Implementing Lindahl allocations in a warm-glow economy*

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Abstract

We provide a mechanism that delivers efficient provision of public goods in an economy that may display warm-glow preferences. No information about the fundamentals of the economy is required on the mechanism designer's side, so the mechanism is informationally decentralized. The mechanism is individually rational, single-valued, feasible and continuous, and the Nash equilibria of the mechanism coincide with the Lindahl equilibria of the economy. When no warm-glow preferences are present in the economy, the mechanism fully implements the classical Lindahl allocations.

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1 Introduction

Is it possible to achieve Pareto efficiency by institution design when public goods are present in the economy? Economists believed the answer was negative. Samuelson [26] characterized the conditions for Pareto efficiency in the classical public goods model, and showed that competitive markets fail these conditions due to free riding. Lindahl (Lindahl [22], Foley [13]) offered a solution to achieve the efficiency of competitive markets: to charge each consumer differential taxes (Lindahl prices) proportional to her benefit. As Arrow [5] pointed out, however, Lindahl's solution does not satisfy the incentive compatibility constraints of Hurwicz [19], and consumers have no incentive to truthfully reveal their valuation for the public goods. In a breaking through paper, Groves and Ledyard [16] gave an affirmative answer. Groves and Ledyard [16] invented a mixed competitive mechanism, in which the private goods are allocated by competitive markets and the government designs a game to allow consumers (strategically reporting messages concerning their preferences) to determine the aggregate supply of public goods and the taxes imposed on each consumer. The equilibria of this mechanism are Pareto optimal, though not necessarily Lindahl equilibria. This original work of Groves and Ledyard [16] has been improved by later research. In an abstract setting, Maskin [23] characterized the allocations attained in Nash equilibria, explicitly recognizing the incentive compatibility constraint. Walker [32] provided a simpler mixed competitive mechanism, the equiliria of which attain Lindahl allocations. Other following work includes Hurwicz [20], Postlewaite and Wettstein [25], Tian [27, 28], Varian [31] and Healy and Jain [17]. In particular, Tian [27] designed a game form to allocate both private and public goods to achieve Lindahl equilibria, and the mechanism is single-valued, feasible and continuous. All these earlier contributions assume the classical public goods model.

The motivation of this paper is that the classical public goods model is not enough to capture consumers' public goods contribution behavior. In the classical public goods model, consumers are assumed to care about the aggregate level of public goods provision only. Although it is widely used in public economics due to its lucidity, its theoretical implications have been challenged by field and experimental evidence. As demonstrated by Bergstrom et al [7], the classical public goods model implies that government provision will crowd out private contribution dollar for dollar, and the public goods provision is neutral to the income redistribution up to the set of contributors not changing. Andreoni [2] showed that the classical public goods model implies virtually no one makes contribution in a large economy. These model predictions are in contrast with documented evidence (Andreoni [3]). Based on the earlier work by Becker [6] and Cornes and Sandler [12], Andreoni [3, 4] proposed the "warm-glow" model to explain consumers' public goods contribution behavior better. In the warm-glow model, consumers' altruism is impure, and the reason they make public goods contribution can be social pressure, guilt, sympathy or simply a desire for the warm-glow feeling. From the perspective of a consumer with warm-glow preferences, her contribution is not a perfect substitute for other consumers' contribution, which implies that the crowding-out is not complete, income redistribution effect is not neutral and free-riding is less severe.

The warm-glow model is considered to be more consistent with the observed consumer behavior, and is widely used in public economics. Allouch [1] extended Lindahl's solution to achieve Pareto efficient provision of public goods in a warm-glow economy. In the Lindahl equilibrium of a warm-glow economy, each consumer has two personalized Lindahl prices for each public good, instead of one as in the classical public goods model, with one Lindahl price for her own contribution and the other for other consumers' contribution.

In this paper, inspired by Walker [32] and Tian [27], we propose a mechanism to achieve Lindahl equilibria in an economy where the agents may have warm-glow. We allow for warm-glow effects but do not assume that they are present in the economy.[1] Following the Nash implementation literature of Hurwicz [19] and Maskin [23], the mechanism is fully decentralized and does not require the mechanism designer (or the planner) to know the fundamentals of the economy.[2] As in Walker [32] and Tian [27], consumers strategically report messages concerning their preference, and the mechanism designer simply collects these messages and implements the outcome according to the mechanism and the messages collected from consumers. The Nash equilibria of the mechanism coincide with the Lindahl equilibria of the economy. Our mechanism possesses nice properties: it is individually rational, single-valued, feasible and continuous. As argued by Tian [27] and tested in laboratory experiments (Chen and Plott [11] and Chen [10]), these properties are important for the mechanism to work in practice. Different from Walker [32] and Tian [27], we have to account for two new issues: the first is to decide two Lindahl prices for each consumer, instead of one Lindahl price; the second is to decide the individual contribution of public goods, instead of the aggregate level only. The mechanisms in Walker [32] and Tian [27] are not straightforward to deal with these issues, and the Lindahl prices, individual contributions and taxes levied on consumers have to be redefined to align consumers' incentive in the right way. When no warm-glow preference is present in the economy, our mechanism fully implements the classical Lindahl allocations.

Our paper is related, also, to studies that try to make the mechanisms dynamically stable and to have nice out-of-equilibrium properties in the classical public goods model. Muench and Walker [24] studied the convergence of the Groves-Ledyard mechanism assuming Cournot best-reply learning dynamics. Kim [21] designed a globally stable mechanism under the gradient adjustment process and implemented Lindahl allocations. Chen [9] provided a mechanism which is a supermodular game under some parameter values, and can converge to Nash equilibrium under a wide class of learning dynamics. Van Essen et al [30] investigated the out-of-equilibrium performance of mechanisms in Walker [32], Kim [21], and Chen [9]

[1] Importantly, as shown by Carvajal and Song [8], both Lindahl and Nash-Walras equilibria of a warm-glow economy have strong non-parametric testable implications. Therefore, the warm-glow effect can be identified.

[2] To present the mechanism in the clearest way, we will initially consider a very simple case and assume the mechanism designer possesses some information of the economy–the production technology and consumers' endowment. However, it is not necessary for our mechanism to work, and no knowledge of the fundamentals is required in the general case that we address later.

using laboratory experiments, and found that the Kim mechanism exhibits better out-of-equilibrium efficiency properties. All this work has been confined to a quasi-linear utility environment. Recent progress along this line includes Furusawa and Konishi [14], Healy and Mathevet [18], and Van Essen and Walker [29]. Although our mechanism focuses on establishing static properties of Nash equilibria, our construction can shed light on designing a dynamically stable mechanism, which is the subject of future research.

To present our mechanism in the clearest possible way, we will first define the mechanism for a simple case with one private good and one public good. In this simple case, we assume the mechanism designer has partial information about the economy (the production technology) and allow for discontinuity in the mechanism. All these assumptions are for simplicity of exposition, and are not necessary for the mechanism to work. Later, in the general case with multiple private and public goods, the mechanism is shown to be fully informationally decentralized, in the sense that the mechanism designer knows nothing about the fundamentals of the economy, and to be continuous.

2 A simple case

Consider an economy with two goods, one of which is public. The public good is produced using the private good, according to a given aggregate production function.

2.1 The environment

There are finitely many consumers in the economy, indexed by $i = 1, \ldots, I$ with $I \geqslant 3$. Consumer i is endowed with ω^i units of the private good, and there are no endowments of the public good. Following Andreoni [3, 4], we assume she derives utility from her consumption of the private good, from her individual provision of the public good, and from the aggregate provision of the public good, according to a utility function $v^i(x, y, Y)$, where $x \geqslant 0$ is her private consumption, $y \geqslant 0$ her provision of the public good, and $Y \geqslant y$ the aggregate provision of the latter.[3]

Each individual is completely defined by her endowment and utility function,[4] so the economy is completely described by the set of individuals and the production technology. Throughout the paper we will maintain the following assumptions:

(a) Each individual's utility function:

   i. is continuous;

[3] Sometimes we will write $v(x, y, y+Y)$, in which case Y is to be understood as the provision of the public good that is made in addition to y.

[4] Under our assumptions, it won't be necessary to specify a distribution for the ownership of the firm.

ii. is strictly increasing and strictly quasi-concave in private consumption and in the aggregate provision of the public good (namely in its first and third arguments);

iii. is non-decreasing in individual provision of the public good, and satisfies the following quasi-concavity property: for any (x, Y), if $v^i(x, y, Y) > v^i(x, \bar{y}, Y)$, then

$$v^i(x, \alpha y + (1-\alpha)\bar{y}, Y) > v^i(x, \bar{y}, Y)$$

for all $\alpha \in (0, 1]$; and

iv. satisfies the following interiority condition: for any $(\bar{x}, \bar{Y}) \in \partial\mathbb{R}^2_+$, any $(x, Y) \in \mathbb{R}^2_{++}$, any $y \leqslant Y$ and any $\bar{y} \leqslant \bar{Y}$, $v^i(x, y, Y) > v^i(\bar{x}, \bar{y}, \bar{Y})$.

(b) each individual's endowment of the private good is strictly positive; and

(c) the production function is linear: $Y = \rho X$, where $\rho > 0$.

With these assumptions, our setting *allows for* warm glow effects but *does not assume* that they are present in the economy. The quasi-concavity assumption imposed on y, in particular, is weaker than strict quasiconcavity and allows for the possibility that the individual does not derive utility from the public good other than from its aggregate supply.

Throughout the paper, we will use capital letters to denote the aggregate variables of the economy, and lower case for the ones corresponding to an individual; we will use bold-face to denote profiles of individual variables.[5] That is, for instance: $\mathbf{y} = (y^1, \ldots, y^I) \in \mathbb{R}^I$ is a profile of provision levels of the public good, whereas Y is the amount of supplied by the firm. For the sake of simplicity, from now on we will denote $Y^{\neg i} = \sum_{j \neq i} y^j$, whenever a profile $\mathbf{y}$ is clear in the context.

2.2 Lindahl equilibrium

An allocation in this simple economy is a tuple $(\mathbf{x}, \mathbf{y}, X, Y)$; it is feasible if

$$\textstyle\sum_i x^i + X \leqslant \sum_i \omega^i \quad \text{and} \quad \sum_i y^i \leqslant Y \leqslant \rho X. \tag{1}$$

If private markets were to operate in the economy, a competitive equilibrium would consist of a price for the public good, Q, and an allocation that satisfies Eq. (1) with equalities and such that: (a) pair (X, Y) maximizes the profits of production at the given prices, subject to the technology; and (b) each pair (x^i, y^i) solves the program

$$\max_{x,y} \left\{ v^i(x, y, y + Y^{\neg i}) : x + Qy \leqslant \omega^i \right\}.$$

[5] A second convention is that Greek characters are reserved for exogenous variables.

Because the second commodity is public, the competitive equilibrium allocation would not be efficient as demonstrated by Samuelson [26].[6] The following solution, given by Allouch [1] for the present context, requires the introduction of two personalized prices for each individual, q^i and $q^{\neg i}$.

Definition 1. *A Lindahl equilibrium is a tuple* $[(q^i, q^{\neg i})_{i=1}^I, Q, \bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{X}, \bar{Y}]$ *such that:*

(a) pair $(\bar{X}, \bar{Y})$ *maximizes the profits of production at the given prices,* $QY - X$*, subject to* $Y \leqslant \rho X$*;*

(b) each triple $(\bar{x}^i, \bar{y}^i, \bar{Y}^{\neg i})$ *solves the program*

$$\max_{x,y,Y} \left\{ v^i(x, y, y + Y) : x + q^i y + q^{\neg i} Y \leqslant \omega^i \right\};$$

(c) Eq. (1) *holds with equalities; and*

(d) for each individual i*,* $q^i + \sum_{j \neq i} q^{\neg j} = Q$*.*

Remark 1. *In the classical public goods model, each consumer regards her contribution and others' contribution as perfect substituite, and this entails* $q^i = q^{\neg i}$*. Therefore, this Lindahl equilibrium accommodates the classical one as a special case.*

2.3 Mechanisms

A mechanism (or game form) fixes a message space for each individual and an outcome function defined over the product of the message spaces. Let M^i denote the message space of individual i, with generic element m^i. Let $M = \prod_i M^i$ and denote the outcome function $\varphi : M \to \mathbb{R}^{2n}$ as

$$\varphi(\mathbf{m}) = \left(x^i(\mathbf{m}), y^i(\mathbf{m})\right)_{i=1}^I.$$

The outcome function determines, for each profile of messages, an allocation for economy: the individual variables are given by the function, usage of the private good by the firm is $\sum_i [\omega^i - x^i(\mathbf{m})]$, and its output is $\rho \sum_i [\omega^i - x^i(\mathbf{m})]$. The mechanism is said to be feasible if this allocation satisfies Eq. (1), namely if $\rho \sum_i [\omega^i - x^i(\mathbf{m})] \geqslant \sum_i y^i(\mathbf{m})$, for all profiles of messages.

2.4 Nash implementation

Given the mechanism, we further denote, for each individual,

$$\varphi^i(\mathbf{m}) = \left(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})\right).$$

[6] Actually it can be shown that the competitive allocation is generically constrained suboptimal (Geanakoplos and Polemarchakis [15]).

This function represents the terms of the outcome function that affect individual i's utility. The mechanism defines a game, with strategy spaces given by the message spaces and payoff functions $\pi^i(\mathbf{m}) = v^i(\varphi^i(\mathbf{m}))$.

The set of allocations that are attained at the Nash equilibria of the game defined by the mechanism, given the economy, is the set *implemented* by the mechanism (in Nash equilibrium).

In what follows, we will further denote $Y^{\neg i}(\mathbf{m}) = \sum_{j \neq i} y^j(\mathbf{m})$ and $\mathbf{m}^{\neg i} = (m^j)_{j \neq i}$.

2.5 The Walker-Tian mechanism

Consider the following extension of the ideas of Walker [32] and Tian [27]. The mechanism designer does not know consumers' preference. For the simplicity of exposition, we assume the mechanism designer knows production technology and consumers' endowments. This assumption is not necessary and will be dropped in the general case in Section 3. The message space for all individuals is $M^i = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++}$, with typical element $m^i = (a^i, b^i, c^i, d^i)$. (Intuitively, c^i denotes a *claim* about agent i's public good contribution.)

The definition of the outcome function, which makes the mechanism single-valued, is as follows: given any profile $\mathbf{m}$ of messages,

1. the producer's price of the public good is $Q = 1/\rho$;[7]
2. for each individual, there are two personalized prices:
$$q^i(\mathbf{m}) = \frac{Q}{I} - \sum_{j \neq i} a^j \tag{2}$$
and
$$q^{\neg i}(\mathbf{m}) = \frac{Q}{I} - \sum_{j \neq i} a^j + \frac{1}{I-1} \sum_{j \neq i} b^j; \tag{3}$$
3. the maximum amount of public good that can be afforded by all individuals, as a result of the prices, is
$$\bar{Y}(\mathbf{m}) = \max \left\{ Y \in \mathbb{R} \mid \exists \mathbf{y} \in \mathbb{R}^I_+ : Y = \textstyle\sum_i y^i \ \text{ and } \ q^i(\mathbf{m}) y^i + q^{\neg i}(\mathbf{m}) Y^{\neg i} \leqslant \omega^i \right\};$$
(We allow $\bar{Y}(\mathbf{m})$ to take the value of $+\infty$, and this happens when all $q^i(\mathbf{m})$ and $q^{\neg i}(\mathbf{m})$ are non-positive.)
4. if for some individual one has $q^i(\mathbf{m}) + \sum_{j \neq i} q^{\neg j}(\mathbf{m}) < Q$, the actual supply of public good is $Y(\mathbf{m}) = 0$; otherwise, it is $Y(\mathbf{m}) = \min \left\{ \sum_i c^i, \bar{Y}(\mathbf{m}) \right\}$;

[7] Since the mechanism designer knows the technology, the price is set to equilibrium price; in the general case, the price is determined by consumers' reported messages, and can be out of equilibrium.

5. the allocation of the public good is $y^i(\mathbf{m}) = d^i Y(\mathbf{m})/\sum_j d^j$;

6. there is a *punishment tax* levied on individual i at a rate $\tau^i(\mathbf{m}) = 1 - \exp\{-|b^i - \sum_j a^j|\}$;

7. the allocation of the private good is

$$x^i(\mathbf{m}) = [1 - \tau^i(\mathbf{a}, b^i)] \cdot [\omega^i - q^i(\mathbf{m})y^i(\mathbf{m}) - q^{\neg i}(\mathbf{m})Y^{\neg i}(\mathbf{m}^{\neg i})].$$

Importantly, note that the personalized prices of individual i do *not* depend on her message, so we can denote them as $q^i(\mathbf{m}^{\neg i})$ and $q^{\neg i}(\mathbf{m}^{\neg i})$. Similarly, note that the punishment tax paid by i depends on $\mathbf{a} = (a^1, \ldots, a^I)$ and b^i, but not on other arguments, so we denote it as $\tau^i(\mathbf{a}, b^i)$. By construction, $0 \leqslant \tau^i(\mathbf{a}, b^i) < 1$, and the tax it is null when, and only when, $b^i = \sum_j a^j$.[8]

The following lemmata will be useful in the proof of the implementation theorem. The first one regards properties of the mechanism, the second is on properties of the Nash equilibria of the game it induces.

Lemma 1. *The mechanism gives non-negative consumption to all individuals, is feasible and satisfies:*

(a) If

$$\omega^i - q^i(\mathbf{m}^{\neg i})c^i - q^{\neg i}(\mathbf{m}^{\neg i}) \sum_{j \neq i} c^j \geqslant 0$$

and

$$q^i(\mathbf{m}^{\neg i}) + \sum_{j \neq i} q^{\neg j}(\mathbf{m}^{\neg j}) \geqslant Q$$

for all individuals,

then $Y(\mathbf{m}) = \sum_i c^i$.

(b) Suppose that $x^i(\mathbf{m}) > 0$ *for all individuals and* $Y(\mathbf{m}) > 0$. *If for some* i *there exists* (x, y, Y) *such that*

$$x \leqslant [1 - \tau^i(\mathbf{a}, b^i)] \cdot [\omega^i - q^i(\mathbf{m}^{\neg i})y - q^{\neg i}(\mathbf{m}^{\neg i})Y]$$

and

$$v^i(x, y, y + Y) > v^i(x^i(\mathbf{m}), y^i(\mathbf{m}), y^i(\mathbf{m}) + Y^{\neg i}(\mathbf{m})),$$

then there exists a message that i *can send,* $\hat{m} \in M^i$, *such that*

$$\pi^i(\hat{m}, \mathbf{m}^{\neg i}) > \pi^i(\mathbf{m}).$$

[8] The total tax payed by individual i is

$$q^i(\mathbf{m}^{\neg i})y^i(\mathbf{m}) + q^{\neg i}(\mathbf{m}^{\neg i})Y^{\neg i}(\mathbf{m}^{\neg i}) + \tau^i(\mathbf{a}, b^i)[\omega^i - q^i(\mathbf{m}^{\neg i})y^i(\mathbf{m}) - q^{\neg i}(\mathbf{m}^{\neg i})Y^{\neg i}(\mathbf{m}^{\neg i})].$$

Proof. Non-negativity of the individual bundles follows from the properties of the punishment function τ^i and the construction of $Y(\mathbf{m})$. To see that the mechanism is feasible, if $Y(\mathbf{m}) = 0$, it is obviously feasible. Note that $Y(\mathbf{m}) > 0$ implies that $q^i(\mathbf{m}) + \sum_{j \neq i} q^{\neg j}(\mathbf{m}) \geqslant Q$ for all individuals. Then,

$$\begin{aligned}
\rho \textstyle\sum_i [\omega^i - x^i(\mathbf{m})] &\geqslant \rho \textstyle\sum_i \left[q^i(\mathbf{m}) y^i(\mathbf{m}) + q^{\neg i}(\mathbf{m}) Y^{\neg i}(\mathbf{m})\right] \\
&= \rho \textstyle\sum_i \left[q^i(\mathbf{m}) y^i(\mathbf{m}) + q^{\neg i}(\mathbf{m}) \sum_{j \neq i} y^j(\mathbf{m})\right] \\
&= \rho \textstyle\sum_i \left[q^i(\mathbf{m}) + \sum_{j \neq i} q^{\neg j}(\mathbf{m})\right] y^i(\mathbf{m}) \\
&\geqslant \rho Q \textstyle\sum_i y^i(\mathbf{m}) \\
&= \textstyle\sum_i y^i(\mathbf{m}).
\end{aligned}$$

For property (a), suppose, by way of contradiction, that $\sum_i c^i > \bar{Y}(\mathbf{m})$. By definition of $\bar{Y}(m)$, then, for all profiles $\mathbf{y}$ such that $\sum_i y^i = \sum_i c^i$, we have that $q^i y^i + q^{\neg i} Y^{\neg i} > \omega^i$ for some i, which contradicts the hypothesis in the lemma. For property (b), fix a profile of strategies such that $x^i(\mathbf{m}) > 0$ for all i and $Y(\mathbf{m}) > 0$, and note that this implies, by property (a), that $\sum_i y^i(\mathbf{m}) = Y(\mathbf{m}) < \bar{Y}(\mathbf{m})$. Suppose that for some i, one such (x, y, Y) exists. Define, for $\lambda \in (0, 1)$, the convex combination

$$(x_\lambda, y_\lambda, Y_\lambda) = \lambda(x, y, Y) + (1 - \lambda)[x^i(\mathbf{m}), y^i(\mathbf{m}), Y^{\neg i}(\mathbf{m})],$$

and note that

$$x_\lambda \leqslant [1 - \tau^i(\mathbf{a}, b^i)] \cdot [\omega^i - q^i(\mathbf{m}) y_\lambda - q^{\neg i}(\mathbf{m}) Y_\lambda],$$

by construction (given the definition of $x^i(\mathbf{m})$). Define the message $\hat{m} = (\hat{a}, \hat{b}, \hat{c}, \hat{d})$ as follows: $\hat{a} = a^i$, $\hat{b} = b^i$, $\hat{c} = Y_\lambda + y_\lambda - \sum_{j \neq i} c^j$, and $\hat{d}$ such that

$$\frac{\hat{d}}{\hat{d} + \sum_{j \neq i} d^j} (Y_\lambda + y_\lambda) = y_\lambda.$$

Consider the allocation induced by profile of messages $(\hat{m}, \mathbf{m}^{\neg i})$. If λ is small enough, by continuity $x^j(\hat{m}, \mathbf{m}^{\neg i}) \geqslant 0$ for all j and, hence, again by property (a), $Y(\hat{m}, \mathbf{m}^{\neg i}) = \sum_j c^j = Y_\lambda + y_\lambda$. This further implies that $y^i(\hat{m}, \mathbf{m}^{\neg i}) = y_\lambda$ and hence that $x^i(\hat{m}, \mathbf{m}^{\neg i}) \geqslant x_\lambda$. To prove that message $\hat{m}$ is a beneficial deviation for agent i, it suffices to notice that

$$\pi^i(\hat{m}, \mathbf{m}^{\neg i}) = v^i(x_\lambda, y_\lambda, y_\lambda + Y_\lambda) > v^i(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})) = \pi^i(\mathbf{m}),$$

by strict quasi-concavity of v^i.[9] □

Lemma 2. *Let $\bar{\mathbf{m}}$ be a Nash equilibrium of the game induced by the mechanism.*

[9] The argument is a little more subtle than it seems. If $(x, y + Y) \neq [x^i(\mathbf{m}), Y(\mathbf{m})]$, strict quasi-concavity of v^i in its first and third arguments yields the inequality. Else, the result follows from the (weaker) quasi-concavity property imposed on the second argument of v^i, namely item (iii) in assumption (a).

(a) *Private consumption of all individuals and the supply of public good are interior:* $x^i(\bar{\mathbf{m}}) > 0$ *for all* i, *and* $Y(\bar{\mathbf{m}}) > 0$.

(b) *Aggregate supply of the public good satisfies the individual claims:* $Y(\bar{\mathbf{m}}) = \sum_i c^i < \bar{Y}(\bar{\mathbf{m}})$.

(c) *Both personalized prices are positive for all individuals:* $q^i(\bar{\mathbf{m}}) > 0$ *and* $q^{\neg i}(\bar{\mathbf{m}}) > 0$.

(d) *For all* i, $\bar{b}^i = \sum_{j=1}^{I} \bar{a}^j$, $Q = q^i(\bar{\mathbf{m}}^{\neg i}) + \sum_{j \neq i} q^{\neg j}(\bar{\mathbf{m}}^{\neg j})$ *and no individual is punished:* $\tau^i(\bar{\mathbf{m}}) = 0$.

Proof. For property (a), suppose, by way of contradiction, that either $x^i(\bar{\mathbf{m}}) = 0$ for some i or $Y(\bar{\mathbf{m}}) = 0$. Consider the following message $\hat{m} = (\hat{a}, \hat{b}, \hat{c}, \hat{d})$ that individual i can play:[10] $\hat{a}$ and $\hat{b}$ are such that

$$q^j(\hat{m}, \bar{\mathbf{m}}^{\neg i}) + \sum_{j' \neq j} q^{\neg j'}(\hat{m}, \bar{\mathbf{m}}^{\neg i}) \geqslant Q + \varepsilon,$$

where $\varepsilon > 0$, for all j, and let $\hat{d} = \varepsilon$. This guarantees that

$$Y(\hat{m}, \bar{\mathbf{m}}^{\neg i}) = \min\left\{\hat{c} + \textstyle\sum_{j \neq i} \bar{c}^j, \bar{Y}(\hat{m}, \bar{\mathbf{m}}^{\neg i})\right\},$$

where $\bar{Y}(\hat{m}, \bar{\mathbf{m}}^{\neg i}) > 0$. Now, define $\hat{c} = \varepsilon - \sum_{j \neq i} \bar{c}^j$ and note that, if ε is small enough, then

$$Y(\hat{m}, \bar{\mathbf{m}}^{\neg i}) = \hat{c} + \sum_{j \neq i} \bar{c}^j = \varepsilon > 0$$

(and $y^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) = \varepsilon^2/(\varepsilon + \sum_{j \neq i} \bar{d}^j) > 0$). Moreover,

$$x^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) = [1 - \tau^i(\hat{a}, \bar{\mathbf{a}}^{\neg i}, \hat{b})] \cdot [\omega^i - q^i(\bar{\mathbf{m}}^{\neg i})y^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) - q^{\neg i}(\bar{\mathbf{m}}^{\neg i})Y^{\neg i}(\hat{m}, \bar{\mathbf{m}}^{\neg i})] > 0.$$

By the interiority assumption, $\pi^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) > \pi^i(\bar{\mathbf{m}})$, so $\bar{\mathbf{m}}$ cannot be a Nash equilibrium. For property (b), note that property (a) implies, right away that $Y(\bar{\mathbf{m}}) = \sum_i y^i(\bar{\mathbf{m}}) > 0$ and

$$\omega^i > q^i(\hat{m}, \bar{\mathbf{m}}^{\neg i})y^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) + q^{\neg i}(\hat{m}, \bar{\mathbf{m}}^{\neg i})Y^{\neg i}(\hat{m}, \bar{\mathbf{m}}^{\neg i})$$

for all i. The latter is possible, by definition, only if $Y(\bar{\mathbf{m}}) < \bar{Y}(\bar{\mathbf{m}})$. By property (b) in Lemma 1 and property (a) here, note that if $\bar{\mathbf{m}}$ is a Nash equilibrium, then each $(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), Y^{\neg i}(\bar{\mathbf{m}}))$ must solve the problem

$$\max_{x,y,Y} \left\{ v^i(x, y, y + Y) : x \leqslant [1 - \tau^i(\bar{\mathbf{a}}, \bar{b}^i)] \cdot [\omega^i - q^i(\bar{\mathbf{m}}^{\neg i})y - q^{\neg i}(\bar{\mathbf{m}}^{\neg i})Y] \right\}.$$

Since function v^i is strictly increasing in all arguments, the latter implies property (c). Finally, for property (d), suppose otherwise: $\bar{b}^i \neq \sum_{j=1}^{I} \bar{a}^j$ for some individual, at some equilibrium. By property (a), it follows that

$$\omega^i - q^i(\bar{\mathbf{m}}^{\neg i})y^i(\bar{\mathbf{m}}) - q^{\neg i}(\bar{\mathbf{m}}^{\neg i}) \sum_{j \neq i} y^j(\bar{\mathbf{m}}) > 0$$

[10] If $x^i(\bar{\mathbf{m}}) > 0$ for all i, then let $i = 1$.

and $\rho \sum_i \omega^i > \sum_i y^i(\bar{\mathbf{m}})$. A message $\hat{m}$ where $\hat{b}$ and $\hat{a}^i$ are chosen to reduce the difference $|\hat{b} - \sum_j a^j|$ and to keep

$$\sum_i \left[q^i(\hat{\mathbf{m}}^{\neg i}) + \textstyle\sum_{j \neq i} q^{\neg j}(\hat{\mathbf{m}}^{\neg j})\right] y^i(\bar{\mathbf{m}}) \geqslant Q \sum_i y^i(\bar{\mathbf{m}})$$

will reduce $\tau^i(\hat{m}, \bar{\mathbf{m}}^{\neg i})$ and increase $x^i(\hat{m}, \bar{\mathbf{m}}^{\neg i})$, while leaving $y^i(\hat{m}, \bar{\mathbf{m}}^{\neg i})$ and $Y^{\neg i}(\hat{m}, \bar{\mathbf{m}}^{\neg i})$ unchanged. Since υ is strictly increasing in x, such deviation would increase $\pi^i(\hat{m}, \bar{\mathbf{m}}^{\neg i})$, which is impossible as $\bar{\mathbf{m}}$ is an equilibrium. □

2.6 Nash implementation of Lindahl allocations

The previous two lemmata yield the following theorem.

Theorem 1. *The set of allocations implemented by the mechanism is the (complete) set of Lindahl equilibrium allocations.*

Proof. We first argue that if $\bar{\mathbf{m}}$ is a Nash equilibrium, the resulting allocation corresponds to a Lindahl equilibrium. This is now straightforward: by properties (a) and (d) of Lemma 2 and (b) of Lemma 1, each $(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), Y^{\neg i}(\bar{\mathbf{m}}))$ must solve the problem

$$\max_{x,y,Y} \left\{ \upsilon^i(x, y, y + Y) : x + q^i(\bar{\mathbf{m}}^{\neg i})y + q^{\neg i}(\bar{\mathbf{m}}^{\neg i})Y \leqslant \omega^i \right\}. \tag{4}$$

This yields condition (b) in Definition 1. Property (c) follows from property (d) in Lemma 2 and the fact that the mechanism is feasible. Eqs. (2) and (3) and property (d) in Lemma 2 also imply that

$$q^i(\bar{\mathbf{m}}^{\neg i}) + \sum_{j \neq i} q^{\neg j}(\bar{\mathbf{m}}^{\neg j}) = Q,$$

namely property (d) in Definition 1. This, in turn, implies property (a) in the definition, since the technology has been assumed to be $f(X) = \rho X$ and $Q = 1/\rho$.

We now argue that for any Lindahl equilibrium, $[(q^i, q^{\neg i})_{i=1}^I, Q, \bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{X}, \bar{Y}]$ there is a Nash equilibrium that implements its allocation. To see this, first let $(\mathbf{a}, \mathbf{c})$ solve the following linear equation system: $\sum_i c^i = \bar{Y}$, for all i

$$\sum_{j \neq i} a^j = \frac{1}{\rho I} - q^i,$$

and $c^i \bar{Y} - \sum_j c^j \bar{y}^i = 0$, also for each i. Besides, let $b^i = \sum_j a^j$ and $d^i = c^i$. With these numbers, construct the profile of strategies $\bar{\mathbf{m}}$. By construction,

$$(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), q^i(\bar{\mathbf{m}}^{\neg i}), q^{\neg i}(\bar{\mathbf{m}}^{\neg i})) = (\bar{x}^i, \bar{y}^i, q^i, q^{\neg i})$$

for all individuals. Condition (b) in Definition 1 implies that for all deviations $\hat{m}$,

$$\begin{aligned}\pi^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) &= v^i(x^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}), y^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}), y^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) + Y^{\neg i}(\hat{m}, \bar{\mathbf{m}}^{\neg i})) \\ &\leqslant v^i(\bar{x}^i, \bar{y}^i, \bar{Y}^i) \\ &= \pi^i(\bar{\mathbf{m}}),\end{aligned}$$

since, in particular, the choice of $\hat{m}$ cannot affect the individual's personalized prices. □

Remark 2. *Since the Nash equilibria attain Lindahl allocations, the mechanism is individually rational, i.e., each consumer will become better off from this mechanism.*

2.7 Continuity

A technical difficulty with the mechanism above is the discontinuity introduced in the definition of the aggregate supply of the public good. In order to recover continuity, consider the following modification of the mechanism: given the personalized prices $q^i(\mathbf{m}^{\neg i})$ and $q^{\neg i}(\mathbf{m}^{\neg i})$, define the set $\mathcal{B}(\mathbf{m})$ of aggregate levels Y of the public public good such that:

(a) there exists a profile $\mathbf{y}$ such that $Y = \sum_i y^i$,

$$\omega^i - q^i(\mathbf{m}^{\neg i})y^i - q^{\neg i}(\mathbf{m}^{\neg i}) \sum_{j \neq i} y^j \geqslant 0,$$

and

$$\sum_i \left[q^i(\mathbf{m}^{\neg i}) + \textstyle\sum_{j\neq i} q^{\neg j}(\mathbf{m}^{\neg j})\right] y^i \geqslant QY;$$

(b) $\rho \sum_i \omega^i \geqslant Y$.

This "budget" set captures the maximal level of the public good that can be afforded by all individuals (under some distribution), allows for compensation of the firm, and can be produced under the available aggregate endowment of the private good in the economy. Notice that it may well happen that the individuals express an aggregate claim ($\sum_i \mathbf{c}^i$) that does not lie in this set. In the previous section, such problem was dealt with by inducing that no public good is offered in such case. A better solution is to let the aggregate supply be given by

$$Y(\mathbf{m}) = \text{argmin}_Y \left\{ |\textstyle\sum_i \mathbf{c}^i - Y| : Y \in \mathcal{B}(\mathbf{m}) \right\}.$$

That this modification gives a continuous mechanism is immediate. To see that the other properties remain valid, consider for instance Property (a) in Lemma 2, and suppose that $Y(\mathbf{m}) = 0$. Consider the following message $\hat{m} = (\hat{a}, \hat{b}, \hat{c}, \hat{d})$ that individual $i = 1$ can play: $\hat{a}$ and $\hat{b}$ are such that

$$q^j(\hat{m}, \bar{\mathbf{m}}^{\neg 1}) + \sum_{k \neq j} q^{\neg k}(\hat{m}, \bar{\mathbf{m}}^{\neg 1}) \geqslant Q + \varepsilon,$$

where $\varepsilon > 0$, for all j. This guarantees that there is some $Y \in \mathcal{B}(\hat{m}, \bar{\mathbf{m}}^{\neg 1})$, $Y > 0$. As before, define $\hat{c} = \varepsilon - \sum_{j \neq 1} \bar{c}^j$ and $\hat{d} = \varepsilon$. If ε is small enough, then $Y(\hat{m}, \bar{\mathbf{m}}^{\neg 1}) = \hat{c} + \sum_{j \neq 1} \bar{c}^j = \varepsilon > 0$, and, moreover,

$$x^1(\hat{m}, \bar{\mathbf{m}}^{\neg 1}) = [1 - \tau^1(\hat{a}, \bar{\mathbf{a}}^{\neg 1}, \hat{b})] \cdot [\omega^1 - q^1(\bar{\mathbf{m}}^{\neg 1}) y^1(\hat{m}, \bar{\mathbf{m}}^{\neg 1}) - q^{\neg 1}(\bar{\mathbf{m}}^{\neg 1}) Y^{\neg 1}(\hat{m}, \bar{\mathbf{m}}^{\neg 1})] > 0.$$

By the interiority assumption, again, $\pi^1(\hat{m}, \bar{\mathbf{m}}^{\neg 1}) > \pi^1(\bar{\mathbf{m}})$, so $\bar{\mathbf{m}}$ cannot be a Nash equilibrium.

3 The general case

Now assume the economy has L private goods and K public goods, which we will index by $\ell = 1, \ldots, L$ and $k = 1, \ldots, K$, respectively. The aggregate production function of public goods $f : \mathbb{R}^L_+ \mapsto \mathbb{R}^K_+$ is understood to be to transform input bundles of the private commodities into output bundles of the public goods.

Consumer i's endowment, ω^i, is now a bundle of the private goods, and there remain to be no endowments of the public good. She derives utility from her consumption of the private goods, (potentially) from her individual provision of the public goods, and from the aggregate provision of the public goods, according to the utility function $v^i(x, y, Y)$ defined over

$$\mathbb{R}^L_+ \times \{(y, Y) \in \mathbb{R}^{2K}_+ \mid Y \geqslant y\}.$$

The following assumptions, which simply extend the ones we used before, are maintained from now on: each function $v^i(x, y, Y)$ is continuous, strictly increasing and strictly quasi-concave in (x, Y) and non-decreasing in y, and satisfies that (i) for any (x, Y), if $v^i(x, y, Y) > v^i(x, \bar{y}, Y)$, then

$$v^i(x, \alpha y + (1 - \alpha)\bar{y}, Y) > v^i(x, \bar{y}, Y)$$

for all $\alpha \in (0, 1]$, while (ii) for any $(\bar{x}, \bar{Y}) \in \partial \mathbb{R}^{L+K}_+$, any $(x, Y) \in \mathbb{R}^{L+K}_{++}$, any $y \leqslant Y$ and any $\bar{y} \leqslant \bar{Y}$, $v^i(x, y, Y) > v^i(\bar{x}, \bar{y}, \bar{Y})$; each $\omega^i \gg 0$; and function f displays constant returns to scale.

3.1 Lindahl equilibrium

An allocation is now a tuple $(\mathbf{x}, \mathbf{y}, X, Y)$; it is feasible if

$$\textstyle\sum_i x^i + X \leqslant \sum_i \omega^i \quad \text{and} \quad \sum_i y^i \leqslant Y \leqslant f(X). \tag{5}$$

Again if private markets were to operate in this economy, the competitive equilibrium allocation would not be efficient.[11] Instead, efficiency is restored by the introduction of two personalized prices for each individual, q^i and $q^{\neg i}$.

[11] A competitive equilibrium would consist of a price for the private goods, p, a price for the public goods, q, and an allocation that satisfies Eq. (5) with equalities and such that: (X, Y) solves $\max_{\bar{x}, \bar{Y}} \{q \cdot \bar{Y} - p \cdot \bar{X} : \bar{Y} \leqslant f(\bar{X})\}$; and each (x^i, y^i) solves $\max_{x,y} \{v^i(x, y, y + Y^{\neg i}) : p \cdot x + q \cdot y \leqslant \omega^i\}$.

Definition 2. *A Lindahl equilibrium is a tuple* $[p, (q^i, q^{\neg i})_{i=1}^I, Q, \bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{X}, \bar{Y}]$ *such that:*

(a) pair $(\bar{X}, \bar{Y})$ *solves the program* $\max_{X,Y}\{Q \cdot Y - p \cdot X : Y \leqslant f(X)\}$;

(b) each triple $(\bar{x}^i, \bar{y}^i, \bar{Y}^{\neg i})$ *solves the program*

$$\max_{x,y,Y} \left\{v^i(x, y, y+Y) : p \cdot x + q^i \cdot y + q^{\neg i} \cdot Y \leqslant p \cdot \omega^i\right\};$$

(c) Eq. (5) *holds with equalities; and*

(d) for each individual i, $q^i + \sum_{j \neq i} q^{\neg j} = Q$.

3.2 Mechanisms and Nash implementation

Keeping the same notation as in the simple case for the message spaces, we only need to generalize the outcome function to $\varphi : M \to \mathbb{R}^{(K+L)I}$, with

$$\varphi(\mathbf{m}) = \left(x^i(\mathbf{m}), y^i(\mathbf{m})\right)_{i=1}^I$$

and

$$\varphi^i(\mathbf{m}) = \left(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})\right),$$

as before. Again, usage of the private good by the firm is $X(\mathbf{m}) = \sum_i[\omega^i - x^i(\mathbf{m})]$, its output is bundle $Y(\mathbf{m}) = f(X(\mathbf{m}))$, and the mechanism is feasible if $Y(\mathbf{m}) \geqslant \sum_i y^i(\mathbf{m})$. The mechanism again defines a game,[12] and the set of allocations that are attained at its Nash equilibria is the set *implemented* by the mechanism (in Nash equilibrium).

3.3 The Walker-Tian mechanism

As in the simple case, the mechanism designer does not know whether there exists warm-glow effect or not, and now we also drop the unnecessary assumption that she knows the technology of the firm. The message space for all individuals is now

$$M^i = \mathbb{R}^L_{++} \times \mathbb{R}^K_{++} \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^K_{++} \times \mathbb{R}^L_{+} \times \mathbb{R}_{++},$$

with typical element $m^i = (p^i, Q^i, a^i, b^i, c^i, d^i, e^i, n^i)$. Vectors a^i, b^i, c^i and d^i are multi-dimensional analogous of the corresponding scalars in the simplified case. Vector e^i is agent i's demand for private goods, while vectors p^i and Q^i can be understood as the agent's recommendation of the prices of the private goods and the prices paid to the firm for the public goods; their entries are denoted e^i_ℓ, p^i_ℓ and Q^i_k, respectively. Number n^i, finally, will be used in a "shouting" game.

For the definition of the outcome function, given any profile $\mathbf{m}$ of messages:

[12] With sets M^i as strategy spaces and and payoff functions $\pi^i = v^i \circ \varphi^i : M \to \mathbb{R}$.

1. The market prices of the private goods and the producer prices of the public goods are constructed as follows: first, let

$$s^i(\mathbf{m}) = \sum_{j,j' \neq i} \|(p^j, Q^j) - (p^{j'}, Q^{j'})\|$$

and $s(\mathbf{m}) = \sum_i s^i(\mathbf{m})$; then, if $s(\mathbf{m}) = 0$, let $p(\mathbf{m}) = p^1$ and $Q(\mathbf{m}) = Q^1$; otherwise,

$$[p(\mathbf{m}), Q(\mathbf{m})] = \frac{1}{s(\mathbf{m})} \textstyle\sum_i s^i(\mathbf{m})(p^i, Q^i).$$

2. For each individual, there are two personalized prices for each public good k:

$$q_k^i(\mathbf{m}) = \frac{1}{I} Q_k(\mathbf{m}) - \sum_{j \neq i} a_k^j \tag{6}$$

and

$$q_k^{-i}(\mathbf{m}) = \frac{1}{I} Q_k(\mathbf{m}) - \sum_{j \neq i} a_k^j + \frac{1}{I-1} \sum_{j \neq i} b_k^j. \tag{7}$$

3. The set of bundles of public goods that can be produced and is affordable by all individuals, as a result of the prices, is the set of $Y \in R_+^K$ for which

   (a) there exists a profile $\mathbf{y}$ such that $Y = \sum_i y^i$,

   $$p(\mathbf{m}) \cdot \omega^i - q^i(\mathbf{m}^{-i}) \cdot y^i - q^{-i}(\mathbf{m}) \cdot Y^{-i} \geqslant 0,$$

   and

   $$\sum_i \left[q^i(\mathbf{m}) + \textstyle\sum_{j \neq i} q^{-j}(\mathbf{m})\right] \cdot y^i \geqslant Q(\mathbf{m}) \cdot Y$$

   for all i; and

   (b) $f(\sum_i \omega^i) \geqslant Y$.

   Denoting this set by $\mathcal{B}(\mathbf{m})$, the actual supply of public goods is

   $$Y(\mathbf{m}) = \text{argmin}_Y \left\{\| \textstyle\sum_i c^i - Y\| : Y \in \mathcal{B}(m)\right\},$$

   and its allocation is, for each good k,

   $$y_k^i(\mathbf{m}) = \frac{d_k^i}{\sum_j d_k^j} Y_k(\mathbf{m}).$$

4. There is a *punishment tax* levied on individual i at a rate

$$\tau^i(\mathbf{m}) = 1 - \exp\{-\|b^i - \textstyle\sum_j a^j\|\},$$

so the total tax paid by her is the sum of

$$q^i(\mathbf{m}) \cdot y^i(\mathbf{m}) + q^{\neg i}(\mathbf{m}) \cdot [Y(\mathbf{m}) - y^i(\mathbf{m})]$$

and

$$\tau^i(\mathbf{m})[p(\mathbf{m}) \cdot \omega^i - q^i(\mathbf{m}) \cdot y^i(\mathbf{m}) - q^{\neg i}(\mathbf{m}) \cdot [Y(\mathbf{m}) - y^i(\mathbf{m})].$$

We denote this sum $T^i(\mathbf{m})$.

5. The set of bundles of private goods that can be produced and is affordable by individual i is $\mathcal{B}^i(\mathbf{m})$, defined as

$$\{x^i \in R^L_+ \mid p(\mathbf{m}) \cdot x^i \leqslant p(\mathbf{m}) \cdot \omega^i - T^i(\mathbf{m}) \text{ and } f(\textstyle\sum_j \omega^j - x^i) \geqslant Y(\mathbf{m})\},$$

and the bundle of private goods for individual i closest to her claim is

$$\hat{x}^i(\mathbf{m}) = \text{argmin}_x \left\{\|x - e^i\| : x \in \mathcal{B}^i(\mathbf{m})\right\}. \tag{8}$$

In order to determine the actual allocation of the private good define a "shrinking factor"

$$N(\mathbf{m}) = \min\left\{N \in R_{++} : f\left(\textstyle\sum_i \omega^i - \sum_i \frac{n^i}{N}\hat{x}^i(\mathbf{m})\right) \geqslant Y(\mathbf{m}) \text{ and } N \geqslant n^i \text{ for all } i\right\}, \tag{9}$$

and then let

$$x^i(\mathbf{m}) = \frac{n^i}{N(\mathbf{m})}\hat{x}^i(\mathbf{m}). \tag{10}$$

Note that, so constructed, the outcome function is single-valued and continuous. Other features of it include the facts that $p(\mathbf{m})$ and $q(\mathbf{m})$ are functions of the component $(p^i, Q^i)_{i=1}^I$ of the messages only, and that, as before, τ^i depends on $\mathbf{a} = (a^1, \ldots, a^I)$ and b^i only, so we again denote it as $\tau^i(\mathbf{a}, b^i)$. Importantly, the punishment tax is once again null when, and only when, $b^i = \sum_j a^j$.

Remark 3. *In the above construction, consumers' endowment is assumed to be known to the mechanism designer. This is just for simplicity of the argument. Instead, we can ask consumers to report their endowment as in Tian [27], and consumers will truthfully report their endowment in equilibria. This does not bring more insight.*

The following two result extends Lemmas 1 and 2 to the general case. Much of the logic of their arguments is the same as in Section 2, so we can omit many details.

Lemma 3. *The mechanism gives non-negative consumption to all individuals, is feasible and satisfies:*

(a) If $v^i(\hat{x}^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m}))) > v^i(x, y, Y)$*, then individual* i *can choose a message* $\hat{m}$ *such that* $\pi^i(\hat{m}, \mathbf{m}^{\neg i}) > v^i(x, y, Y)$.

(b) *Suppose that* $x^i(\mathbf{m}) \in R^L_{++}$ *for all individuals and* $Y(\mathbf{m}) \in R^L_{++}$*. If for some* i *there exists* (x, y, Y) *with* $Y = \sum_{j\neq i} y^j$ *such that* $p(\mathbf{m}) \cdot x \leqslant p(\mathbf{m}) \cdot \omega^i - T^i(\mathbf{m})$,

$$[q^i(\mathbf{m}) + \textstyle\sum_{j\neq i} q^{-j}(\mathbf{m})] \cdot y + \sum_{j\neq i}[q^j(\mathbf{m}) + \sum_{j'\neq j} q^{-j'}(\mathbf{m})] \cdot y^j \geqslant Q(\mathbf{m})(y + Y),$$

and

$$v^i(x, y, y + Y) > v^i(x^i(\mathbf{m}), y^i(\mathbf{m}), y^i(\mathbf{m}) + Y^{-i}(\mathbf{m})),$$

then there exists a message that i *can send,* $\hat{m}$*, such that* $\pi^i(\hat{m}, \mathbf{m}^{-i}) > \pi^i(\mathbf{m})$.

Proof. Non-negativity of individual consumption and feasibility of the individual bundles follow, once again, from the properties of the punishment function τ^i and the construction of $Y(\mathbf{m})$ and $x^i(\mathbf{m})$. For property (a), since $N(\hat{n}, \mathbf{m}^{-i}) \geqslant \hat{n}$, individual i can announce a large enough $\hat{n}$ to make $x^i(\hat{m}, \mathbf{m}^{-i}) \approx \hat{x}^i(\mathbf{m})$, which, suffices for the result, by continuity of v^i. For property (b), define the convex combination $(x_\lambda, y_\lambda, Y_\lambda)$ as in the proof of Lemma 3, and note that $p(\mathbf{m}) \cdot x_\lambda \leqslant p(\mathbf{m}) \cdot \omega^i - T^i(\mathbf{m})$,

$$[q^i(\mathbf{m}) + \textstyle\sum_{j\neq i} q^{-j}(\mathbf{m})] \cdot y_\lambda + \sum_{j\neq i}[q^j(\mathbf{m}) + \sum_{j'\neq j} q^{-j'}(\mathbf{m})] \cdot y^j_\lambda \geqslant Q(\mathbf{m})(y_\lambda + Y_\lambda),$$

and

$$v^i(x_\lambda, y_\lambda, y_\lambda + Y_\lambda) > v^i(x^i(\mathbf{m}), y^i(\mathbf{m}), y^i(\mathbf{m}) + Y^{-i}(\mathbf{m})).$$

Then, using the quasi-concavity properties of v^i, the result follows if we define the message $\hat{m}$ as: $\hat{p} = p^i$, $\hat{Q} = Q^i$, $\hat{a} = a^i$, $\hat{b} = b^i$, $\hat{c} = Y_\lambda + y_\lambda - \sum_{j\neq i} c^j$, $\hat{e} = x_\lambda$, and $\hat{d}$ such that

$$\frac{\hat{d}_k}{\hat{d}_k + \sum_{j\neq i} d^j_k}(Y_{\lambda,k} + y_{\lambda,k}) = y_{\lambda,k}$$

for each public good. If λ is small enough and $\hat{n}$ large enough, by continuity $\bar{x}^i(\hat{m}, \mathbf{m}^{-i}) = x_\lambda$, and $Y(\hat{m}, \mathbf{m}^{-i}) = Y_\lambda + y_\lambda$, which further implies that $y^i(\hat{m}, \mathbf{m}^{-i}) = y_\lambda$. □

Lemma 4. *Let* $\bar{\mathbf{m}}$ *be a Nash equilibrium of the game induced by the mechanism. Then* $Y(\bar{\mathbf{m}}) \gg 0$ *and for all individuals:*

(a) $x^i(\bar{\mathbf{m}}) \gg 0$*;*

(b) $q^i(\bar{\mathbf{m}}) \gg 0$ *and* $q^{-i}(\bar{\mathbf{m}}) \gg 0$*;*

(c) $\bar{b}_i = \sum_{j=1}^{I} \bar{a}_j$*,* $q(\bar{\mathbf{m}}) = q_i(\bar{\mathbf{m}}) + \sum_{j\neq i} q_{\neg j}(\bar{\mathbf{m}})$ *and* $\tau^i(\bar{\mathbf{m}}) = 0$.

(d) $p(\bar{\mathbf{m}}) \cdot x^i(\bar{\mathbf{m}}) + q_i(\bar{\mathbf{m}}) \cdot y^i(\bar{\mathbf{m}}) + q_{-i}(\bar{\mathbf{m}}) \cdot Y^{-i}(\bar{\mathbf{m}}) = p(\bar{\mathbf{m}}) \cdot \omega^i$*; and*

(e) $N(\bar{\mathbf{m}}) = n^i$ *and thus* $x^i(\bar{\mathbf{m}}) = \hat{x}^i(\bar{\mathbf{m}})$.

Proof. Only the last property is new to the general case. For all others, the logic is the same as in Lemma 2, keeping in mind that, when constructing a deviation $\hat{m}$ for an individual, number $\hat{n}$ must be sufficiently large.[13] For property (e), suppose $N(\bar{\mathbf{m}}) > n^i$ for some individual i. Then

$$x^i(\bar{\mathbf{m}}) = \frac{n^i}{N}\hat{x}^i(\bar{\mathbf{m}}) < \hat{x}^i(\bar{\mathbf{m}}),$$

and therefore

$$p(\bar{\mathbf{m}}) \cdot x^i(\bar{\mathbf{m}}) + q^i(\bar{\mathbf{m}}) \cdot y^i(\bar{\mathbf{m}}) + q^{-i}(\bar{\mathbf{m}}) \cdot Y^{-i}(\bar{\mathbf{m}}) < p(\bar{\mathbf{m}}) \cdot \omega^i,$$

which contradicts property (d). □

3.4 Nash implementation of Lindahl allocations

Again, the two lemmas yield the following theorem, which is the main result in this note.

Theorem 2. *The set of allocations implemented by the mechanism is the (complete) set of Lindahl equilibrium allocations.*

Proof. We first argue that if $\bar{\mathbf{m}}$ is a Nash equilibrium, the resulting allocation corresponds to a Lindahl equilibrium. This is now straightforward: by properties (a) and (d) of Lemma 2 and (b) of Lemma 1,[14] each $(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), Y^{-i}(\bar{\mathbf{m}}))$ must solve the problem

$$\max_{x,y,Y} \left\{ v^i(x, y, y+Y) : p(\bar{\mathbf{m}}) \cdot x + q^i(\bar{\mathbf{m}}^{-i}) \cdot y + q^{-i}(\bar{\mathbf{m}}^{-i}) \cdot Y \leqslant p(\bar{\mathbf{m}}) \cdot \omega^i \right\}. \quad (11)$$

This yields condition (b) in Definition 2. Property (c) follows from property (d) in Lemma 4 and the fact that the mechanism is feasible. Property (c) in Lemma 4 also imples that

$$q^i(\bar{\mathbf{m}}) + \sum_{j \neq i} q^{-j}(\bar{\mathbf{m}}) = Q(\bar{\mathbf{m}}),$$

namely property (d) in Definition 2. Property (d) in Lemma 4 implies property (a) in the definition. We now argue that for any Lindahl equilibrium, $[P, (q^i, q^{-i})_{i=1}^I, Q, \bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{X}, \bar{Y}]$ there is a Nash equilibrium that implements its allocation. To see this, first let $(\mathbf{a}, \mathbf{c})$ solve the following linear equation system:

$$\textstyle\sum_i c^i = \bar{Y},$$

for all i

$$\sum_{j \neq i}^{I} a^j = \frac{1}{I} Q - q^i,$$

[13] This applies, in particular, in the argument that $Y(\bar{\mathbf{m}}) \gg 0$ and in property (a).

[14] Note that the condition $[q^i(\mathbf{m}) + \sum_{j \neq i} q^{-j}(\mathbf{m})] \cdot y + \sum_{j \neq i}[q^j(\mathbf{m}) + \sum_{j' \neq j} q^{-j'}(\mathbf{m})] \cdot y^j \geqslant Q(\mathbf{m})(y + Y)$ in (b) of Lemma 1 automatically holds at Nash equilibrium $\bar{\mathbf{m}}$, since $q^i(\bar{\mathbf{m}}) + \sum_{j \neq i} q^{-j}(\bar{\mathbf{m}}) = Q(\bar{\mathbf{m}}$ for each i.

and

$$c_k^i \bar{Y}_k - \sum_j c_k^j \bar{y}_k^i = 0,$$

also for each i and each k. Besides, let $p^i = p$, $Q^i = Q$, $b^i = \sum_j a^j$, $d^i = c^i$, $e^i = \bar{x}^i$, and $n^i = 1$. With these numbers, construct the profile of strategies $\bar{\mathbf{m}}$. By construction,

$$[p(\bar{\mathbf{m}}), Q(\bar{\mathbf{m}})] = (p, Q),$$

while

$$(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), q^i(\bar{\mathbf{m}}), q^{\neg i}(\bar{\mathbf{m}})) = (\bar{x}^i, \bar{y}^i, q^i, q^{\neg i}),$$

and

$$q^i(\bar{\mathbf{m}}) + \sum_{j \neq i} q^{\neg j}(\bar{\mathbf{m}}) = Q$$

for all individuals. Condition (b) in Definition 1 implies that for all deviations $\hat{m}$,

$$\begin{aligned} \pi^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) &= v^i(x^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}), y^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}), y^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) + Y^{\neg i}(\hat{m}, \bar{\mathbf{m}}^{\neg i})) \\ &\leqslant v^i(\bar{x}^i, \bar{y}^i, \bar{Y}) \\ &= \pi^i(\bar{\mathbf{m}}), \end{aligned}$$

since, in particular, the choice of $\hat{m}$ cannot affect the individual's personalized prices. □

4 Conclusion

Significant progress has been made in terms of the study of institutions that achieve Pareto efficiency of public goods provision since the original work of Groves and Ledyard [16]. However, all these relevant work assume the classical public goods model, which has theoretical predictions that have been rejected by empirical evidence. The warm-glow model has been regarded to be more appealing and successful in describing consumers' public goods provision behavior. How to achieve Pareto efficiency in the presence of warm-glow effect is addressed in this paper.

The mechanism we define is informationally decentralized and respects the incentive compatibility constraint: the mechanism designer does not know the characteristics of the economy (he does not even know whether there exists warm-glow effects in the economy) and relies on the information provided by consumers in the economy to make allocation according to the outcome function. Consumers strategically report messages concerning their preferences and production technology, and can manipulate the messages for their own benefit. However, in equilibria, all consumers will report truthfully, and the Nash equilibria fully implement Lindahl equilbria of the economy. Our mechanism possess good properties: it is individually rational, single-valued, feasible and continuous. When there is no warm-glow effect, the mechanism implements the classical Lindahl allocations.

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