A simple(r) Lindahl solution to the provision of public goods with warm-glow: efficiency and implementation[1]

Andrés Carvajal[2] Xinxi Song[3]

November 24, 2020

[1] We thank seminar participants at Liaoning University. Xinxi Song acknowledges funding by NSFC NO.71703110.

[2] University of California, Davis, and EPGE-FGV; e-Mail: `acarvajal@ucdavis.edu`.

[3] *Corresponding author*, International School of Economics and Management, Capital University of Economics and Business; e-Mail: `songxinxi@cueb.edu.cn`.

Abstract

We provide a simple solution to the problem of efficiently providing public goods in a warm-glow economy. Compared with Allouch [2013], our solution is closer to Lindahl [1958], which requires only one personalized price for each consumer. As an application, we show that under our solution concept, the implementation mechanism of Tian [1989] can be modified to implement Lindahl allocations.

Keywords: public goods, warm-glow, efficiency, implementation.

JEL Classification Number: C72, D64, D82, H41.

1 Introduction

In the classic model of public goods provision, consumers care about the aggregate amount of public goods only, and the identity is irrelevant. Samuelson (1954) characterized the conditions for Pareto efficiency in the classical public goods model, and showed that competitive markets fail these conditions due to free riding.[1] Lindahl (1958) and Foley (1970) offered a solution to achieve the efficiency of competitive markets: to charge each consumer differential taxes (Lindahl prices) proportional to her benefit. That is, the Pareto efficient level of public goods can be achieved through Lindahl equilibrium with each consumer having one personalized price for the public goods.

However, as Warr (1983) and Bergstrom, Blume, and Varian (1986) among others demonstrated, the classical public goods model implies that government provision will crowd out private contribution dollar for dollar, and the public goods provision is neutral to the income redistribution up to the set of contributors not changing. Andreoni (1988) showed that the classical public goods model implies virtually no one makes contribution in a large economy. These model predictions are in contrast with evidence as docomented by Andreoni (1989).

Based on the earlier work by Becker (1974) and Cornes and Sandler (1984), Andreoni (1989, 1990) proposed the "warm-glow" model to explain consumers' public goods contribution behavior better. In the warm-glow model, consumers' altruism is impure, and the reason they make public goods contribution can be social pressure, guilt, sympathy or simply a desire for the warm-glow feeling. From the perspective of a consumer with warm-glow preferences, her contribution is not a perfect substitute for other consumers' contribution, which implies that the crowding-out is not complete, income redistribution effect is not neutral and free-riding is less severe. The warm-glow model is considered to be more consistent with the observed consumer behavior, and is widely used in public economics.

Allouch (2013) extended Lindahl's solution to achieve Pareto efficient provision of public goods in a warm-glow economy. In his *warm-glow equilibrium*, each consumer has two personalized Lindahl prices for each public good, instead of one as in the classical public goods model, with one Lindahl price for her own contribution and the other for other consumers' contribution.[2]

In this paper, we provide an alternative solution to the problem of efficient public goods provision in a decentralized competitive market *à la* Lindahl when there is warm-glow. Our argument is based on the following observation: a consumer's provision of the impure public goods enters her utility twice—both as private goods and as public goods; for private goods, every consumer consumes different bundles and pays the same prices; and for public goods, every consumer consumes the same bundle and pays different prices. Therefore, in

[1] Actually the competitive allocation is generically constrained suboptimal as showed by Geanakoplos and Polemarchakis (2008).

[2] Carvajal and Song (2018) showed that the warm-glow model under the equilibrium concept of Allouch (2013) has strong nonparametric testable implications.

our *warm-glow Lindahl equilibrium* the impure public goods have two market prices: one as private goods and the other as public goods. We interpret the first price as the price for the appropriation right of each unit of the impure public goods, and the second one as the price of contributing to their communal provision. Each consumer pays the same price for appropriation rights. Each consumer also has a personalized price, the sum of which is the production price for the supply of public good. We show that warm-glow Lindahl equilibrium achieves Pareto efficiency, and that any Pareto efficient allocation can be supported as a warm-glow Lindahl equilibrium. Under standard assumptions, a warm-glow Lindahl equilibrium exists and lies in the core. Compared with Allouch (2013), our solution requires less prices to support efficient allocations: for each consumer, only one personalized price is needed. In this sense our solution is closer to Lindahl (1958), and many techniques developed to deal with the pure public goods provision can be applied to the warm-glow case.

As Arrow (1970) pointed out, however, Lindahl's solution does not satisfy the incentive compatibility constraints of Hurwicz (1972), and consumers have no incentive to truthfully reveal their valuation for the public goods. Then the question is how to design an incentive compatible mechanism to achieve the Pareto efficient provision of public goods. Groves and Ledyard (1977) invented a mixed competitive mechanism, in which the private goods are allocated by competitive markets and the government designs a game to allow consumers (strategically reporting messages concerning their preferences) to determine the aggregate supply of pure public goods and the taxes imposed on each consumer. The equilibria of this mechanism are Pareto optimal, though not necessarily Lindahl equilibria. This original work of Groves and Ledyard (1977) has been improved by later research. In particular, Tian (1989) designed a game form to allocate both private and pure public goods to achieve Lindahl equilibria, and the mechanism is single-valued, feasible and continuous.

Since our solution to efficient public goods provision with warm-glow is close to Lindahl (1958), we show that Tian's mechanism for pure public goods provision can be modified to implement Pareto efficent provision of public goods with warm-glow. The trick is to treat each consumer's personal contribution of public goods as private goods, but with three caveats: first, unlike private goods, each consumer's personal contribution of public goods does not need satisfy a boundary condition; second, each consumer's personal contribution of public goods should be determined by a different mechanism from that of private goods, since the sum of each consumer's personal contribution has to equal the aggregate public goods level; third, only private goods are used as input to produce public goods, and consumers' personal contribution of public goods will not affect production feasibility as private goods. Therefore, Tian's mechanism has to be modified accordingly.

The paper is organized as follows. Section 2 defines the economic environment and introduces the solution concept. Section 3 shows the solution concept satisfies the two welfare properties, and the existence of the equilibrium and the core property follow from the standard argument. Section 4 provides a mechanism to implement Pareto efficient provision of public goods with warm-glow, by modifying Tian's mechanism.

2 Environment

Consider an economy with L private goods and K public goods. There are $I \geq 2$ consumers in the economy, denoted by $i = 1, \ldots, I$. Denote by $x^i \in \mathbb{R}^L_+$ consumer i's private goods consumption, and by $y^i \in \mathbb{R}^K_+$ her public goods provision.

Each consumer is endowed with a warm-glow utility function, $u^i(x^i, y^i, \sum_j y^j)$, and an initial bundle of private goods, $w^i \in \mathbb{R}^L_{++}$. There are no initial endowments of public goods, which instead are produced from private goods. The public goods production set is denoted by $\mathbb{Y}$. Its typical element is (X, Y), where $X \in \mathbb{R}^L_-$ is the private goods input, and $Y \in \mathbb{R}^K_+$ is the public goods output.

For each individual i, the utility function u^i is assumed to be continuous, strictly increasing, and quasi-concave on $\mathbb{R}^{L+2K}_+$. The production set $\mathbb{Y}$ is assumed to be a closed, convex cone, satisfying that $(\mathbb{R}^L_-, 0) \subseteq \mathbb{Y}$ (free disposal and possibility of inaction); and that for any $Y \in \mathbb{R}^K_+$, there is an $X \in \mathbb{R}^L_-$ such that $(X, Y) \in \mathbb{Y}$.

Throughout the paper, we will use boldface to denote profiles of variables.[3] Thus, an *allocation* is a triple $(\mathbf{x}, \mathbf{y}, Y) \in \mathbb{R}^{I(L+K)+K}_+$; it is *feasible* if $Y = \sum_i y^i$ and $(\sum_i(x^i - w^i), Y) \in \mathbb{Y}$; it is *Pareto efficient* if it is feasible and there does not exist another feasible allocation $(\tilde{x}, \tilde{y}, \tilde{Y})$ such that $u^i(\tilde{x}^i, \tilde{y}^i, \tilde{Y}) \geq u^i(x^i, y^i, Y)$ for each consumer i, with strict inequality at least for one.

Definition 1 *A* warm-glow Lindahl equilibrium *is a tuple* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{Y}, p, \mathbf{q}, Q, r)$, *where* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{Y})$ *is a feasible allocation,* $p \in \mathbb{R}^L_+$ *denotes the prices of private goods, each* $q^i \in \mathbb{R}^K_+$ *is the vector of personalized consumption price for public goods,* $Q \in \mathbb{R}^K_+$ *is the production price for public goods, and* $r \in \mathbb{R}^K_+$ *is the price for appropriation rights over public goods, such that:*

1. *the firm maximizes profits: for all* $(X, Y) \in \mathbb{Y}$,

$$p \cdot X + (Q + r) \cdot Y \leq p \cdot \textstyle\sum_i(\bar{x}^i - w^i) + (Q + r) \cdot \bar{Y},$$

   where $Q = \sum_i q^i$; *and*

2. *each consumer maximizes her utility:*

$$p \cdot \bar{x}^i + r \cdot \bar{y}^i + q^i \cdot \bar{Y} \leq p \cdot w^i,$$

   and

$$u^i(x, y, Y) > u^i(\bar{x}^i, \bar{y}^i, \bar{Y}) \Rightarrow p \cdot x + r \cdot y + q^i \cdot Y > p \cdot w^i,$$

[3] For example, $\mathbf{x} = (x^1, \ldots, x^I)$.

In economies with warm-glow effects, the public goods contain both a proprietary component, y^i, and a communal component, Y. In the latter decentralized competitive market, these two components are traded separately: all individuals pay a common price r_k for the appropriation of each unit of public good k, whereas each i contributes q^i_k per unit of the whole communal provision of that good.[4] The firm receives the aggregate of the personalized contributions, Q_k, as well as the appropriation price, r_k, per unit of good k.

3 Pareto efficiency

We now show that the concept of warm-glow Lindahl equilibrium retains the two fundamental theorems of welfare economics.

Theorem 1 *Any warm-glow Lindahl equilibrium allocation is Pareto efficient.*

Proof. Suppose, by way of contradiction, that there exists a warm-glow Lindahl equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{Y}, p, \mathbf{q}, Q, r)$ whose allocation is not efficient. Let $(\tilde{x}, \tilde{y}, \tilde{Y})$ be feasible and Pareto superior.

From utility maximization, it must be true that

$$p \cdot \tilde{x}^i + r \cdot \tilde{y}^i + q^i \cdot \tilde{Y} \geq p \cdot w^i$$

for each consumer i, with strict inequality at least for one of them. Summing the above inequalities across consumers, we have

$$p \cdot \textstyle\sum_i \tilde{x}^i + r \cdot \sum_i \tilde{y}^i + \sum_i q^i \cdot \tilde{Y} > p \cdot \sum_i w^i.$$

Given that $\sum_i \tilde{y}^i = \tilde{Y}$ and $\sum_i q^i = Q$, the above inequality implies that

$$p \cdot \textstyle\sum_i \tilde{x}^i + (r + Q) \cdot \tilde{Y} > p \cdot \sum_i w^i,$$

contradicting the firms' profit maximization. ■

Theorem 2 *If the feasible allocation $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{Y})$ is Pareto efficient, then there exists a price system $(p, \mathbf{q}, Q, r)$ such that:*

1. *for all $(X, Y) \in \mathbb{Y}$,*

$$p \cdot X + (Q + r) \cdot Y \leq p \cdot \textstyle\sum_i (\bar{x}^i - w^i) + (Q + r) \cdot \bar{Y},$$

where $Q = \sum_i q^i$; and

[4] In the classic public goods model, consumers derive utility from using the communal provision of public goods only, and therefore $r = 0$.

2. for each consumer i,

$$u^i(x, y, Y) > u^i(\bar{x}^i, \bar{y}^i, \bar{Y}) \Rightarrow p \cdot x^i + r \cdot y^i + q^i \cdot Y > p \cdot \bar{x}^i + r \cdot \bar{y}^i + q^i \cdot \bar{Y}.$$

Proof. As with the previous theorem, the proof is standard. Let $\mathcal{F}$ denote the set of arrays $(X, Y, \ldots, Y)$, where Y appears $I + 1$ times, for some $(X, Y) \in \mathbb{Y}$. This is a convex cone, since $\mathbb{Y}$ is a convex cone, and is non-empty since $(0, 0) \in \mathbb{Y}$.

Define also as $\mathcal{S}$ the set of arrays $(X, Y, Y^1, \ldots, Y^I) \in \mathbb{R}^{L+K(I+1)}$ such that there exist $\mathbf{x}$ and $\mathbf{y}$ for which: $\sum_i (x^i - w^i) = X$; $\sum_i y^i = Y$; and for all i, $u^i(x^i, y^i, Y^i) > u^i(\bar{x}^i, \bar{y}^i, \bar{Y})$. This set is convex since each utility function is quasi-concave, and non-empty since each utility function is strictly increasing.

Since $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{Y})$ is efficient, set $\mathcal{F}$ and $\mathcal{S}$ are disjoint, so, by the separating hyperplane theorem, there exist a vector $(p, \mathbf{q}, r) \neq 0$ and a scalar α such that

$$\forall (X, Y, Y, \ldots, Y) \in \mathcal{F},\ p \cdot X + (r + \textstyle\sum_i q^i) \cdot Y \leq \alpha, \tag{1}$$

while

$$\forall (X, Y, Y^1, \ldots, Y^I) \in \bar{\mathcal{S}},\ p \cdot X + r \cdot Y + \textstyle\sum_i q^i \cdot Y^i \geq \alpha, \tag{2}$$

where $\bar{\mathcal{S}}$ denotes the closure of $\mathcal{S}$.

By continuity of the utility functions,

$$(\bar{X}, \bar{Y}, \bar{Y}, \ldots, \bar{Y}) \in \mathcal{F} \cap \bar{\mathcal{S}},$$

so

$$p \cdot \textstyle\sum_i (\bar{x}^i - w^i) + (\sum_{i=1}^n q^i + r) \cdot \bar{Y} = \alpha,$$

which, together with Eq. (1), yields the first claim of the theorem.

Since $(0, 0, \ldots, 0) \in \mathcal{F}$, which is a cone, we can take $\alpha = 0$ in Eqs (1) and (2). Moreover, by monotonicity of all utility functions one can prove that $p > 0$.

Now, suppose $u^j(\tilde{x}^j, \tilde{y}^j, \tilde{Y}^j) > u^j(\bar{x}^j, \bar{y}^j, \bar{Y}^j)$ for some individual j. Let $(\tilde{x}^i, \tilde{y}^i) = (\bar{x}^i, \bar{y}^i)$ for all $i \neq j$, and note that

$$\left(\textstyle\sum_{i=1}^I (\tilde{x}^i - w^i), \sum_{i=1}^I \tilde{y}^i, \tilde{Y}^1, ..., \tilde{Y}^n\right) \in \bar{\mathcal{S}}.$$

This implies that

$$p \cdot \textstyle\sum_{i=1}^I (\tilde{x}^i - w^i) + r \cdot \sum_{i=1}^I \tilde{y}^i + \sum_{i=1}^I \bar{q}^i \cdot \tilde{Y}^i \geq p \cdot \sum_{i=1}^I (\bar{x}^i - w^i) + (\sum_{i=1}^n q^i + r) \cdot \bar{Y}.$$

Since all terms are the same on both sides except those corresponding to consumer j, it must be true that

$$p \cdot \tilde{x}^j + r \cdot \tilde{y}^j + q^j \cdot \tilde{Y}^y \geq p \cdot \bar{x}^j + r \cdot \bar{y}^j + \bar{q}^j \cdot \bar{Y}. \tag{3}$$

Suppose equality in this equation held. By the assumptions on set $\mathbb{Y}$, there is another input-output point with all private goods smaller, so with lower value since $p > 0$. Along the line between this point and $(\tilde{x}^j, \tilde{y}^j, \tilde{Y}^j)$, all points have lower value than $(\bar{x}^j, \bar{y}^j, \bar{j})$. But near $(\tilde{x}^j, \tilde{y}^j, \tilde{Y}^j)$, by continuity of u^j, there will be a point preferred to $(\bar{x}^j, \bar{y}^j, \bar{Y})$. This corresponds to a point in $\bar{\mathcal{S}}$ that has a smaller inner product with $(p, r, \mathbf{q})$ than

$$(\textstyle\sum_{i=1}^{I} \bar{x}^i, \sum_{i=1}^{n} \bar{y}^i, \bar{Y}, ..., \bar{Y}),$$

contradicting Eq. (2). Therefore, Eq. (3) must hold with equality, which gives the second claim of the theorem. ■

As usual, supporting the Pareto efficient allocation of this theorem requires a transfer policy whereby $p \cdot \bar{x}^i + r \cdot \bar{x}^i + \bar{q}^i \cdot \bar{Y} - p \cdot w^i$ is given to consumer i, lump-sum.

The existence of the warm-glow Lindahl equilibrium can by proved by modifying the argument of Foley (1970). The strategy is to construct a private economy in which the equilibrium exists by Debreu (1962); then to show the quasi-equilibrium of this economy is a warm-glow Lindahl equilibrium.

Foley (1970) defined the core of an economy e as the set of feasible allocations which cannot be blocked by any non-empty coalition. Foley's argument can be applied to show that the warm-glow Lindahl equilibrium allocation is in the core of the economy.

4 Implementation

The warm-glow Lindahl equilibrium we proposed in last section achieves efficient allocations in a warm-glow economy. However, we cannot rely on such market machanism to operate in real world, since, as Arrow (1970) pointed out, Lindahl's solution does not satisfy the incentive compatibility constraints of Hurwicz (1972), and consumers have no incentive to truthfully reveal their valuation for the public goods. Groves and Ledyard (1977) invented a mixed competitive mechanism, in which the private goods are allocated by competitive markets and the government designs a game to allow consumers (strategically reporting messages concerning their preferences) to determine the aggregate supply of pure public goods and the taxes imposed on each consumer. The equilibria of this mechanism are Pareto optimal, though not necessarily Lindahl equilibria. This original work of Groves and Ledyard (1977) has been improved by later research. In particular, Tian (1989) designed a game form to allocate both private and pure public goods to achieve Lindahl equilibria, and the mechanism is single-valued, feasible and continuous.

A natural question is how to design a mechanism to implement Pareto efficient applications in a warm-glow economy. As we argued, one advantage of our solution is that it is more close to Lindahl (1958), and many techniques developed to deal with the pure public goods provision can be applied to the warm-glow case. In this section, we show that Tian's mechanism can be modified to fully implement warm-glow Lindahl equilibria. For implementation, we maintain the following assumptions from now on: $I \geq 3$; each function $u^i(x, y, Y)$ is continuous, strictly increasing and strictly quasi-concave in (x, Y), non-decreasing and strictly quasi-concave in y, and satisfies that for any $\bar{x} \in \partial R_+^L$, any $x \in R_{++}^L$, any $y \leq Y$ and any $\bar{y} \leq \bar{Y}$,

$$u^i(x, y, Y) > u^i(\bar{x}, \bar{y}, \bar{Y}).$$

4.1 Mechanisms

A mechanism (or game form) fixes a message space for each individual and an outcome function defined over the product of the message spaces. Let M^i denote the message space of individual i, with generic element m^i. Let $M = \prod_i M^i$ and denote the outcome function $\varphi : M \to \mathbb{R}^{(K+L)I}$ as

$$\varphi(\mathbf{m}) = \left(x^i(\mathbf{m}), y^i(\mathbf{m})\right)_{i=1}^{i}.$$

The outcome function determines, for each profile of messages, an allocation for economy: the individual variables are given by the function, usage of the private good by the firm is $\sum_i [\omega^i - x^i(\mathbf{m})]$, and its output is $\sum_i y^i(\mathbf{m})$. The mechanism is said to be *feasible* if this allocation satisfies

$$(\textstyle\sum_i [x^i(\mathbf{m}) - \omega^i], \sum_i y^i(\mathbf{m})) \in \mathbb{Y}$$

for all profiles of messages.

4.2 Nash implementation

Given the mechanism, we further denote, for each individual,

$$\varphi^i(\mathbf{m}) = \left(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})\right).$$

This function represents the terms of the outcome function that affect individual i's utility. The mechanism defines a game, with strategy spaces given by the message spaces and payoff functions $\upsilon^i(\mathbf{m}) = u^i(\varphi^i(\mathbf{m}))$.

The set of allocations that are attained at the Nash equilibria of the game defined by the mechanism, given the economy, is the set *implemented* by the mechanism (in Nash equilibrium).

In what follows, we will further denote $Y^{\neg i}(\mathbf{m}) = \sum_{j \neq i} y^j(\mathbf{m})$ and $\mathbf{m}^{\neg i} = (m^j)_{j \neq i}$.

4.3 The Tian mechanism

The mechanism designer does not know the underlying fundamentals: neither whether there exists warm-glow effect or not, nor the technology of the firm.

An $I \times I$ matrix β is pre-fixed, i.e.,

$$\beta = \begin{pmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & 1 & 0 & 2-I \\ 2-I & 2-I & \cdots & \cdots & 2-I & 2-I & 0 \end{pmatrix}_{I\times I}$$

where $\sum_{i=1}^{I} \beta_{ij} = 0$, $\beta_{ii} = 0$, and the matrix has rank $I-1$.

The message space for all individuals is

$$M^i = \mathbb{R}^L_{++} \times \mathbb{R}^K_{++} \times \mathbb{R}^K_{++} \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^K_{++} \times \mathbb{R}^L_{+} \times \mathbb{R}_{++},$$

with typical element $m^i = (p^i, r^i, Q^i, a^i, b^i, c^i, d^i, n^i)$. Vector a^i is used to determine other consumers' personalized prices (different from i), b^i is used to determine the aggregate level of public goods, and c^i is used to determine $i's$ personal contribution of public goods. Vector d^i is agent i's demand for private goods, while vectors p^i, r^i and Q^i can be understood as the agent's recommendation of the prices of the private goods, the price for proprietary right over the public goods and the prices for the appropriate right over the public goods; their entries are denoted d^i_ℓ, p^i_ℓ, r^i_k and Q^i_k, respectively. Number n^i, finally, will be used in a "shouting" game.

Then:

1. The market prices of the private goods, the market price for proprietary right over the public goods and the market prices for the appropriate right over the public goods are constructed as follows: first, let

$$s^i(\mathbf{m}) = \sum_{j,h\neq i} \|(p^j, r^j, Q^j) - (p^h, r^h, Q^h)\|$$

and $s(\mathbf{m}) = \sum_i s^i(\mathbf{m})$; then, if $s(\mathbf{m}) = 0$, let $p(\mathbf{m}) = p^1$, $r(\mathbf{m}) = r^1$ and $Q(\mathbf{m}) = Q^1$; otherwise,

$$[p(\mathbf{m}), r(\mathbf{m}), Q(\mathbf{m})] = \frac{1}{s(\mathbf{m})} \sum_i s^i(\mathbf{m})(p^i, r^i, Q^i).$$

2. For each individual, there is one personalized price for each public good k:

$$q_k^i(\mathbf{m}) = \frac{1}{I} Q_k(\mathbf{m}) - \sum_{j=1}^{i} \beta_{ij} a_k^j. \tag{4}$$

   Then, by construction, $\sum_i q_k^i(\mathbf{m}) = Q_k(\mathbf{m})$.

3. The set of bundles of public goods that can be produced and is affordable by all individuals, as a result of the prices, is the set of $Y \in R_+^K$ for which

   (a) there exists a profile $\mathbf{y}$ such that $Y = \sum_i y^i$,

$$p(\mathbf{m}) \cdot \omega^i - r(\mathbf{m}) \cdot y^i - q^i(\mathbf{m}) \cdot Y \geq 0,$$

   for all i; and

   (b) $(-\sum_i \omega^i, Y) \in \mathbb{Y}$.

   Denoting this set by $\mathcal{B}(\mathbf{m})$, the actual supply of public goods is

$$Y(\mathbf{m}) = \text{argmin}_Y \left\{ || \textstyle\sum_i b^i - Y|| : Y \in \mathcal{B}(m) \right\},$$

   and its allocation is, for each good k,

$$y_k^i(\mathbf{m}) = \frac{c_k^i}{\sum_j c_k^j} Y_k(\mathbf{m}).$$

4. There is a *tax* levied on individual i at a rate

$$T^i(\mathbf{m}) = r(\mathbf{m}) \cdot y^i(\mathbf{m}) + q^i(\mathbf{m}) \cdot Y(\mathbf{m}).$$

5. The set of bundles of private goods that can be produced and is affordable by individual i is $\mathcal{B}^i(\mathbf{m})$, defined as

$$\{x^i \in R_+^L \mid p(\mathbf{m}) \cdot x^i \leq p(\mathbf{m}) \cdot \omega^i - T^i(\mathbf{m}) \text{ and } (x^i - \textstyle\sum_j \omega^j, Y(\mathbf{m})) \in \mathbb{Y}\},$$

   and the bundle of private goods for individual i closest to her claim is

$$\hat{x}^i(\mathbf{m}) = \text{argmin}_x \left\{ \|x - d^i\| : x \in \mathcal{B}^i(\mathbf{m}) \right\}. \tag{5}$$

   In order to determine the actual allocation of the private good define a "shrinking factor"

$$N(\mathbf{m}) = \min \left\{ N \in \mathbb{R}_{++} : (\textstyle\sum_i \frac{n^i}{N} \hat{x}^i(\mathbf{m}) - \sum_i \omega^i, Y(\mathbf{m})) \in \mathbb{Y} \text{ and } N \geq n^i \text{ for all } i \right\}, \tag{6}$$

and then let

$$x^i(\mathbf{m}) = \frac{n^i}{N(\mathbf{m})}\hat{x}^i(\mathbf{m}). \quad (7)$$

Remark 1 *In the above construction, consumers' endowment is assumed to be known to the mechanism designer. This is just for simplicity of the argument. Instead, we can ask consumers to report their endowment as in Tian (1989), and consumers will truthfully report their endowment in equilibria. This does not bring more insight.*

Lemma 1 *The mechanism gives non-negative consumption to all individuals, is feasible and satisfies:*

1. *If $u^i(\hat{x}^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})) > u^i(x, y, Y)$, then individual i can choose a message $\hat{m}$ such that $\upsilon^i(\hat{m}, \mathbf{m}^{\neg i}) > u^i(x, y, Y)$.*
2. *Suppose that $x^i(\mathbf{m}) \in R^L_{++}$ for all individuals. If for some i there exists (x, y, Y) with $Y = \sum_j y^j$ such that $p(\mathbf{m}) \cdot x \leq p(\mathbf{m}) \cdot \omega^i - T^i(\mathbf{m})$, and*

   $$u^i(x, y, Y) > u^i(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})),$$

   then there exists a message that i can send, $\hat{m}$, such that $\upsilon^i(\hat{m}, \mathbf{m}^{\neg i}) > \upsilon^i(\mathbf{m})$.

Proof. Non-negativity of individual consumption and feasibility of the individual bundles follow from the construction of $Y(\mathbf{m})$ and $x^i(\mathbf{m})$. For property 1, since $N(\hat{n}, \mathbf{m}^{\neg i}) \geq \hat{n}$, individual i can announce a large enough $\hat{n}$ to make $x^i(\hat{m}, \mathbf{m}^{\neg i}) \approx \hat{x}^i(\mathbf{m})$, which, suffices for the result, by continuity of υ^i.

For property 2, define, for $\lambda \in (0, 1)$, the convex combination

$$(x_\lambda, y_\lambda, Y_\lambda) = \lambda(x, y, Y) + (1 - \lambda)[x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})],$$

and note that

$$p(\mathbf{m}) \cdot x_\lambda \leq p(\mathbf{m}) \cdot \omega^i - T^i(\mathbf{m}),$$

by construction (given the definition of $x^i(\mathbf{m})$).

Define the message $\hat{m}$ as: $\hat{p} = p^i$, $\hat{r} = r^i$, $\hat{Q} = Q^i$, $\hat{a} = a^i$, $\hat{b} = Y_\lambda - \sum_{j \neq i} b^j$, $\hat{d} = x_\lambda$, and $\hat{c}$ such that

$$\frac{\hat{c}_k}{\hat{c}_k + \sum_{j \neq i} c^j_k} Y_{\lambda,k} = y_{\lambda,k}$$

for each public good. If $x^j(\mathbf{m}) \in R^L_{++}$ for all individuals, and λ is small enough, then $p(\mathbf{m}) \cdot \omega^j - r(\mathbf{m}) \cdot y^j_\lambda - q^j(\mathbf{m}) \cdot Y_\lambda > 0$ for all j and $(x^i_\lambda - \sum_j \omega^j, Y_\lambda) \in \mathbb{Y}$. And if $\hat{n}$ is large enough, by continuity $\bar{x}^i(\hat{m}, \mathbf{m}^{\neg i}) = x_\lambda$, and $Y(\hat{m}, \mathbf{m}^{\neg i}) = Y_\lambda$, which further implies that $y^i(\hat{m}, \mathbf{m}^{\neg i}) = y_\lambda$.

To prove that message $\hat{m}$ is a beneficial deviation for agent i, it suffices to notice that

$$v^i(\hat{m}, \mathbf{m}^{\neg i}) = u^i(x_\lambda, y_\lambda, Y_\lambda) > u^i(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})) = v^i(\mathbf{m}),$$

by strict quasi-concavity of u^i.[5] ■

Lemma 2 *Let* $\bar{\mathbf{m}}$ *be a Nash equilibrium of the game induced by the mechanism. Then for all individuals:*

1. $x^i(\bar{\mathbf{m}}) \gg 0$*;*
2. $p(\bar{\mathbf{m}}) \cdot x^i(\bar{\mathbf{m}}) + r(\bar{\mathbf{m}}) \cdot y^i(\bar{\mathbf{m}}) + q^i(\bar{\mathbf{m}}) \cdot Y(\bar{\mathbf{m}}) = p(\bar{\mathbf{m}}) \cdot \omega^i$*; and*
3. $N(\bar{\mathbf{m}}) = n^i$ *and thus* $x^i(\bar{\mathbf{m}}) = \hat{x}^i(\bar{\mathbf{m}})$*.*

Proof. For property 1, suppose, by way of contradiction, that $x^i(\bar{\mathbf{m}}) = 0$ for some i. Consider the following message $\hat{m} = (\hat{p}, \hat{r}, \hat{Q}, \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{n})$ that individual i can play: $\hat{p} = \bar{p}$, $\hat{r} = \bar{r}$, $\hat{Q} = \bar{Q}$, $\hat{a} = \bar{a}$, $\hat{c} = \bar{c}$, $\hat{n} = \bar{n}$, and

$$\hat{b} = -\sum_{j \neq i} y^j(\bar{\mathbf{m}}),$$

$$\hat{d} = x^i,$$

where $x^i \gg 0$ and $p(\bar{\mathbf{m}}) \cdot x^i \leq p(\bar{\mathbf{m}}) \cdot \omega^i$. Then $\hat{x}^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) = x^i$ and $Y(\hat{m}, \bar{\mathbf{m}}^{\neg i}) = 0$. By the interiority assumption, $v^i(\hat{m}, \bar{\mathbf{m}}^{\neg i}) > v^i(\bar{\mathbf{m}})$, so $\bar{\mathbf{m}}$ cannot be a Nash equilibrium.

For property 2, suppose, by way of contradiction, that $p(\bar{\mathbf{m}}) \cdot x^i(\bar{\mathbf{m}}) + r(\bar{\mathbf{m}}) \cdot y^i(\bar{\mathbf{m}}) + q^i(\bar{\mathbf{m}}) \cdot Y(\bar{\mathbf{m}}) < p(\bar{\mathbf{m}}) \cdot \omega^i$ for some i. Then there exists (x^i, y^i, Y) which is budget feasible, and strictly preferred. But property 2 in Lemma 1 implies $\bar{\mathbf{m}}$ is not a Nash equilibrium.

For property 3, suppose $N(\bar{\mathbf{m}}) > n^i$ for some individual i. Then

$$x^i(\bar{\mathbf{m}}) = \frac{n^i}{N} \hat{x}^i(\bar{\mathbf{m}}) < \hat{x}^i(\bar{\mathbf{m}}),$$

and therefore

$$p(\bar{\mathbf{m}}) \cdot x^i(\bar{\mathbf{m}}) + r(\bar{\mathbf{m}}) \cdot y^i(\bar{\mathbf{m}}) + q^i(\bar{\mathbf{m}}) \cdot Y(\bar{\mathbf{m}}) < p(\bar{\mathbf{m}}) \cdot \omega^i,$$

which contradicts property 2. ■

[5] The argument is a little more subtle than it seems. If $(x, Y) \neq [x^i(\mathbf{m}), Y(\mathbf{m})]$, strict quasi-concavity of u^i in its first and third arguments yields the inequality. Else, the result follows from the (weaker) quasi-concavity property imposed on the second argument of u^i.

Theorem 3 *The set of allocations implemented by the mechanism is the (complete) set of Lindahl equilibrium allocations.*

Proof. We first argue that if $\bar{\mathbf{m}}$ is a Nash equilibrium, the resulting allocation corresponds to a Lindahl equilibrium. This is now straightforward: by property 2 in Lemma 2, the firm's zero profit condition holds; by construction, the resulting allocations are feasible, and by property 2 in Lemma 1, each $(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), Y(\bar{\mathbf{m}}))$ must solve the problem

$$\max_{x,y,Y} \left\{ u^i(x, y, Y) : p(\bar{\mathbf{m}}) \cdot x + r(\bar{\mathbf{m}}) \cdot y + q^i(\bar{\mathbf{m}}) \cdot Y \leq p(\bar{\mathbf{m}}) \cdot \omega^i \right\}. \tag{8}$$

We now argue that for any Lindahl equilibrium, $[P, (q^i)_{i=1}^I, r, Q, \bar{x}, \bar{y}, \bar{X}, \bar{Y}]$ there is a Nash equilibrium that implements its allocation. To see this, first let $(\mathbf{a}, \mathbf{b})$ solve the following linear equation system:

$$\textstyle\sum_i b^i = \bar{Y},$$

for all i

$$q_k^i(\mathbf{m}) = \frac{1}{I} Q_k(\mathbf{m}) - \textstyle\sum_{j=1}^i \beta_{ij} a_k^j,$$

and

$$b_k^i \bar{Y}_k - \textstyle\sum_j b_k^j \bar{y}_k^i = 0,$$

also for each i and each k. Besides, let $p^i = p$, $r^i = r$, $Q^i = Q$, $c^i = b^i$, $d^i = \bar{x}^i$, and $n^i = 1$. With these numbers, construct the profile of strategies $\mathbf{m}$. By construction,

$$[p(\mathbf{m}), r(\mathbf{m}), Q(\mathbf{m})] = (p, r, Q),$$

while

$$(x^i(\mathbf{m}), y^i(\mathbf{m}), q^i(\mathbf{m})) = (\bar{x}^i, \bar{y}^i, q^i),$$

and $\sum_i q^i(\mathbf{m}) = Q$ for all individuals. For all deviations $\hat{m}$,

$$\begin{aligned} v^i(\hat{m}, \mathbf{m}^{\neg i}) &= u^i(x^i(\hat{m}, \mathbf{m}^{\neg i}), y^i(\hat{m}, \mathbf{m}^{\neg i}), Y(\hat{m}, \mathbf{m}^{\neg i})) \\ &\leq u^i(\bar{x}^i, \bar{y}^i, \bar{Y}) \\ &= v^i(\mathbf{m}), \end{aligned}$$

since, in particular, the choice of $\hat{m}$ cannot affect the individual's personalized prices. ■

Carvajal and Song (2020) construct a mechanism to implement efficient public goods provision with warm-glow under Allouch's equilibrium concept. That construction is quite different from the current mechanism since it has to take two personalized prices into account. Besides, there is resource waste out of equilibrium, which is not the case in the current mechanism.

References

N. Allouch. A competitive equilibrium for a warm-glow economy. *Economic Theory*, 53: 269–282, 2013.

J. Andreoni. Privately provided public goods in a large economy: the limits of altruism. *Journal of Public Economics*, 35:57–73, 1988.

J. Andreoni. Giving with impure altruism: applications to charity and Richardian equivalence. *Journal of Political Economy*, 97:1447–1458, 1989.

J. Andreoni. Impure altruism and donations to public goods: a theory of warm-glow giving. *Economic Journal*, 100:464–477, 1990.

K. J. Arrow. Political and economic evaluation of social effects and externalities. In M. Intriligator, editor, *Frontiers of quantitative economics*. Amsterdam, North Holland., 1970.

G. S. Becker. A theory of social interactions. *Journal of Political Economy*, 82:1063–1093, 1974.

T. C. Bergstrom, L. Blume, and H. Varian. On the private provision of public goods. *Journal of Public Economics*, 29:25–49, 1986.

A. Carvajal and X. Song. Testing Pareto efficiency and competitive equilibrium in economies with public goods. *Journal of Mathematical Economics*, 75:19–30, 2018.

A. Carvajal and X. Song. Implementing lindahl allocations in a warm-glow economy. Unpublished manuscript, 2020.

R. Cornes and T. Sandler. Easy riders, joint production, and public goods. *Economic Journal*, 94:1063–1093, 1984.

G. Debreu. New concepts and techniques for equilibrium analysis. *International Economic Review*, 3:257–273, 1962.

D. Foley. Lindahl's solution and the core of an economy with public goods. *Econometrica*, 38:66–72, 1970.

J. Geanakoplos and H. Polemarchakis. Pareto improving taxes. *Journal of Mathematical Economics*, 44:682–696, 2008.

T. Groves and J. Ledyard. Optimal allocation of public goods: a solution to the free rider problem. *Econometrica*, 45:783–809, 1977.

L. Hurwicz. On informationally decentralized systems. In C. McGuire and R. Radner, editors, *Decision and organization*. Amsterdam, North Holland., 1972.

E. Lindahl. Just taxation-a positive solution. In R.A. Musgrave and A.T Peacock, editors, *Classics in the theory of public finance*. London, Macmillan., 1958.

P. A. Samuelson. The pure theory of public expenditure and taxation. *Review of Economics and Statistics*, 36:387–389, 1954.

G. Tian. Implementation of the Lindahl correspondence by a single-valued, feasible, and continuous mechanism. *Review of Economic Studies*, 56:613–621, 1989.

P. G. Warr. The private provision of a public good is independent of the distribution of income. *Economics Letters*, 13:207–211, 1983.