Journal of Mathematical Economics 75 (2018) 19–30

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Testing Pareto efficiency and competitive equilibrium in economies with public goods☆

Andrés Carvajal [a,b], Xinxi Song [c,*]

[a] University of California, Davis, United States
[b] FGV, EPGE Escola Brasileira de Economia e Finanças, Brazil
[c] International School of Economics and Management, Capital University of Economics and Business, China

ARTICLE INFO

Article history:
Received 1 May 2017
Received in revised form 20 September 2017
Accepted 12 November 2017
Available online 28 December 2017

Keywords:
Public goods
Warm-glow
Pareto efficiency
Nash–Walras equilibrium

ABSTRACT

We characterize the nonparametric testable implications of Pareto efficiency and competitive equilibrium in economies with public goods, with and without warm-glow preferences, using mixed integer programming (MIP). Compared with tests based on the Tarski–Seidenberg algorithm, our tests are linear with respect to real and integer variables, and therefore operational, i.e., applicable to real data with multiple individuals and multiple observations. Monte Carlo simulation shows our tests can be implemented within reasonable time and have reasonable power when individual consumption can be (partially) observed.

In an influential paper, Brown and Matzkin (1996) prove that the model of general competitive equilibrium imposes testable implications in pure exchange economies, upon observation of aggregate commodity endowments, the distribution of nominal income, and the prices of the commodities. These results have been extended to models of Pareto efficient provision of public goods (Snyder, 1999), financial markets (Kübler, 2003), random preferences (Carvajal, 2004b), production (Carvajal, 2005), Pareto efficient and individually rational allocations (Bachmann, 2006) and models with externalities (Deb, 2009; Carvajal, 2010). This literature typically applies a three-step method: First, by using Afriat's theorem to characterize the individual rationality of the agents' decisions, it is shown that there exist utility functions that rationalize the data if and only if there exist some unobservable variables (utility levels, marginal utilities of income and individual consumptions) satisfying a family of polynomial inequalities. Second, from the Tarski–Seidenberg theorem of quantifier elimination, it is demonstrated that these unobservable variables can be solved out of the system of inequalities, and the resulting polynomial inequalities only include observable variables. Third, through a constructed example of observations that cannot be rationalized, it is shown that the derived polynomial inequalities are not tautologies.

A problem with the application of these results is that the Tarski–Seidenberg theorem does not offer an efficient algorithm for its implementation. In the words of Snyder (1999), "the lack of practical quantifier elimination algorithms makes [the actual derivation of testable restrictions for an economy] difficult". Cherchye et al. (2011b), a fortiori, prove that there does not exist an efficient (polynomial time) algorithm for the implementation of the results of Brown and Matzkin.[1] By exploiting the equivalence between the Afriat inequalities and the Generalized Axiom of Revealed Preference (GARP), they however provide a characterization of the restrictions that transforms their polynomial inequalities into a collection of linear restrictions with mixed integer variables. This is much easier to implement, and can be used to test data with more than two individuals.

In this paper, we focus on the empirical implications of Pareto optimal provision of public goods and of competitive equilibrium with public goods. The data set to test the empirical implications does not require full information on individual private goods consumption, and only involves market prices, aggregate endowments and production, government tax revenue and individual incomes.[2]

☆ Xinxi Song gratefully acknowledges funding by NSFC No. 71703110, financial support from the UK–China Scholarships for Excellence Programme, and financial support from the Research and Innovation Center of Metropolis Economic and Social Development, Capital University of Economics and Business. We also thank John Quah and two anonymous referees for very useful comments to an earlier draft, as well as Felix Kübler, Herakles Polemarchakis, and participants at the CRETA conference on The Identification of Rationality in Markets and Games at the University of Warwick.

* Corresponding author.
E-mail addresses: acarvajal@ucdavis.edu (A. Carvajal), songxinxi@cueb.edu.cn (X. Song).

1 Unless, in the language of computer science, $P = NP$.

2 In the warm-glow case, we assume individual contributions of public goods are also observed.

https://doi.org/10.1016/j.jmateco.2017.11.001
0304-4068/

Our research is based on Snyder (1999) and Carvajal (2010), where the Tarski–Seidenberg algorithm is used to derive necessary and sufficient conditions for observable data to be consistent with Pareto optimal provision of public goods and competitive equilibrium with public goods, respectively. Here, following the insight of Cherchye et al. (2011b), we characterize these conditions by Mixed Integer Programming (MIP) and transform the polynomial inequalities of Snyder (1999) and Carvajal (2010) into linear inequalities with mixed integer variables. Then, the consistency of data with the hypotheses is reduced to checking the feasibility of these linear inequalities with integer variables. We show that the hypotheses of Pareto efficiency and competitive equilibrium are independent, and extend the analysis to count for unobserved price for the public goods and public expenditure in markets.

We also extend the analysis to the type of economies proposed by Andreoni (1989, 1990), often referred to as warm-glow economies. We demonstrate through examples that considering warm-glow effects makes a difference when testing for Pareto efficiency and competitive equilibrium in economies with public goods. A simple Monte Carlo study shows that data with multiple observations, individuals and commodities can be tested, and the tests have reasonable power when individual consumption can be (partially) observed. One point to note is that with public goods, full observation of individual consumption cannot eliminate the computational complexity due to non-observability of personalized prices. Our results complement those of Deb et al. (2014) on testing intrinsic motivations for charitable giving at the individual level. A difference, though, is that we impose less structure on the unobserved preferences of the individuals, as our results do not rely on convexity properties.

Our paper is related, also, to studies that try to extend the approach of revealed preferences to game theory. Obviously related are the results of Deb (2009) and Carvajal (2010) showing that competitive and Nash behavior in economies with externalities yield very little empirical content. This last fact had already been suggested by Carvajal (2004a), for games played in continuous domains, for both the Nash Equilibrium and the Pareto efficiency hypotheses. Later on, it has been clear that the structure of public good games does impose empirical restrictions, even under extremely mild assumptions on the functional form of unobserved fundamental; an instance of this is Carvajal et al. (2013, 2014): the public-good nature of aggregate output in a Cournot oligopoly generates testable restrictions for the Nash equilibrium hypothesis.

1. The setup

1.1. The economy

An analyst has access to observations of an economy populated by I individuals, who are indexed by $i = 1, \ldots, I$. There are $L + 1$ commodities, to be consumed in non-negative amounts. Consumption of the last commodity, $L + 1$, is public.

The preference of individual i is represented by a utility function, $u^i : \mathbb{R}^L_+ \times \mathbb{R}_+ \to \mathbb{R}$, so that if she consumes a private bundle x^i and there is an amount y of public good available, her utility is $u^i(x^i, y)$. There is a production technology, $\Gamma \subseteq \mathbb{R}^L \times \mathbb{R}$, for the transformation of commodities.

We denote by $u := (u^1, \ldots, u^I)$ the profile of individual preferences, and by $\{u, \Gamma\}$ the set of exogenous objects that define the economy. They are not observable by the analyst.

1.2. An observation

Let bundle $(E, K) \in \mathbb{R}^L_+ \times \mathbb{R}_+$ be the aggregate endowment of commodities. Individual i's nominal income is m^i, which is understood to include any dividends she collects from production. We denote by $m := (m^1, \ldots, m^I)$ the profile of individual nominal incomes.

A production plan is $(X, Y) \in \mathbb{R}^L \times \mathbb{R}$. The entries in the production plan are interpreted as net-puts of the aggregate technology.[3]

Prices are (p, q), with $p \in \mathbb{R}^L_+$ denoting the prices at which private commodities are traded, while $q \in \mathbb{R}_+$ represents the price at which the firm sells (or buys) the public good.

An *observation* is (E, K, m, X, Y, p, q). It contains both exogenous and endogenous variables that are observable to the analyst. For aggregate consistency, it must be true that $\sum_i m^i = p \cdot (E + X) + q \cdot (K + Y)$.

1.3. Pareto efficiency

We say that, given $\{u, \Gamma\}$, an observation *is consistent with Pareto efficiency* if there exist individual private bundles, $x^i \in \mathbb{R}^L_+$, and Lindahl prices, $q^i \in \mathbb{R}_+$, such that:

1. for each individual, her consumption plan is rational in the sense of Lindahl equilibrium:
$$(x^i, K + Y) \in \underset{(x,y)}{\operatorname{argmax}} \left\{u^i(x, y) : p \cdot x + q^i \cdot y \leqslant m^i\right\}; \quad (1)$$
2. for the firm, its production plan is rational:
$$(X, Y) \in \underset{(x,y)}{\operatorname{argmax}} \{p \cdot x + q \cdot y : (x, y) \in \Gamma\};$$
3. private markets clear: $\sum_i x^i = E + X$; and
4. the public good is fully funded by the Lindahl pricing mechanism: $\sum_i q^i = q$.

1.4. Nash–Walras equilibrium

Alternatively, the analyst may be interested in whether the data is generated by the private allocation of both the private commodities and the public good.

Given $\{u, \Gamma\}$, we say that an observation *is consistent with Nash–Walras equilibrium* if there exist individual bundles, $(x^i, y^i) \in \mathbb{R}^L_+ \times \mathbb{R}_+$, consisting of both the private commodities and the public good, such that:

1. for each individual, her consumption plan is rational in the sense of Nash–Walras behavior:
$$(x^i, y^i) \in \underset{(x,y)}{\operatorname{argmax}} \left\{u^i\left(x, y + \sum_{j \neq i} y^j\right) : p \cdot x + q \cdot y \leqslant m^i\right\}; \quad (2)$$
2. for the firm, its production plan is rational:
$$(X, Y) \in \underset{(x,y)}{\operatorname{argmax}} \{p \cdot x + q \cdot y : (x, y) \in \Gamma\};$$
3. private markets clear: $\sum_i x^i = E + X$; and,
4. the public good market clears: $\sum_i y^i = K + Y$.

1.5. The data set

Suppose that the analyst has access to a data set, $\mathcal{D}$, consisting of T observations which we index by $t = 1, \ldots, T$. Observation t is denoted by $(E_t, K_t, m_t, X_t, Y_t, p_t, q_t)$.

[3] We assume that usage of commodity $L + 1$ for production purposes is *not* public, i.e., the amount of public good used in the production process of the firm does not enter individuals' utility functions. All results extend to relaxations of this assumption.

We shall say that the data set is *Pareto rationalized* by economy $\{u, \Gamma\}$ if every observation $(E_t, K_t, m_t, X_t, Y_t, p_t, q_t)$ is consistent with Pareto efficiency, given $\{u, \Gamma\}$. It is *Nash–Walras rationalized* by $\{u, \Gamma\}$ if each observation is consistent with Nash–Walras equilibrium given $\{u, \Gamma\}$.

An important feature of this literature is the desideratum to impose as little structure as possible on the (unobserved) fundamentals of the economy—in particular, no functional form or parameterization is to be used. In keeping with this tradition, we say that the data set is *Pareto rationalizable* if there exists an economy $\{u, \Gamma\}$ that Pareto rationalizes it and where: (i) for each individual i, function u^i is continuous and monotone; and (ii) technology Γ exhibits constant returns to scale.[4] Similarly, the data set is *Nash–Walras rationalizable* if there exists $\{u, \Gamma\}$, satisfying these two conditions, that Nash–Walras rationalizes it.

2. Testing Pareto efficiency

The following proposition presents an MIP algorithm for testing the hypothesis of Pareto efficiency.

Proposition 1. *Given data set $\mathcal{D}$, define, for each individual, $B^i = \max_t \left\{m_t^i\right\} + 1$. The following two statements are equivalent:*

[A] *$\mathcal{D}$ is Pareto rationalizable, and each individual's utility function can be chosen to be quasiconcave.*

[B] *There exist, for each individual: for each observation, a private bundle $x_t^i \in \mathbb{R}_+^L$, and a Lindahl price $q_t^i \in \mathbb{R}_+$, and for each pair of distinct observations a (binary) number $V_{t,s}^i \in \{0, 1\}$, such that the following conditions are satisfied:*

(1) *for each observation, $p_t \cdot x_t^i + q_t^i \cdot (K_t + Y_t) = m_t^i$;*

(2) *for each pair of distinct observations,*

$$m_t^i - \left[p_t \cdot x_s^i + q_t^i \cdot (K_s + Y_s)\right] < V_{t,s}^i \cdot B^i$$

and

$$m_t^i - \left[p_t \cdot x_s^i + q_t^i \cdot (K_s + Y_s)\right] \leqslant (1 - V_{s,t}^i) \cdot B^i.$$

(3) *for each triple of distinct observations, $V_{t,s}^i + V_{s,r}^i \leqslant 1 + V_{t,r}^i$;*

(4) *for each pair of observations, $p_t \cdot X_t + q_t \cdot Y_t = 0$ and $p_t \cdot X_s + q_t \cdot Y_s \leqslant 0$;*

(5) *for each observation, $\sum_i x_t^i = E_t + X_t$ and $\sum_i q_t^i = q_t$.*

Proof. The proof is a straightforward application of Cherchye et al. (2011b). It suffices to show that conditions 1, 2 and 3, together, are equivalent to the existence of a continuous, quasiconcave and monotone utility function, $u^i : \mathbb{R}_+^L \times \mathbb{R}_+ \to \mathbb{R}$, such that, at each t, Eq. (1) holds true.[5] As in Snyder (1999), it suffices, thus, to show that the three conditions are equivalent to the requirement that the individual demand data $\{[(p_t, q_t^i), (x_t^i, K_t + Y_t), m_t^i] : t = 1, \ldots, T\}$ satisfy Varian's Generalized Axiom of Revealed Preference, GARP. In Lemma 1, which is in Appendix A, we restate and follow Cherchye et al. (2011b) in arguing that the latter is indeed the case. That lemma, hence, completes the proof. □

Proposition 1 is not novel and similar results have appeared in the working paper version of Cherchye et al. (2011b) and in Cherchye et al. (2011a) for testing collective consumption behavior. We include it here for completeness. It is important to note that the system introduced in the proposition is linear. Also, following Cherchye et al. (2015) and Talla-Nobibon et al. (2016) it is useful to note that, for the case of data sets with a large number of observations, the following modification of the system yields gains in terms of computational efficiency. Note that condition 3 in statement [B] involves triples of observations. For a similar problem, Cherchye et al. (2015) and Talla-Nobibon et al. (2016) propose the following equivalent system: dispense with condition 3, and require instead the existence of numbers $\upsilon_t^i \in [0, 1)$ such that

$$\upsilon_t^i - \upsilon_s^i < V_{t,s}^i \text{ and } V_{t,s}^i - 1 \leqslant \upsilon_t^i - \upsilon_s^i,$$

for each t and s, $t \neq s$. This system contains two more variables but only requires computations for pairs of observations, which is computationally less demanding when the number of observations is large.

3. Testing Nash–Walras equilibrium

The case of competitive equilibrium is more complicated. The following result, which is analogous to what was done for the hypothesis of efficiency, rids the problem of a first non-linearity.

Proposition 2. *Given a data set $\mathcal{D}$, define, for each individual, $C^i = \max_t \left\{m_t^i\right\} + 1$. The following two statements are equivalent:*

[A] *$\mathcal{D}$ is Nash–Walras rationalizable.*

[B] *There exist, for each individual: for each observation, a private bundle, $x_t^i \in \mathbb{R}_+^L$, and a level of public good, $y_t^i \in \mathbb{R}_+$; and for each pair of distinct observations, a (binary) number $W_{t,s}^i \in \{0, 1\}$, such that the following conditions are satisfied:*

(1) *for each observation, $p_t \cdot x_t^i + q_t \cdot y_t^i = m_t^i$;*

(2) *for each pair of distinct observations,*

$$p_t \cdot (x_t^i - x_s^i) - q_t \cdot \max\left\{\sum_j (y_s^j - y_t^j), -y_t^i\right\} < W_{t,s}^i \cdot C^i$$

and

$$p_t \cdot (x_t^i - x_s^i) - q_t \cdot \max\left\{\sum_j (y_s^j - y_t^j), -y_t^i\right\} \leqslant (1 - W_{s,t}^i) \cdot C^i;$$

(3) *for each triple of distinct observations, $W_{t,s}^i + W_{s,r}^i \leqslant 1 + W_{t,r}^i$;*

(4) *for each pair of observations, $p_t \cdot X_t + q_t \cdot Y_t = 0$ and $p_t \cdot X_s + q_t \cdot Y_s \leqslant 0$;*

(5) *for each observation, $\sum_i x_t^i = E_t + X_t$ and $\sum_i y_t^i = K_t + Y_t$.*

Proof. The logic of the argument is the same as before. We need to show that conditions 1, 2 and 3, together, are equivalent to the existence of a continuous and monotone utility function, $u^i : \mathbb{R}_+^L \times \mathbb{R}_+ \to \mathbb{R}$ such that Eq. (2) holds true at every t.

We first claim that the existence of such a utility function is equivalent to the requirement that the individual demand data $\{[(p_t, q_t), (x_t^i, y_t^i, y_t^{-i}), m_t^i] : t = 1, \ldots, T\}$, where $y_t^{-i} = \sum_{j \neq i} y_t^j$, satisfies the following generalization of GARP to non-linear budget sets, due to Forges and Minelli (2009): first, define the function

$$g_t^i(x, y) = \max\left\{p_t \cdot x + q_t \cdot y - m_t^i - q_t \cdot y_t^{-i}, p_t \cdot x - m_t^i\right\}; \quad (3)$$

then, the axiom requires that for any finite sequence $(t_n)_{n=1}^N$ of indices in $\{1, \ldots, T\}$,

$$\left[g_{t_n}^i(x_{t_{n+1}}^i, y_{t_{n+1}}^i + y_{t_{n+1}}^{-i}) \leqslant 0, \forall n = 1, \ldots, N-1\right] \Rightarrow g_{t_N}^i(x_{t_1}^i, y_{t_1}^i + y_{t_1}^{-i}) \geqslant 0. \quad (4)$$

[4] Following Carvajal (2005), it is not difficult to extend these results to other assumptions on the production technology.

[5] It is well known that the Axiom of Profit Maximization, condition 4, is equivalent to the existence of a cone Γ such that $(X_t, Y_t) \in \operatorname{argmax}_{(x,y)}\{p_t \cdot x + q_t \cdot y : (x, y) \in \Gamma\}$ at each t.

To see that this is the case, note that, as in the proof of Lemma 2 in Carvajal (2010), we can re-write the individual rationality condition of Nash–Walras rationalizability as the requirement that each $(x_t^i, y_t^i + y_t^{-i})$ solve the program

$$\max_{x,y} \left\{ u^i(x,y) : p_t \cdot x + q_t \cdot y \leqslant m_t^i + q_t \cdot y_t^{-i} \text{ and } y \geqslant y_t^{-i} \right\}.$$

Since u^i is monotone, Eq. (2) is, hence, equivalent to the requirement that $(x_t^i, y_t^i + y_t^{-i})$ solve[6]

$$\max_{x,y} \left\{ u^i(x,y) : g_t^i(x,y) \leqslant 0 \right\}.$$

Since each function g_t is continuous and increasing, and satisfies that $g_t(x_t^i, y_t^i + y_t^{-i}) = 0$, it follows from Proposition 3 in Forges and Minelli (2009) that there exists u^i such that

$$(x_t^i, y_t^i) \in \operatorname*{argmax}_{(x,y)} \left\{ u^i \left(x, y + \sum_{j \neq i} y_t^j \right) : p_t \cdot x + q_t \cdot y \leqslant m_t^i \right\}$$

for each t, as required by Nash–Walras rationalizability, if and only if the generalization of GARP stated above holds true.

Now, using Lemma 2 of Carvajal (2010), to conclude the argument we only need to show that such generalization of GARP is in fact equivalent to the first three conditions. In Lemma 2, we generalize the result of Cherchye et al. (2011b) to argue again that the latter is indeed the case. This lemma is presented in Appendix A. □

As in the case of Proposition 1, the system of statement [B] can be simplified, following Cherchye et al. (2015) and Talla-Nobibon et al. (2016), by replacing the third condition for one that does not involve triples of observations. Unlike that proposition, however, Proposition 2 does not yield a linear system, given the presence of the maximum operator used on the left-hand sides of the inequalities in the second condition. The following proposition now presents a linear MIP algorithm for testing the hypothesis of Nash–Walras equilibrium.

Proposition 3. *Given a data set $\mathcal{D}$, define C^i, for each i, as in Proposition 2. The following two statements are equivalent:*

[A] *$\mathcal{D}$ is Nash–Walras rationalizable.*

[B] *There exist, for each individual: for each observation, a private bundle, $x_t^i \in \mathbb{R}_+^L$, and a level of public good, $y_t^i \in \mathbb{R}_+$; and for each pair of distinct observations, (binary) numbers $W_{t,s}^i, d_{t,s}^i, e_{t,s}^i \in \{0, 1\}$; such that the following conditions are satisfied:*

(1) *for each observation, $p_t \cdot x_t^i + q_t \cdot y_t^i = m_t^i$;*

(2.a) *for each pair of distinct observations,*

$$p_t \cdot (x_s^i - x_t^i) + q_t \cdot \sum_j (y_s^j - y_t^j) + \left[W_{t,s}^i - 2(d_{t,s}^i - 1) \right] \cdot C^i > 0$$

and

$$p_t \cdot (x_s^i - x_t^i) + q_t \cdot \sum_j (y_s^j - y_t^j) + \left[1 - W_{s,t}^i - 2(d_{t,s}^i - 1) \right] \cdot C^i \geqslant 0;$$

(2.b) *for each pair of distinct observations,*

$$p_t \cdot (x_s^i - x_t^i) + q_t \cdot y_t^i + \left[W_{t,s}^i - 2(e_{t,s}^i - 1) \right] \cdot C^i > 0$$

and

$$p_t \cdot (x_s^i - x_t^i) + q_t \cdot y_t^i + \left[1 - W_{s,t}^i - 2(e_{t,s}^i - 1) \right] \cdot C^i \geqslant 0$$

(2.c) *for each pair of distinct observations, $d_{t,s}^i + e_{t,s}^i = 1$;*

(3) *for each triple of distinct observations, $W_{t,s}^i + W_{s,r}^i \leqslant 1 + W_{t,r}^i$;*

(4) *for each pair of observations, $p_t \cdot X_t + q_t \cdot Y_t = 0$ and $p_t \cdot X_s + q_t \cdot Y_s \leqslant 0$;*

(5) *for each observation, $\sum_i x_t^i = E_t + X_t$ and $\sum_i y_t^i = K_t + Y_t$.*

Proof. It suffices to show that condition 2 in Proposition 2 is equivalent to the existence of numbers $d_{t,s}^i, e_{t,s}^i \in \{0, 1\}$ such that conditions 2.a, 2.b and 2.c hold true. In order to simplify notation, write $\min \{W_{t,s}^i, 1 - W_{s,t}^i\} = w_{t,s}^i \in \{0, 1\}$.

If $w_{t,s}^i = 1 - W_{s,t}^i$, we can write condition 2 in Proposition 2 as the requirement the largest of the following two numbers be non-negative:

$$p_t \cdot (x_s^i - x_t^i) + q_t \cdot \sum_j (y_s^j - y_t^j) + w_{t,s}^i \cdot C^i \qquad (*)$$

and

$$p_t \cdot (x_s^i - x_t^i) - q_t \cdot y_t^i + w_{t,s}^i \cdot C^i. \qquad (**)$$

Note now that, by construction, $p_t \cdot (x_s^i - x_t^i) + q_t \cdot \sum_j (y_s^j - y_t^j) + w_{t,s}^i \cdot C^i \geqslant -2C^i$ and $p_t \cdot (x_s^i - x_t^i) - q_t \cdot y_t^i + w_{t,s}^i \cdot C^i \geqslant -2C^i$, so we can invoke Lemma 3, which appears in Appendix A, to conclude the proof.

If, on the other hand, $w_{t,s}^i = W_{t,s}^i$, the argument is identical except for the fact that Condition (2) in Proposition 2 requires that both of the numbers $(*)$ and $(**)$ be strictly positive. □

Importantly, it follows from Lemma 2 in Carvajal (2010) that for any individual for whom $y_t^i \geqslant \sum_j (y_t^j - y_s^j)$ at all pairs of distinct observations, her utility function can be constructed so that it displays quasiconcavity.

4. Independence of the two hypotheses

The following examples show that the hypotheses of Pareto efficiency and competitive equilibrium are independent: they show data that can be rationalized by one and only one of the two models.

Example 1 (*Data that cannot be Pareto Rationalizable, but can be Nash–Walras Rationalizable*)**.** There are two individuals, one private commodity and one public good. There are two observations, where the price of the private commodity is 2 and 1/2, respectively, while the price of the public good is the same at both, equal to unity. Aggregate consumption of the private commodity is 10 at the first observation, and 2 at the second. Aggregate supply of the public good is 1 and 9, respectively. The nominal income of the two individuals at the first observation is the same at 10 1/2, while at the second observation it is the same at 5.

In our notation, this is:

Variable	$t = 1$	$t = 2$
p_t	2	1/2
q_t	1	1
$x_t^1 + x_t^2$	10	2
$K_t + Y_t$	1	9
$m_t = (m_t^1, m_t^2)$	(10 1/2, 10 1/2)	(5, 5)

In Appendix B we show that these data cannot be rationalized as Pareto efficient, since it violates WARP condition required by Lindahl equilibrium. However, we show that the data can be rationalized as Nash–Walras equilibrium.

[6] Here, monotonicity implies that, at the optimum, $g_t(x_t^i, y_t^i + y_t^{-i}) = 0$. If this equality holds true with $p_t \cdot x_t^i - m_t^i = 0$, it is immediate that $y_t^i = y_t^{-i}$, as $q_t > 0$. Alternatively, suppose that $p_t \cdot x_t^i - m_t^i < 0$. Then, $p_t \cdot x_t^i + q_t \cdot y_t^i - m_t^i - q_t \cdot y_t^{-i} = 0$, in which case $q_t \cdot (y_t^{-i} - y_t^i) = p_t \cdot x_t^i - m_t^i < 0$. Again, $q_t > 0$ implies that $y_t^{-i} < y_t^i$.

Example 2 (*Data that cannot be Nash–Walras Rationalizable, but can be Pareto Rationalizable*). The context is the same as in Example 1, with the following observed data:

Variable	$t = 1$	$t = 2$
p_t	2	1/2
q_t	1	1
$x_t^1 + x_t^2$	8	4
$K_t + Y_t$	4	7 3/4
$m_t = (m_t^1, m_t^2)$	(6, 14)	(3 3/4, 6)

As verified in Appendix B, this data set cannot be Nash–Walras rationalizable, but it can be explained by a Lindahl equilibrium, so it is Pareto rationalizable.

5. Two extensions

5.1. Unobserved price for the public commodity

If the public good is provided through a mechanism other than a market, a price need not be observed and the analyst might need to impute it. Suppose, for example, that what she observes is the public expenditure in the public good, M. Then, an observation would be of the form (E, K, m, X, Y, p, M), and the imputation for the price would be $q = M/(K + Y)$. Assuming that i's nominal income, m^i, is net of any taxes she pays, the condition required for consistency would then be that $\sum_i m^i = p \cdot (E + X)$.

For Pareto rationalizability, the definition given before has to be extended, as the individuals' income is used only for their private expenditure and not for any contributions to the public good. In order to do this, one only needs to change the first condition in the definition of consistency with Pareto efficiency, Eq. (1), to the requirement that

$$(x^i, K+Y) \in \underset{(x,y)}{\operatorname{argmax}} \left\{ u^i(x,y) : p \cdot x + q^i \cdot [y - (K+Y)] \leqslant m^i \right\}, \quad (5)$$

so that her consumption plan is still rational in the sense of Lindahl pricing, but with her budget constraint augmented to make the public expenditure affordable.

Extending the MIP test to this case is not very difficult: given a data set, define $B^i = \max_t \left\{ m_t^i + M_t \right\} + 1$. Pareto rationalizability (with each individual's utility function chosen to be quasiconcave) is equivalent to the existence of a solution to the following adaptation of the system in Proposition 1:

(1) $p_t \cdot x_t^i = m_t^i$;
(2) $m_t^i + q_t^i \cdot (K_t + Y_t) - \left[p_t \cdot x_s^i + q_t^i \cdot (K_s + Y_s) \right] < V_{t,s}^i \cdot B^i$, and
$$m_t^i + q_t^i \cdot (K_t + Y_t) - \left[p_t \cdot x_s^i + q_t^i \cdot (K_s + Y_s) \right] \leqslant (1 - V_{s,t}^i) \cdot B^i;$$
(3) $V_{t,s}^i + V_{s,r}^i \leqslant 1 + V_{t,r}^i$;
(4) $p_t \cdot X_t + (M_t \cdot Y_t)/(K_t + Y_t) = 0$ and $p_t \cdot X_s + (M_t \cdot Y_s)/(K_t + Y_t) \leqslant 0$;
(5) $\sum_i x_t^i = E_t + X_t$ and $\sum_i q_t^i \cdot (K_t + Y_t) = M_t$.

5.2. Public expenditure in markets

Of course, it may also occur that a government is funding part of the public good, even when there is an operational market for it and a price is observed. In such case, when the analyst has available information on both M and q, the imputation $q = M/(K+Y)$ is not necessary and may even be incorrect.

Suppose instead that $M/q < K+Y$, so that there must be private contributions to the public good provision. Then, understanding that the nominal incomes observed are all net of any taxes, and assuming that $\sum_i m_i + M = p \cdot (E + X) + q \cdot (K + Y)$, the analyst must extend the definition of Nash–Walras rationalizability in two respects: the definition of individual rationality must be written as

$$(x^i, y^i) \in \underset{(x,y)}{\operatorname{argmax}} \left\{ u^i \left(x, y + \sum_{j \neq i} y^j + \frac{M}{q} \right) : p \cdot x + q \cdot y \leqslant m^i \right\},$$

instead of Eq. (2); and the market clearing condition for the public good must be replaced by $\sum_i y^i + M/q = K + Y$.

As before, we can extend the MIP test of Proposition 3: given the data set, define $C^i = \max_t \left\{ m_t^i \right\} + 1$. Nash–Walras rationalizability is equivalent to the existence of a solution to the system that appears in statement [B] of Proposition 3, only with the terms $\sum_j (y_s^j - y_t^j)$ in condition 2.*a* replaced by $\sum_j (y_s^j + M_s/q_s - y_t^j - M_t/q_t)$ and the last condition replaced by the requirement that $\sum_i x_t^i = E_t + X_t$ and $\sum_i y_t^i + M_t/q_t = K_t + Y_t$.

6. Warm-glow effects

Andreoni (1989, 1990) proposes that consumers may be *impurely altruistic*: they do not consider their contribution of public goods as a perfect substitute for other consumers' contributions. Following this "warm-glow" motivation, suppose that individual i's preference can be represented by a utility function $u^i : \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, so that

$$u^i = u^i \left(x^i, y^i, y^i + \sum_{j \neq i} y^j \right). \quad (6)$$

In words, consumer i's demand of the public commodity enters her utility function both as a private good and as part of the provision of the public good.

For the purposes of testing our efficiency and equilibrium hypotheses under this model, we assume that individual contributions to the provision of the public good are observable, so that the data set is augmented to

$$\mathcal{D} = \{(E_t, K_t, m_t, y_t, X_t, Y_t, p_t, q_t) : t = 1, \ldots, T\},$$

where $y_t = (y_t^1, \ldots, y_t^I)$ is the profile of individual contributions at observation t, which satisfies that $\sum_i y_t^i = K_t + Y_t$ and $q_t \cdot y_t^i \leqslant m_t^i$, for all t and all i.

6.1. Efficiency

Based on Allouch (2013), we use the following definition: given warm-glow economy $\{u, \Gamma\}$, an observation *is consistent with Pareto efficiency* if there exist a private bundle, $x^i \in \mathbb{R}_+^L$, and two Lindahl prices, $q^i, Q^i \in \mathbb{R}_+$, for each individual, such that:

1. individual consumption plans are rational in the sense of Lindahl equilibrium:
$$(x^i, y^i, K + Y - y^i) \in \underset{(x,y,\bar{y})}{\operatorname{argmax}} \left\{ u^i(x, y, y + \bar{y}) : p \cdot x + q^i \cdot y + Q^i \cdot \bar{y} \leqslant m^i \right\}; \quad (7)$$
2. for the firm, its production plan is rational:
$$(X, Y) \in \underset{(x,y)}{\operatorname{argmax}} \{ p \cdot x + q \cdot y : (x, y) \in \Gamma \};$$
3. private markets clear: $\sum_i x^i = E + X$; and
4. the public good is fully funded by the Lindahl pricing mechanism: for each individual i, $q^i + \sum_{j \neq i} Q^j = q$.

As before, we shall say that a data set is *Pareto rationalized* by a warm-glow economy $\{u, \Gamma\}$ if every observation is consistent with Pareto efficiency, given economy $\{u, \Gamma\}$.

Proposition 4. *Given data set $\mathcal{D}$, define B^i, for each individual, as in Proposition 1. The following two statements are equivalent:*

[A] *$\mathcal{D}$ is Pareto rationalizable by a warm-glow economy, and each individual's utility function can be chosen to be quasiconcave.*

[B] *There exist, for each individual: for each observation, a private bundle, $x_t^i \in \mathbb{R}_+^L$, and two Lindahl prices, $q_t^i, Q_t^i \in \mathbb{R}_+$, and for each pair of distinct observations a (binary) number $V_{t,s}^i \in \{0, 1\}$, such that the following conditions are satisfied:*

(1) *for each observation, $p_t \cdot x_t^i + q_t^i \cdot y_t^i + Q_t^i \cdot (K_t + Y_t - y_t^i) = m_t^i$;*

(2) *for each pair of distinct observations,*

$$m_t^i - \left[p_t \cdot x_s^i + q_t^i \cdot y_s^i + Q_t^i \cdot (K_s + Y_s - y_s^i)\right] < V_{t,s}^i \cdot B^i$$

and

$$m_t^i - \left[p_t \cdot x_s^i + q_t^i \cdot y_s^i + Q_t^i \cdot (K_s + Y_s - y_s^i)\right] \leqslant (1 - V_{s,t}^i) \cdot B^i;$$

(3) *for each triple of distinct observations, $V_{t,s}^i + V_{s,r}^i \leqslant 1 + V_{t,r}^i$;*

(4) *for each pair of observations, $p_t \cdot X_t + q_t \cdot Y_t = 0$ and $p_t \cdot X_s + q_t \cdot Y_s \leqslant 0$;*

(5) *for each observation, $\sum_i x_t^i = E_t + X_t$ and $q_t^i + \sum_{j \neq i} Q_t^j = q_t$.*

Proof. The logic of this result is the same as in Proposition 1, so a detailed argument can be omitted. □

Importantly, the introduction of warm-glow effects weakens the testable implications of Pareto efficiency, strictly, but it does not make the hypothesis of Pareto efficiency unfalsifiable. This follows from the next corollary and examples.

Corollary 1. *Any data set that is Pareto rationalizable in an economy without warm-glow effects is Pareto rationalizable in one with warm-glow effects.*

Proof. By necessity of statement [B] in Proposition 1, one can find a solution to the system of that statement. Letting $Q_t^i = q_t^i$, one obtains a solution to the system in statement [B] of Proposition 4. This suffices for statement [A] of that proposition. □

Example 3 (*Data that cannot be Pareto Rationalizable by a warm-glow economy*)**.** The context is the same as in Example 1, with the following observed data:[7]

Variable	$t = 1$	$t = 2$
p_t	2	1/2
q_t	1	1
$x_t^1 + x_t^2$	7 1/2	1
$K_t + Y_t$	2	6
$m_t = (m_t^1, m_t^2)$	(8, 7)	(1/4, 1/4)

In Appendix B we show that if the data shows an invariant contribution of public good by individual 1, $y_1^1 = y_2^1$, it cannot be Pareto rationalizable by a warm-glow economy, as the WARP system of Proposition 4 would not have a solution.

Example 4 (*Data that can be Pareto Rationalizable with warm-glow effects, but not without them*)**.** Consider again the setting of Example 1, with the following observed data:

Variable	$t = 1$	$t = 2$
p_t	2	1/2
q_t	1	1
$x_t^1 + x_t^2$	10	1
$K_t + Y_t$	2	8
$m_t = (m_t^1, m_t^2)$	(8, 12)	(1/4, 1/4)

As verified in Appendix B, this data set cannot be rationalized as Pareto efficient without warm-glow effects since it violates the WARP condition required by its Lindahl equilibrium. On the other hand, we show also that these data can be rationalized by a Lindahl equilibrium with warm-glow effects.

6.2. Competitive equilibrium

Given a warm-glow economy $\{u, \Gamma\}$, we say that an observation *is consistent with Nash–Walras equilibrium* if there exist individual private bundles, $x^i \in \mathbb{R}_+^L$, such that:

1. for each individual, her consumption plan is rational in the sense of Nash–Walras competitive behavior:

$$(x^i, y^i) \in \underset{(x,y)}{\operatorname{argmax}} \left\{ u^i\left(x, y, y + \sum_{j \neq i} y^j\right) : p \cdot x + q \cdot y \leqslant m^i \right\}; \quad (8)$$

2. for the firm, its production plan is rational:

$$(X, Y) \in \underset{(x,y)}{\operatorname{argmax}} \{p \cdot x + q \cdot y : (x, y) \in \Gamma\};$$

3. private markets clear: $\sum_i x^i = E + X$; and,
4. the public good market clears: $\sum_i y^i = K + Y$.

And, as before, we say that a data set is *Nash–Walras rationalized* by a warm-glow economy $\{u, \Gamma\}$ if every observation is consistent with Nash–Walras equilibrium given $\{u, \Gamma\}$.

Proposition 5. *Given a data set $\mathcal{D}$, define C^i, for each i, as in Proposition 2. The following two statements are equivalent:*

[A] *$\mathcal{D}$ is Nash–Walras rationalizable by a warm-glow economy.*

[B] *There exist, for each individual: a private bundle, $x_t^i \in \mathbb{R}_+^L$, two shadow prices for the public good, $q_t^i, Q_t^i \in \mathbb{R}_+$; and (binary) numbers $W_{t,s}^i \in \{0, 1\}$, such that the following conditions are satisfied:*

(1) *for each observation, $p_t \cdot x_t^i + q_t \cdot y_t^i = m_t^i$ and $q_t^i + Q_t^i = q_t$;*

(2) *for each pair of distinct observations,*

$$p_t \cdot (x_t^i - x_s^i) + q_t^i \cdot (y_t^i - y_s^i) - Q_t^i \cdot \max\left\{\sum_j (y_s^j - y_t^j), -y_t^i\right\} < W_{t,s}^i \cdot C^i$$

and

$$p_t \cdot (x_t^i - x_s^i) + q_t^i \cdot (y_t^i - y_s^i) - Q_t^i \cdot \max\left\{\sum_j (y_s^j - y_t^j), -y_t^i\right\} \leqslant (1 - W_{s,t}^i) \cdot C^i;$$

(3) *for each triple of distinct observations, $W_{t,s}^i + W_{s,r}^i \leqslant 1 + W_{t,r}^i$;*

(4) *for each pair of observations, $p_t \cdot X_t + q_t \cdot Y_t = 0$ and $p_t \cdot X_s + q_t \cdot Y_s \leqslant 0$;*

[7] In Examples 3 and 4, the nominal incomes are assumed to be expenditure on the private goods, while in Examples 3 to 6 individual contributions of the public goods are assumed to be observable. Both assumptions make the computations more straightforward, but are not necessary for Pareto efficiency and competitive equilibrium to be testable.

(5) *for each observation,* $\sum_i x_t^i = E_t + X_t$.[8]

Proof. The logic of this result is the same as in Proposition 2, once one has a suitable extension of the revealed preference results for the case of warm-glow preferences, which we provide in Lemma 4. □

The maximum operator in Proposition 5 can be eliminated as in Proposition 3, and the result is not repeated here. As with Nash–Walras without warm-glow effects, for any individual for whom one has $y_t^i \geqslant \sum_j (y_t^j - y_s^j)$ at all pairs of distinct observations, her utility function can be constructed so that it displays quasiconcavity.

Importantly, it is again true that the introduction of warm-glow effects weakens the testable implications of Nash–Walras equilibrium, but does not make the hypothesis of competitive equilibrium unfalsifiable.

Corollary 2. *Any data set that is Nash–Walras rationalizable in an economy without warm-glow effects is Nash–Walras rationalizable in one with warm-glow effects.*

Proof. As in Corollary 1, it suffices to show that existence of a solution to statement [B] in Proposition 2 implies existence of a solution to the system of statement [B] in Proposition 5. In this case, one just needs to define $q_t^i = 0$ and $Q_t^i = q_t$. □

Example 5 (*Data that cannot be Nash–Walras Rationalizable by a warm-glow economy*). The context is the same as in Example 1, with the observed information as follows:

Variable	$t = 1$	$t = 2$
p_t	2	1/2
q_t	1	1
$x_t^1 + x_t^2$	11	4
$K_t + Y_t$	2	8
$m_t = (m_t^1, m_t^2)$	(12, 12)	(3 1/2, 6 1/2)

As verified in Appendix B, the system of Proposition 5 cannot have a solution, and hence the data cannot be Nash–Walras rationalizable by a warm-glow economy.

Example 6 (*Data that can be Nash–Walras Rationalizable with warm-glow effects, but not without them*). Finally, suppose that the observed data include:

Variable	$t = 1$	$t = 2$
p_t	2	1/2
q_t	1	1
$x_t^1 + x_t^2$	12	4
$K_t + Y_t$	2	8
$m_t = (m_t^1, m_t^2)$	(22, 4)	(6 1/2, 3 1/2)

As verified in Appendix B, this data set cannot be rationalized as Nash–Walras equilibrium without warm-glow effects since it violates the required WARP condition. On the other hand, the data can be rationalized by a Nash–Walras equilibrium with warm-glow effects, as the system of Proposition 5 does have a solution.

7. Performance of the tests

We now evaluate the performance of our tests using Monte Carlo simulations.[9] First, Table 1 gives the time needed to implement the tests of Pareto efficiency and Nash–Walras equilibrium for different number of observations, agents, and private commodities: in each cell, the first number is for the test without warm-glow effects, the second number is for the test with them. As the table shows, the MIP method can accommodate multiple observations, agents and commodities, and the test can be implemented in reasonable time.

Table 1
Computation times, in seconds.

		Pareto efficiency		N-W equilibrium	
		$L = 1$	$L = 3$	$L = 1$	$L = 3$
$T = 3$	$I = 2$	(8, 18)	(8, 18)	(17, 23)	(17, 23)
	$I = 3$	(8, 18)	(8, 18)	(17, 23)	(17, 23)
	$I = 4$	(8, 18)	(8, 18)	(17, 23)	(17, 23)
$T = 6$	$I = 2$	(8, 18)	(9, 18)	(17, 23)	(18, 23)
	$I = 3$	(9, 18)	(9, 19)	(18, 23)	(22, 24)
	$I = 4$	(9, 19)	(15, 21)	(21, 23)	(33, 24)

Table 2
Power of the tests, for $I = 2$ & $L = 1$.

	Pareto efficiency	N-W equilibrium
$T = 3$	(0.003, 0.036, 0.253)	(0, 0.016, 0.102)
$T = 4$	(0.004, 0.079, 0.381)	(0.002, 0.047, 0.194)
$T = 5$	(0.004, 0.125, 0.560)	(0.003, 0.112, 0.548)

Without public goods, the test of Walras equilibrium has been implemented by Cherchye et al. (2011b).[10] Although the data passes their test, they also observe that the power of that test is null.[11] They show, however, that if individual consumption can be (at least partially) observed, then the power can be reasonably high. The scenario they consider is that the lower bound of individual consumption is known, i.e., $x_t^i \geqslant \kappa \hat{x}_t^i$, where $\kappa \in [0, 1]$, and $\hat{x}_t^i$ is the true consumption level.

In the following, we compare the power of our two MIP tests, without warm-glow effects, using Monte Carlo simulations based on the procedure in Bronars (1987). We draw price data from the uniform distribution $U[50, 100]$, and individual income data from $U[10000, 11000]$, to generate high price variation and low income variation. Then, we randomly choose a share s_t^i from $U[0, 1]$ and construct "artificial" variables

$$\hat{x}_t^i = \frac{s_t^i m_t^i}{p_t} \text{ and } \hat{y}_t^i = \frac{(1 - s_t^i) m_t^i}{q_t}.$$

Finally, we construct the aggregate endowments $E_t = \sum_i \hat{x}_t^i$ and $K_t = \sum_i \hat{y}_t^i$, guaranteeing consistency between nominal and real variables in the simulations.

The MIP tests are implemented one thousand times for each hypothesis. The pure test of Propositions 1 and 3 implies that there is absolutely no information on the individual consumption of commodities, in the spirit of Brown and Matzkin (1996). Following the ideas of Cherchye et al. (2011b), we then run the tests again, imposing the constraints that $x_t^i \geqslant \kappa \hat{x}_t^i$, considering two cases: "partial observation", where $\kappa = 0.9$; and "full observation", where $\kappa = 1$. As in Bronars (1987), the alternative hypothesis is that individuals exhaust their budgets randomly.

Table 2 records the power of the tests: in each cell, the first number is the power when there is no information on individual private consumption, the second number is the power under partial observation, and the third number is the power under full observation. As in the case of pure competitive equilibrium first

---

8 That the public market clears too, $\sum_i y_t^i = K_t + Y_t$, is assumed to be observed in the data.

9 All the simulations in this section were implemented using the Rglpk package in R on a PC with Intel CPU at 3.30 GHz and RAM of 4.00 GB, namely a pretty average computer, at best.

10 They assume that all agents within the same US region are the same type of tastes and incomes, and use regional data from the US economy (8 regions, 12 annual observations, and 18 commodities) to implement their test.

11 Cherchye et al. (2011b) attribute the low test power to low price variation and high income increase across observations, and conjecture that their test "may effectively have reasonable power if the data show sufficient price variation together with low income variation".

proposed by Brown and Matzkin (1996), the power of our tests is very low when no information of individual consumption is known. However, knowing some lower bounds for individual consumption will significantly increase the power, especially when the number of observations is large.[12]

Note that even the full observation of individual consumption does not make our tests vacuous. Without public goods as in Cherchye et al. (2011b), GARP or Afriat inequalities can be checked efficiently when individual consumption can be fully observed, and the MIP method does not have any advantage. Nevertheless, when there are public goods (with or without warm-glow effects), the personalized prices cannot be observed and the difficulty of nonlinearity does not disappear. In this case, our tests are easy to implement and our simulation exercises suggest that they are suitable for examining efficiency and competition in economies with public goods. We see our tests as complementary to other methods used in applied work.

Appendix A. Lemmata

Lemma 1 (*Cherchye et al., 2011b*). *Consider a finite set* $\{(\pi_t, \chi_t) : t = 1, \dots, T\} \subseteq \mathbb{R}^{\lambda}_{+} \times \mathbb{R}^{\lambda}_{+}$.[13] *The following statements are equivalent:*

1. *The set satisfies GARP: for any finite sequence* $(t_n)_{n=1}^N$ *of indices in* $\{1, \dots, T\}$,
$$(\pi_{t_n} \cdot \chi_{t_{n+1}} \leqslant \pi_{t_n} \cdot \chi_{t_n}, \forall n = 1, \dots, N-1) \Rightarrow \pi_{t_N} \cdot \chi_{t_1} \geqslant \pi_{t_N} \cdot \chi_{t_N}.$$
2. *There exists a solution,* $\{\upsilon_{t,s} \in \{0,1\} \mid t, s = 1, \dots, T; t \neq s\}$, *to Cherchye et al.'s Integer Consumer System, CS.I: letting* $\beta = \max_t \{\pi_t \cdot \chi_t\} + 1$,
   (a) *for each pair of distinct observations,*
$$\pi_t \cdot \chi_t - \pi_t \cdot \chi_s < \upsilon_{t,s} \cdot \beta \text{ and}$$
$$\pi_t \cdot \chi_t - \pi_t \cdot \chi_s \leqslant (1 - \upsilon_{s,t}) \cdot \beta;$$
   (b) *for each triple of distinct observations,* $\upsilon_{t,s} + \upsilon_{s,r} \leqslant 1 + \upsilon_{t,r}$.

Proof. See Cherchye et al. (2011b). □

Lemma 2. *Consider a finite set*
$$\{(p_t, q_t, x_t, y_t, \bar{y}_t, m_t) \in \mathbb{R}^L_+ \times \mathbb{R}_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ : t = 1, \dots, T\}.$$
The following statements are equivalent:

1. *The set satisfies the Forges–Minelli generalized version of GARP: for any finite sequence* $(t_n)_{n=1}^N$ *of indices in* $\{1, \dots, T\}$,[14]
$$\left[g_{t_n}(x_{t_{n+1}}, y_{t_{n+1}} + \bar{y}_{t_{n+1}}) \leqslant 0, \forall n = 1, \dots, N-1\right] \Rightarrow g_{t_N}(x_{t_1}, y_{t_1} + \bar{y}_{t_1}) \geqslant 0, \tag{9}$$
   where[15]
$$g_t(x, y) = \max\{p_t \cdot x + q_t \cdot y - m_t - q_t \cdot \bar{y}_t, p_t \cdot x - m_t\}; \tag{10}$$
2. *There exists a solution,* $\left\{\omega_{t,s} \in \{0,1\} \mid t, s = 1, \dots, T; t \neq s\right\}$, *to the following generalization of Cherchye et al.'s Integer Consumer System, CS.I: letting*
$$c = \max_t \{p_t \cdot x_t + q_t \cdot (y_t + \bar{y}_t)\} + 1,$$
   (a) *for each pair of distinct observations,*
$$p_t \cdot (x_t - x_s) + q_t \cdot [(y_t + \bar{y}_t) - \max\{y_s + \bar{y}_s, \bar{y}_t\}] < \omega_{t,s} \cdot c$$
   and
$$p_t \cdot (x_t - x_s) + q_t \cdot [(y_t + \bar{y}_t) - \max\{y_s + \bar{y}_s, \bar{y}_t\}] \leqslant (1 - \omega_{s,t}) \cdot c;$$
   (b) *for each triple of distinct observations,* $\omega_{t,s} + \omega_{s,r} \leqslant 1 + \omega_{t,r}$.

Proof. We first argue that statement 1 implies statement 2, by constructing a solution to the system introduced in the latter. Say that $\omega_{t,s} = 1$ if there exists a finite sequence $(t_n)_{n=1}^N$ such that $t_1 = t, t_N = s$ and $g_{t_n}(x_{t_{n+1}}, y_{t_{n+1}} + \bar{y}_{t_{n+1}}) \leqslant 0$. Otherwise, let $\omega_{t,s} = 0$.

First, suppose that $g_t(x_s, y_s + \bar{y}_s) \leqslant 0$. Then, $\omega_{t,s} = 1$ and, therefore,
$$\omega_{t,s} \cdot c \geqslant p_t \cdot x_t + q_t \cdot (y_t + \bar{y}_t) \geqslant p_t \cdot x_t + q_t \cdot (y_t + \bar{y}_t) - p_t \cdot x_s - q_t \cdot \max\{y_s + \bar{y}_s, \bar{y}_t\}.$$
If, on the other hand, $g_t(x_s, y_s + \bar{y}_s) > 0$, then
$$\begin{aligned} 0 &< \max\{p_t \cdot x_s + q_t \cdot (y_s + \bar{y}_s) - m_t - q_t \cdot \bar{y}_t, p_t \cdot x_s - m_t\} \\ &= p_t \cdot x_s - m_t + q_t \cdot \max\{y_s + \bar{y}_s - \bar{y}_t, 0\} \\ &= p_t \cdot x_s - (p_t \cdot x_t + q_t \cdot y_t) + q_t \cdot \max\{y_s + \bar{y}_s - \bar{y}_t, 0\} \\ &= p_t \cdot (x_s - x_t) + q_t \cdot [\max\{y_s + \bar{y}_s - \bar{y}_t, 0\} - y_t] \\ &= p_t \cdot (x_s - x_t) + q_t \cdot [\max\{y_s + \bar{y}_s, \bar{y}_t\} - (y_t + \bar{y}_t)]. \end{aligned}$$
Together, these two inequalities imply that, in any case,
$$p_t \cdot (x_t - x_s) + q_t \cdot [(y_t + \bar{y}_t) - \max\{y_s + \bar{y}_s, \bar{y}_t\}] < \omega_{t,s} \cdot c. \tag{$*$}$$
Now, suppose that $\omega_{s,t} = 1$. By construction, there is a sequence $(t_n)_{n=1}^N$ for which $t_1 = s, t_N = t$ and $g_{t_n}(x_{t_{n+1}}, y_{t_{n+1}} + \bar{y}_{t_{n+1}}) \leqslant 0$. By our generalization of GARP, Eq. (9), then $g_t(x_s, y_s + \bar{y}_s) \geqslant 0$, which implies, right away, that
$$p_t \cdot (x_t - x_s) + q_t \cdot [(y_t + \bar{y}_t) - \max\{y_s + \bar{y}_s, \bar{y}_t\}] \leqslant (1 - \omega_{s,t}) \cdot c. \tag{$**$}$$
If, on the other hand, $\omega_{s,t} = 0$, the latter is immediate, by construction.

Eqs. $(*)$ and $(**)$ together imply condition (a) of statement 2. For condition (b), we show that if $\omega_{t,s} = \omega_{s,r} = 1$, then $\omega_{t,r} = 1$. For this, there exist finite sequences $(t_n)_{n=1}^N$ and $(s_m)_{m=1}^M$, such that $t_1 = t, t_N = s, s_1 = s, s_M = r$,
$$g_{t_n}(x_{t_{n+1}}, y_{t_{n+1}} + \bar{y}_{t_{n+1}}) \leqslant 0 \text{ and } g_{s_m}(x_{s_{m+1}}, y_{s_{m+1}} + \bar{y}_{s_{m+1}}) \leqslant 0.$$
Concatenating these sequences as $(r_l)_{l=1}^{N+M} = (t_1, t_s, \dots, t_N, s_1, s_s, \dots, s_M)$, we get that $\omega_{t,r} = 1$, by definition.

To show that statement 2 implies statement 1, let $(\omega_{t,s})_{t \neq s}$ satisfy conditions (a) and (b) in statement 1, and let finite sequence $(t_n)_{n=1}^N$ be such that for all $n = 1, \dots, N-1$, $g_{t_n}(x_{t_{n+1}}, y_{t_{n+1}} + \bar{y}_{t_{n+1}}) \leqslant 0$. Then, for each $n \leqslant N-1$,
$$\max\left\{p_{t_n} \cdot x_{t_{n+1}} + q_{t_n} \cdot (y_{t_{n+1}} + \bar{y}_{t_{n+1}}) - m_{t_n} - q_{t_n} \cdot \bar{y}_{t_n}, p_{t_n} \cdot x_{t_{n+1}} - m_{t_n}\right\} \leqslant 0.$$

---
[12] In the simulation of both tests with warm-glow effects, not reported here, we get similar results: the power is low without information on individual private consumption, and reasonably high when (partially) observing individual private consumption.
[13] For the purpose of this lemma, the reader may assimilate $\pi_t = (p_t, q_t)$, $\chi_t = (x_t, y_t)$ and $\lambda = L + 1$.
[14] The following equation re-expresses Eq. (4) in general notation.
[15] Consistently, the following is a re-expression of Eq. (3).

This implies, by condition (a), that $\omega_{t_n,t_{n+1}} = 1$. Using condition (b) recursively, we further have that $\omega_{t_1,t_N} = 1$, which implies, by (a) again, that

$$p_{t_N} \cdot (x_{t_N} - x_{t_1}) + q_{t_N} \cdot \left[(y_{t_N} + \bar{y}_{t_N}) - \max\left\{y_{t_1} + \bar{y}_{t_1}, \bar{y}_{t_N}\right\}\right] \leqslant 0.$$

We can rewrite this as

$$p_{t_N} \cdot x_{t_1} + q_{t_N} \cdot \left[\max\left\{y_{t_1} + \bar{y}_{t_1}, \bar{y}_{t_N}\right\} - \bar{y}_{t_N}\right] - (p_{t_N} \cdot x_{t_N} + q_{t_N} \cdot y_{t_N}) \geqslant 0,$$

which implies that $g_{t_N}(x_{t_1}, y_{t_1} + \bar{y}_{t_1}) \geqslant 0$, since $p_{t_N} \cdot x_{t_N} + q_{t_N} \cdot y_{t_N} = m_{t_N}$. □

Lemma 3. *Suppose that $\alpha, \beta \geqslant \nu$. The following two statements are equivalent:*

1. *Equation $\max\{\alpha, \beta\} \geqslant 0$ holds true.*
2. *There exist binary numbers $\delta, \varepsilon \in \{0, 1\}$ such that $\alpha + (\delta - 1)\nu \geqslant 0$, $\beta + (\varepsilon - 1)\nu \geqslant 0$, and $\delta + \varepsilon = 1$.*

The equivalence with all the inequalities replaced by strict inequalities holds true as well.

Proof. To see that the first statement implies the second, note that either $\delta = 1$ or $\varepsilon = 1$. If the former is true, $\alpha \geqslant 0$; otherwise, $\beta \geqslant 0$.

For the other implication, suppose first that $\alpha \geqslant 0$. Let $\delta = 1$ and $\varepsilon = 0$, so that

$$\alpha + (\delta - 1)\nu = \alpha \geqslant 0 \text{ and } \beta + (\varepsilon - 1)\nu = \beta - \nu \geqslant 0.$$

Else, $\beta \geqslant 0$ and the result still holds true, with $\delta = 0$ and $\varepsilon = 1$.

If $\alpha, \beta > \nu$, the argument for existence of a solution to equation $\max\{\alpha, \beta\} > 0$ is identical with the requirement that $\alpha + (\delta - 1)\nu > 0$ and $\beta + (\varepsilon - 1)\nu > 0$. □

Lemma 4. *Consider a finite set*

$$\left\{(p_t, q_t, x_t, y_t, \bar{y}_t, m_t) \in \mathbb{R}^L_+ \times \mathbb{R}_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ : t = 1, \ldots, T\right\}.$$

The following statements are equivalent:

1. *There exists a continuous and monotone function $u : \mathbb{R}^L_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that, at all t,*
$$(x_t, y_t) \in \underset{(x,y)}{\operatorname{argmax}}\{u(x, y, y + \bar{y}_t) : p_t \cdot x + q_t \cdot y \leqslant m_t\}. \quad (11)$$
2. *There exist numbers $\alpha_t, \beta_t \in \mathbb{R}_+$ such that $\alpha_t + \beta_t = q_t$ and, for any finite sequence $(t_n)_{n=1}^N$ of indices in $\{1, \ldots, T\}$,*
$$\left[g_{t_n}(x_{t_{n+1}}, y_{t_{n+1}}, y_{t_{n+1}} + \bar{y}_{t_{n+1}}) \leqslant 0, \forall n = 1, \ldots, N-1\right] \Rightarrow g_{t_N}(x_{t_1}, y_{t_1}, y_{t_1} + \bar{y}_{t_1}) \geqslant 0,$$
where
$$g_t(x, y, Y) = \max\{p_t \cdot x + \alpha_t \cdot y + \beta_t \cdot Y - m_t - \beta_t \cdot \bar{y}_t, p_t \cdot x + \alpha_t \cdot y - m_t\};$$
3. *There exists a solution,*
$$\left\{(\alpha_t, \beta_t, \omega_{t,s}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \{0, 1\} \mid t, s = 1, \ldots, T; t \neq s\right\},$$
to the following system: letting $c = \max_t\{p_t \cdot x_t + q_t \cdot (y_t + \bar{y}_t)\} + 1$,
   (a) *for each pair of distinct observations,*
$$p_t \cdot (x_t - x_s) + \alpha_t \cdot (y_t - y_s) + \beta_t \cdot [(y_t + \bar{y}_t) - \max\{y_s + \bar{y}_s, \bar{y}_t\}] < \omega_{t,s} \cdot c$$
and
$$p_t \cdot (x_t - x_s) + \alpha_t \cdot (y_t - y_s) + \beta_t \cdot [(y_t + \bar{y}_t) - \max\{y_s + \bar{y}_s, \bar{y}_t\}] \leqslant (1 - \omega_{s,t}) \cdot c;$$
   (b) *for each triple of distinct observations, $\omega_{t,s} + \omega_{s,r} \leqslant 1 + \omega_{t,r}$.*

Proof. We first show that the first two statements are equivalent. To see this, first note that Eq. (11) is equivalent to the existence of numbers $\alpha_t, \beta_t \in \mathbb{R}_+$ such that $\alpha_t + \beta_t = q_t$ and

$$(x_t, y_t, y_t + \bar{y}_t) \in \underset{(x,y,z)}{\operatorname{argmax}}\{u(x, y, z) : p_t \cdot x + \alpha_t \cdot y + \beta_t \cdot z \leqslant m_t + \beta_t \cdot \bar{y}_t \text{ and } z \geqslant \bar{y}_t\}.$$

Now, given the array of numbers $\{(\alpha_t, \beta_t) : t = 1, \ldots, T\}$, the latter problem is that, for all t,

$$(x_t, y_t, y_t + \bar{y}_t) \in \underset{(x,y,z)}{\operatorname{argmax}}\{u(x, y, z) : g_t(x, y, z) \leqslant 0\},$$

so that, as in Carvajal (2010), the result follows from Forges and Minelli (2009). That the second and third statements are equivalent follows the same argument as Lemma 2. □

Appendix B. Computations

Example 1. To show the data set is not consistent with Lindahl equilibrium, note that individual rationality of individual 1 (WARP) requires that either

$$2x^1_1 + q^1_1 \leqslant 2x^1_2 + 9q^1_1, \quad (12)$$

or

$$\frac{1}{2}x^1_2 + 9q^1_2 \leqslant \frac{1}{2}x^1_1 + q^1_2; \quad (13)$$

while, for individual 2, either

$$2x^2_1 + q^2_1 \leqslant 2x^2_2 + 9q^2_1, \quad (14)$$

or

$$\frac{1}{2}x^2_2 + 9q^2_2 \leqslant \frac{1}{2}x^2_1 + q^2_2. \quad (15)$$

Note that neither of Eqs. (13) and (15) can hold, and Eqs. (12) and (14) cannot hold simultaneously. It thus follows that WARP condition of the Lindahl equilibrium cannot be satisfied.

Individual rationality of individual 1 in Nash–Walras equilibrium (WARP) requires that either

$$g^1_1(x^1_2, y^1_2) \geqslant 0 : p_1 x^1_1 + q_1 \sum_i y^i_1 \leqslant p_1 x^1_2 + q_1 \sum_i y^i_2 \text{ or } p_1 x^1_2 \geqslant m^1_1,$$

or

$$g^1_2(x^1_1, y^1_1) \geqslant 0 : p_2 x^1_2 + q_2 \sum_i y^i_2 \leqslant p_2 x^1_1 + q_2 \sum_i y^i_1 \text{ or } p_2 x^1_1 \geqslant m^1_2;$$

while, for individual 2, either

$$g^2_1(x^2_2, y^2_2) \geqslant 0 : p_1 x^2_1 + q_1 \sum_i y^i_1 \leqslant p_1 x^2_2 + q_1 \sum_i y^i_2 \text{ or } p_1 x^2_2 \geqslant m^2_1,$$

or

$$g^2_2(x^2_1, y^2_1) \geqslant 0 : p_2 x^2_2 + q_2 \sum_i y^i_2 \leqslant p_2 x^2_1 + q_2 \sum_i y^i_1 \text{ or } p_2 x^2_1 \geqslant m^2_2.$$

By straightforward algebra, these systems of inequalities become that, for individual 1, either

$$x^1_1 \leqslant x^1_2 + 4 \text{ or } x^1_2 \geqslant \frac{21}{4}, \quad (16)$$

or

$$x^1_1 \geqslant x^1_2 + 16 \text{ or } x^1_1 \geqslant 10; \quad (17)$$

while, for individual 2, either

$$x_1^2 \leqslant x_2^2 + 4 \text{ or } x_2^2 \geqslant \frac{21}{4}, \quad (18)$$

or

$$x_1^2 \geqslant x_2^2 + 16 \text{ or } x_1^2 \geqslant 10. \quad (19)$$

Then $x_1^1 = 5, x_1^2 = 5, x_2^1 = 1, x_2^2 = 1$ is a solution, satisfying the first parts of Eqs. (16) and (18).

Example 2. Individual rationality of individual 1 in Nash–Walras equilibrium (WARP) requires that either

$$g_1^1(x_2^1, y_2^1) \geqslant 0 : p_1x_1^1 + q_1 \sum_i y_1^i \leqslant p_1x_2^1 + q_1 \sum_i y_2^i \text{ or } p_1x_2^1 \geqslant m_1^1,$$

or

$$g_2^1(x_1^1, y_1^1) \geqslant 0 : p_2x_2^1 + q_2 \sum_i y_2^i \leqslant p_2x_1^1 + q_2 \sum_i y_1^i \text{ or } p_2x_1^1 \geqslant m_2^1;$$

while, for individual 2, either

$$g_1^2(x_2^2, y_2^2) \geqslant 0 : p_1x_1^2 + q_1 \sum_i y_1^i \leqslant p_1x_2^2 + q_1 \sum_i y_2^i \text{ or } p_1x_2^2 \geqslant m_1^2,$$

or

$$g_2^2(x_1^2, y_1^2) \geqslant 0 : p_2x_2^2 + q_2 \sum_i y_2^i \leqslant p_2x_1^2 + q_2 \sum_i y_1^i \text{ or } p_2x_1^2 \geqslant m_2^2.$$

By straightforward algebra, these systems of inequalities become that, for individual 1, either

$$x_1^1 \leqslant x_2^1 + \frac{15}{8} \text{ or } x_2^1 \geqslant 3, \quad (20)$$

or

$$x_1^1 \geqslant x_2^1 + \frac{15}{2} \text{ or } x_1^1 \geqslant \frac{15}{2}; \quad (21)$$

while, for individual 2, either

$$x_1^2 \leqslant x_2^2 + \frac{15}{8} \text{ or } x_2^2 \geqslant 7, \quad (22)$$

or

$$x_1^2 \geqslant x_2^2 + \frac{15}{2} \text{ or } x_1^2 \geqslant 12. \quad (23)$$

Note that neither Eq. (21) or Eq. (23) can hold, and Eqs. (20) and (22) cannot hold simultaneously. So, these data cannot be rationalized by a Nash–Walras equilibrium.

To show the data set is consistent with Lindahl equilibrium, note that individual rationality of individual 1 (WARP) requires that either

$$2x_1^1 + 4q_1^1 \leqslant 2x_2^1 + \frac{31}{4}q_1^1, \quad (24)$$

or

$$\frac{1}{2}x_2^1 + \frac{31}{4}q_2^1 \leqslant \frac{1}{2}x_1^1 + 4q_2^1; \quad (25)$$

while, for individual 2, either

$$2x_1^2 + 4q_1^2 \leqslant 2x_2^2 + \frac{31}{4}q_1^2, \quad (26)$$

or

$$\frac{1}{2}x_2^2 + \frac{31}{4}q_2^2 \leqslant \frac{1}{2}x_1^2 + 4q_2^2. \quad (27)$$

Simplifying these inequalities, $x_1^1 = 2, x_1^2 = 6, x_2^1 = \frac{15}{4}, x_2^2 = \frac{1}{4}, q_1^1 = \frac{1}{2}, q_2^2 = \frac{47}{62}$ would be a solution, satisfying Eqs. (24) and (27).

Example 3. Suppose $y_1^1 = a$, so that $y_1^2 = 2 - a$, while $y_2^1 = b$ and $y_2^2 = 6 - b$, where $a \in [0, 2]$ and $b \in [0, 6]$. For these data to be consistent with Pareto efficiency with warm-glow effects, the Lindahl condition for individual rationality of individual 1 (WARP) requires that either

$$2x_1^1 + q_1^1a + Q_1^1(2 - a) \leqslant 2x_2^1 + q_1^1b + Q_1^1(6 - b),$$

or

$$\frac{1}{2}x_2^1 + q_2^1b + Q_2^1(6 - b) \leqslant \frac{1}{2}x_1^1 + q_2^1a + Q_2^1(2 - a).$$

For individual 2, by the same argument, either

$$2x_1^2 + q_1^2(2 - a) + Q_1^2a \leqslant 2x_2^2 + q_1^2(6 - b) + Q_1^2b,$$

or

$$\frac{1}{2}x_2^2 + q_2^2(6 - b) + Q_2^2b \leqslant \frac{1}{2}x_1^2 + q_2^2(2 - a) + Q_2^2a.$$

Manipulation of these conditions yields that, for individual 1, either

$$(b - a)q_1^1 + (4 - b + a)Q_1^1 \geqslant 7, \quad (28)$$

or

$$(b - a)q_2^1 + (4 - b + a)Q_2^1 \leqslant \frac{7}{4}; \quad (29)$$

while, for individual 2, either

$$(4 - b + a)q_1^2 + (b - a)Q_1^2 \geqslant 6, \quad (30)$$

or

$$(4 - b + a)q_2^2 + (b - a)Q_2^2 \leqslant \frac{6}{4}. \quad (31)$$

If the data shows an invariant amount of the public good by individual 1, $a = b$, neither Eq. (28) or Eq. (30) can hold, and Eqs. (29) and (31) cannot hold simultaneously. Data set containing the information above cannot, thus, be considered to be consistent with Pareto efficiency in a warm-glow economy.

Example 4. To show the data set is not consistent with Lindahl equilibrium without warm-glow effects, note that individual rationality of individual 1 (WARP) requires that

$$2x_1^1 + 2q_1^1 \leqslant 2x_2^1 + 8q_1^1 \text{ or } \frac{1}{2}x_2^1 + 8q_2^1 \leqslant \frac{1}{2}x_1^1 + 2q_2^1;$$

while, for individual 2,

$$2x_1^2 + 2q_1^2 \leqslant 2x_2^2 + 8q_1^2 \text{ or } \frac{1}{2}x_2^2 + 8q_2^2 \leqslant \frac{1}{2}x_1^2 + 2q_2^2.$$

Simplifying these inequalities, Pareto efficiency requires that either $q_1^1 \geqslant \frac{7}{6}$ or $q_2^1 \leqslant \frac{7}{24}$ *and* either $q_1^2 \geqslant \frac{11}{6}$ or $q_2^2 \leqslant \frac{11}{24}$. It thus follows that there is no solution for $q_1^1 + q_1^2 = 1$ and $q_2^1 + q_2^2 = 1$.

On the other hand, these same data can be rationalized by a Lindahl equilibrium *with* warm-glow effects. Start by letting $y_1^1 = a, y_1^2 = 2 - a, y_2^1 = b$, and $y_2^2 = 8 - b$, where $a \in [0, 2]$ and $b \in [0, 8]$. With warm-glow effects, Lindahl equilibrium requires for individual 1 that either

$$p_1x_1^1 + q_1^1a + Q_1^1(2 - a) \leqslant p_1x_2^1 + q_1^1b + Q_1^1(8 - b),$$

or

$$p_2x_2^1 + q_2^1b + Q_2^1(8 - b) \leqslant p_2x_1^1 + q_2^1a + Q_2^1(2 - a);$$

while, for individual 2, either

$$p_1x_1^2 + q_1^2(2 - a) + Q_1^2a \leqslant p_1x_2^2 + q_1^2(8 - b) + Q_1^2b,$$

or

$$p_2x_2^2 + q_2^2(8 - b) + Q_2^2b \leqslant p_2x_1^2 + q_2^2(2 - a) + Q_2^2a.$$

Manipulation transforms these inequalities into, for individual 1, either

$$(b-a)q_1^1+(6-b+a)Q_1^1\geqslant 7, \tag{32}$$

or

$$(b-a)q_2^1+(6-b+a)Q_2^1\leqslant \frac{7}{4}; \tag{33}$$

while, for individual 2, either

$$(6-b+a)q_1^2+(b-a)Q_1^2\geqslant 11, \tag{34}$$

or

$$(6-b+a)q_2^2+(b-a)Q_2^2\leqslant \frac{11}{4}. \tag{35}$$

If $b=0.05$ and $a=1.95$, then $q_1^1=0.05$, $Q_1^1=0.95$, $q_2^1=0.75$, $Q_2^1=0.75$, $q_1^2=0.05$, $Q_1^2=0.95$, $q_2^2=0.25$, $Q_2^2=0.25$ would be a solution to Eqs. (32) and (35), which suffices.

Example 5. Let $y_1^1=a$, $y_1^2=2-a$, $y_2^1=b$, $y_2^2=8-b$, $q_t^i=(1-c_t^i)q_t$, and $Q_t^i=c_t^iq_t$, where $a\in[0,2]$, $b\in[0,8]$ and $c_t^i\in[0,1]$. In Nash–Walras equilibrium *with* warm-glow effects, individual 2's individual rationality (WARP) would require that either

$$g_1^2(x_2^2,y_2^2)\geqslant 0: p_1x_1^2+(1-c_1^2)q_1(2-a)+c_1^2q_1\sum_i y_1^i\leqslant p_1x_2^2+(1-c_1^2)q_1(8-b)+c_1^2q_1\sum_i y_2^i,$$

or

$$p_1x_2^2+(1-c_1^2)q_1(8-b)\geqslant m_1^2;$$

or

$$g_2^2(x_1^2,y_1^2)\geqslant 0: p_2x_2^2+(1-c_2^2)q_2(8-b)+c_2^2q_2\sum_i y_2^i\leqslant p_2x_1^2+(1-c_2^2)q_2(2-a)+c_2^2q_2\sum_i y_1^i,$$

or

$$p_2x_1^2+(1-c_2^2)q_2(2-a)\geqslant m_2^2.$$

By direct substitution, this means that either

$$x_1^2\leqslant x_2^2+\frac{1-c_1^2}{2}(6-b+a)+3c_1^2 \text{ or } x_2^2+\frac{1-c_1^2}{2}(8-b)\geqslant 6, \tag{36}$$

or

$$x_1^2\geqslant x_2^2+2(1-c_2^2)(6-b+a)+12c_2^2 \text{ or } x_1^2+2c_2^2(2-a)\geqslant 13. \tag{37}$$

When $a=b=\frac{7}{4}$, both Eqs. (36) and (37) are impossible.

Example 6. Without warm-glow effects, individual rationality of individual 1 in Nash–Walras equilibrium (WARP) requires that either

$$g_1^1(x_2^1,y_2^1)\geqslant 0: p_1x_1^1+q_1\sum_i y_1^i\leqslant p_1x_2^1+q_1\sum_i y_2^i \text{ or } p_1x_2^1\geqslant m_1^1,$$

or

$$g_2^1(x_1^1,y_1^1)\geqslant 0: p_2x_2^1+q_2\sum_i y_2^i\leqslant p_2x_1^1+q_2\sum_i y_1^i \text{ or } p_2x_1^1\geqslant m_2^1.$$

By straightforward algebra, these systems of inequalities become that, for individual 1, either

$$x_1^1\leqslant x_2^1+3 \text{ or } x_2^1\geqslant 11, \tag{38}$$

while, for individual 2, either

$$x_1^1\geqslant x_2^1+12 \text{ or } x_1^1\geqslant 13. \tag{39}$$

Note that neither Eq. (38) nor Eq. (39) can hold. So, these data cannot be rationalized by a Nash–Walras equilibrium without warm-glow assumptions.

On the other hand, the data can be rationalized by a Nash–Walras equilibrium *with* warm-glow properties. To see this, let $y_1^1=a$, $y_1^2=2-a$, $y_2^1=b$, and $y_2^2=8-b$, and write $q_t^i=(1-c_t^i)q_t$ and $Q_t^i=c_t^iq_t$, where $a\in[0,2]$, $b\in[0,8]$ and $c_t^i\in[0,1]$. With warm-glow assumptions, individual rationality of individual 1 (WARP) requires that either

$$g_1^1(x_2^1,y_2^1)\geqslant 0: p_1x_1^1+(1-c_1^1)q_1a+c_1^1q_1\sum_i y_1^i\leqslant p_1x_2^1+(1-c_1^1)q_1b+c_1^1q_1\sum_i y_2^i,$$

or

$$p_1x_2^1+(1-c_1^1)q_1b\geqslant m_1^1;$$

or

$$g_2^1(x_1^1,y_1^1)\geqslant 0: p_2x_2^1+(1-c_2^1)q_2b+c_2^1q_2\sum_i y_2^i\leqslant p_2x_1^1+(1-c_2^1)q_2a+c_2^1q_2\sum_i y_1^i,$$

or

$$p_2x_1^1+(1-c_2^1)q_2a\geqslant m_2^1.$$

For individual 2, for the same reason, either

$$g_1^2(x_2^2,y_2^2)\geqslant 0: p_1x_1^2+(1-c_1^2)q_1(2-a)+c_1^2q_1\sum_i y_1^i\leqslant p_1x_2^2+(1-c_1^2)q_1(8-b)+c_1^2q_1\sum_i y_2^i,$$

or

$$p_1x_2^2+(1-c_1^2)q_1(8-b)\geqslant m_1^2;$$

or

$$g_2^2(x_1^2,y_1^2)\geqslant 0: p_2x_2^2+(1-c_2^2)q_2(8-b)+c_2^2q_2\sum_i y_2^i\leqslant p_2x_1^2+(1-c_2^2)q_2(2-a)+c_2^2q_2\sum_i y_1^i,$$

or

$$p_2x_1^2+(1-c_2^2)q_2(2-a)\geqslant m_2^2.$$

That is, for individual 1, either

$$x_1^1\leqslant x_2^1+\frac{1-c_1^1}{2}(b-a)+3c_1^1 \text{ or } x_2^1+\frac{1-c_1^1}{2}q_1b\geqslant 11, \tag{40}$$

or

$$x_1^1\geqslant x_2^1+2(1-c_2^1)(b-a)+12c_2^1 \text{ or } x_1^1+2(1-c_2^1)q_2a\geqslant 13; \tag{41}$$

while, for individual 2, either

$$x_1^2 \leqslant x_2^2 + \frac{1-c_1^2}{2}(6-b+a) + 3c_1^2 \text{ or}$$
$$x_2^2 + \frac{1-c_1^2}{2}q_1(8-b) \geqslant 2, \tag{42}$$

or

$$x_1^2 \geqslant x_2^2 + 2(1-c_2^2)(6-b+a) + 12c_2^2 \text{ or}$$
$$x_1^2 + 2(1-c_2^2)q_2(2-a) \geqslant 7. \tag{43}$$

When $a = \frac{3}{2}, b = \frac{11}{2}, x_1^1 = \frac{41}{4}, x_2^1 = 2, x_1^2 = \frac{7}{4}, x_2^2 = 2, c_2^1 = 0.04$, $c_1^2 = 0.8$, the second part of Eq. (41) and the first part of Eq. (42) hold, which suffices to argue that the data can be rationalized.

References

Allouch, N., 2013. A competitive equilibrium for a warm-glow economy. Econom. Theory 53, 269–282.

Andreoni, J., 1989. Giving with impure altruism: applications to charity and Ricardian equivalence. J. Polit. Econ. 97, 1447–1458.

Andreoni, J., 1990. Impure altruism and donations to public goods: a theory of warm-glow giving. Econ. J. 100, 464–477.

Bachmann, R., 2006. Testable implications of Pareto efficiency and individual rationality. Econom. Theory 29, 489–504.

Bronars, S., 1987. The power of nonparametric tests of preference maximization. Econometrica 55, 693–698.

Brown, D., Matzkin, R., 1996. Testable restrictions on the equilibrium manifold. Econometrica 64, 1249–1262.

Carvajal, A., 2004a. Testable restrictions of Nash Equilibrium in games with continuous domains. Royal Holloway, University of London, Discussion Paper Series 2004–26.

Carvajal, A., 2004b. Testable restrictions on the equilibrium manifold under random preferences. J. Math. Econom. 40, 121–143.

Carvajal, A., 2005. The testable restrictions of general equilibrium in production economies. Discussion paper 2005-01, Royal Holloway College, University of London.

Carvajal, A., 2010. The testable restrictions of competitive equilibrium in economies with externalities. Econom. Theory 45, 349–378.

Carvajal, A., Deb, R., Fenske, J., Quah, J. K.-H., 2013. Revealed preference tests of the Cournot model. Econometrica 81, 2351–2379.

Carvajal, A., Deb, R., Fenske, J., Quah, J. K.-H., 2014. A nonparametric analysis of multi-product oligopolies. Econom. Theory 57, 253–277.

Cherchye, L., De Rock, B., Demuynck, T., 2011a. The revealed preference approach to collective consumption behavior: testing and sharing rule recovery. Rev. Econom. Stud. 78, 176–198.

Cherchye, L., Demuynck, T., De Rock, B., 2011b. Testable implications of general equilibrium models: an integer programming approach. J. Math. Econom. 47, 564–575.

Cherchye, L., Demuynck, T., De Rock, B., Hjertstrand, P., 2015. Revealed preference tests for weak separability: an integer programming approach. J. Econometrics 186, 129–141.

Deb, R., 2009. A testable model of consumption with externalities. J. Econom. Theory 144, 1804–1816.

Deb, R., Gazzale, R.S., Kotchen, M., 2014. Testing motives for charitable giving: a revealed-preference methodology with experimental evidence. J. Public Econ. 120, 181–192.

Forges, F., Minelli, E., 2009. Afriat's theorem for general budget sets. J. Econom. Theory 144, 135–145.

Kübler, F., 2003. Observable restrictions of general equilibrium models with financial markets. J. Econom. Theory 40, 137–153.

Snyder, S., 1999. Testable restrictions of Pareto optimal public good provision. J. Public Econ. 71, 97–119.

Talla-Nobibon, F., Cherchye, L., Crama, Y., Demuynck, T., De Rock, B., Spieksma, F., 2016. Revealed preference tests of collectively rational consumption behavior: formulations and algorithms. Oper. Res. 64, 1197–1216.