{N}\hat{x}^i(\bar{\mathbf{m}}) < \hat{x}^i(\bar{\mathbf{m}}),$$

and therefore

$$P(\bar{\mathbf{m}})\cdot x^i(\bar{\mathbf{m}}) + q^i(\bar{\mathbf{m}})\cdot y^i(\bar{\mathbf{m}}) + q^{-i}(\bar{\mathbf{m}})\cdot Y^{-i}(\bar{\mathbf{m}}) < P(\bar{\mathbf{m}})\cdot\omega^i,$$

which contradicts property (d). □

The previous two lemmas yield the following theorem, which is the main result in this note.

Theorem 1. *The set of allocations implemented by the mechanism is the (complete) set of Lindahl equilibrium allocations.*

Proof. We first argue that if $\bar{\mathbf{m}}$ is a Nash equilibrium, the resulting allocation corresponds to a Lindahl equilibrium. This is now straightforward: by properties (a) and (c) of Lemma 2 and (b) of Lemma 1,[9] each $(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), Y^{-i}(\bar{\mathbf{m}}))$ must solve the problem

$$\max_{x,y,Y}\Big\{\upsilon^i(x, y, y+Y) : P(\bar{\mathbf{m}})\cdot x + q^i(\bar{\mathbf{m}}^{-i})\cdot y + q^{-i}(\bar{\mathbf{m}}^{-i})\cdot Y \leq P(\bar{\mathbf{m}})\cdot\omega^i\Big\}. \quad (7)$$

This yields condition (b) in Definition 1. Condition (c) follows from property (c) in Lemma 2 and the fact that the mechanism is feasible. Property (c) in Lemma 2 also implies that

$$q^i(\bar{\mathbf{m}}) + \sum_{j\neq i} q^{-j}(\bar{\mathbf{m}}) = Q(\bar{\mathbf{m}}),$$

namely condition (d) in Definition 1. Property (d) in Lemma 2 implies condition (a) in the definition. We now argue that for any Lindahl equilibrium, $[P, (q^i, q^{-i})_{i=1}^I, Q, \bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{X}, \bar{Y}]$ there is a Nash equilibrium that implements its allocation. To see this, first let $(\mathbf{a}, \mathbf{c})$ solve the following linear equation system:

$$\textstyle\sum_i c^i = \bar{Y},$$

for all i

$$\sum_{j\neq i}^{I} a^j = \frac{1}{I}Q - q^i,$$

and

$$c_k^i \bar{Y}_k - \textstyle\sum_j c_k^j \bar{y}_k^i = 0,$$

also for each i and each k. Besides, let $P^i = P$, $Q^i = Q$, $b^i = \sum_j a^j$, $d^i = c^i$, $e^i = \bar{x}^i$, and $n^i = 1$. With these numbers, construct the profile of strategies $\bar{\mathbf{m}}$. By construction,

$$[P(\bar{\mathbf{m}}), Q(\bar{\mathbf{m}})] = (P, Q),$$

while

$$(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), q^i(\bar{\mathbf{m}}), q^{-i}(\bar{\mathbf{m}})) = (\bar{x}^i, \bar{y}^i, q^i, q^{-i}),$$

and

$$q^i(\bar{\mathbf{m}}) + \sum_{j\neq i} q^{-j}(\bar{\mathbf{m}}) = Q$$

for all individuals. Condition (b) in Definition 1 implies that for all deviations $\hat{m}$,

$$\begin{aligned}\pi^i(\hat{m}, \bar{\mathbf{m}}^{-i}) &= \upsilon^i(x^i(\hat{m}, \bar{\mathbf{m}}^{-i}), y^i(\hat{m}, \bar{\mathbf{m}}^{-i}), y^i(\hat{m}, \bar{\mathbf{m}}^{-i}) + Y^{-i}(\hat{m}, \bar{\mathbf{m}}^{-i}))\\ &\leq \upsilon^i(\bar{x}^i, \bar{y}^i, \bar{Y})\\ &= \pi^i(\bar{\mathbf{m}}),\end{aligned}$$

since, in particular, the choice of $\hat{m}$ cannot affect the individual's personalized prices. □

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[9] Note that the condition $[q^i(\mathbf{m})+\sum_{j\neq i} q^{-j}(\mathbf{m})]\cdot y+\sum_{j\neq i}[q^j(\mathbf{m})+\sum_{j'\neq j} q^{-j'}(\mathbf{m})]\cdot y^j \geq Q(\mathbf{m})\cdot(y+Y)$ in (b) of Lemma 1 automatically holds at Nash equilibrium $\bar{\mathbf{m}}$, since $q^i(\bar{\mathbf{m}}) + \sum_{j\neq i} q^{-j}(\bar{\mathbf{m}}) = Q(\bar{\mathbf{m}})$ for each i.

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