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SYMPOSIUM

The testable implications of competitive equilibrium in economies with externalities

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Abstract Suppose that one has a data set consisting of prices and individual endowments for some economy. Brown and Matzkin (Econometrica 64:1249–1262, 1996) have shown that there are conditions that the data have to satisfy, if the observed prices are determined by the competitive equilibrium process, given the observed endowments, when there are no external effects in the economy's interactions. The results here show that the same conclusion does not apply, in general, if the economy exhibits externalities. On the other hand: (i) some restrictions exist if there exist at least two commodities on which the individuals' preferences are weakly separable; (ii) although extremely mild, restrictions exist too if one observed individual consumption for the economy that causes the external effects; and (iii) importantly, even if the previous two cases do not apply, restrictions exist when the externalities that exist are in the form of a public good.

Keywords Nash–Walras equilibrium · Externalities · Public goods · Testable restrictions · Empirical implications · Revealed preferences

JEL Classification D12 · D50 · D62

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0 Introduction

Consider a situation where an analyst has a finite data set consisting of the prices and the individual wealths that were observed, for a series of observations, in a given economy. Suppose that the analyst wants to determine whether it is possible to explain these data as the competitive outcomes of the economy, given the underlying, unobserved preferences of its individuals. Specifically, he wants to find what conditions the data have to satisfy if, indeed, there are preferences for which all the observed prices are competitive equilibrium prices, given the corresponding observed endowments. This paper shows that if consumption externalities cannot be ruled out, then, in general, any data set can be explained by the analyst: the hypothesis of Nash–Walras equilibrium imposes no testable restrictions when preferences are allowed to exhibit externalities, unless some structure on these externalities is assumed as part of the hypothesis to be tested. If the analyst assumes that individual preferences are weakly separable in at least two commodities, then the hypothesis of Nash–Walras equilibrium is refutable. Similarly, in a production economy where the externality takes the form of a public good, the hypothesis imposes testable implications. But, in cases more general than that, there are no empirical regularities that the model of competitive equilibrium imposes of aggregate data, even if consumption of the commodity that causes the external effect is observed.[1]

This paper builds upon the results obtained by Brown and Matzkin (1996), who observed that the two principles of competitive equilibrium, namely individual rationality and market clearing, restrict the response of endogenous aggregate variables (prices) to perturbations on individual endowments. The analysis of Brown and Matzkin contrasted with the understanding, derived from the theorem obtained independently by Sonnenschein (1974), Mantel (1974) and Debreu (1974), which was interpreted to imply that the model of general, competitive equilibrium theory imposed no, or hardly any, testable restrictions at the aggregate level.[2] The Sonnenschein–Mantel–Debreu Theorem says, basically, that the response of the aggregate excess demand of a group of competitive individuals to changes in prices is arbitrary; the interpretation was that, hence, anything goes, as far as competitive behavior is concerned, unless one has information about consumption at the individual level.

The results of Brown and Matzkin make it clear that this view of the problem is overly pessimistic. Specifically, they observed that the object under consideration in the Sonnenschein–Mantel–Debreu analysis, the aggregate excess demand function, is not the appropriate object on which restrictions are to be derived, since (i) it is not observable, under the 'null' hypothesis of general equilibrium, except at points at which it is, by definition, zero; and (ii) it measures the response of some endogenous variable (demand) to another endogenous variable (prices). Instead, Brown and Matzkin study

[1] This last point has to be qualified: in this case the model only imposes some extremely weak restrictions.

[2] See Hansen and Heckman (1996), for an observation about the empirical implications of the competitive equilibrium models. For the theorem, see, also, Mas-Colell (1977). Some qualifications are necessary here: firstly, the economy must allow for at least as many consumers as there are commodities; second, the boundary behavior of the aggregate demand function is *not* arbitrary (see Balasko 1986); and thirdly, some very mild conditions, such as Walras's law and homogeneity of degree zero, have to be satisfied.

restrictions on how prices respond to changes in individual endowments: a finite subset of the equilibrium manifold overcomes the observability problem, and provides information on how an endogenous variable responds to perturbations on exogenous variables. The key result is that this response is not arbitrary: there exists a (finite) set of nontautological conditions that is necessary (and sufficient) for the existence of preferences that rationalize a given, finite set of data on prices and individual endowments.

The results of Brown and Matzkin constituted the basis for further developments on the empirical implications of general equilibrium. Chiappori et al. (2000) obtained restrictions in a differential setting, while Kübler (2004) and Kübler and Schmedders (2009) extended Brown and Matzkin (1996) to intertemporal problems under uncertainty, and Carvajal (2004a) to a setting where preferences of individuals are allowed to change randomly. Bachmann (2004) derived a test of the hypothesis that a sequence of allocations can be supported by competitive prices, and Bachmann (2006) derived testable implications of Pareto-efficiency and individual rationality. And, importantly, by an application of the methodology of Brown and Matzkin (1996) to the analysis of public goods via Lindahl prices, Snyder (1999) showed that, for the case where the public good is produced under constant returns to scale, the hypothesis that the observed data correspond to a Pareto-efficient provision of the public goods imposes restrictions if market prices, production levels and individual incomes are observed.[3]

In this paper I study what restrictions, if any, are imposed by the competitive equilibrium model, for economies with externalities. By an immediate extension of the results of Brown and Matzkin (1996), the introduction of externalities will not affect the restrictions they derived, if one imposes additive separability of individual preferences in the consumption of other individuals in the economy: in such case, the externalities welfare effects, but individual behavior should be the same as if there were no external effects. I then study the extent to which one can deviate from this extreme assumption of separability, while maintaining the property that the model imposes empirical implications. Some of the results are negative: even under refined classes of preferences, the hypothesis of Nash–Walras equilibrium is not refutable on the basis of data on prices and individual endowments. Some degree of separability individual preferences, at least between two commodities and the consumption of the rest of the economy, suffices to give some testable restrictions. Importantly, restrictions are imposed also for the particular case where the externality takes the form of a public good.

1 General setting and a basic result

1.1 Nash–Walras equilibrium

Consider a society, denoted by $\mathcal{I}$, populated by at least two individuals, each of whom is denoted by $i = 1, \ldots, I$, and suppose that these agents trade $L+1$ commodities that

[3] Moreover, Kübler et al. (2002), Chiappori et al. (2000), Balasko (2004), Matzkin (2005), Carvajal and Riascos (2005), Carvajal and Riascos (2008) and Balasko and Tvede (2009) further showed conditions under which information on individual fundamentals can be uniquely identified from knowledge of the equilibrium manifold.

are to be consumed in nonnegative amounts. Let a consumption bundle for individual i be denoted by (x^i, y^i), where x^i represents a bundle of the first L commodities and y^i is consumption of the last commodity. Suppose that individuals have private wealth, and let the endowment of agent i be the consumption bundle (e^i, k^i).

In order to consider a simple setting in which there are externalities in the economy, suppose that each individual cares about her own consumption of all commodities and also about the consumption of the last commodity by all the other consumers. In this setting, individual i's preferences are represented by a utility function

$$u^i : \mathbb{R}^L_+ \times \mathbb{R}_+ \times \mathbb{R}^{I-1}_+ \to \mathbb{R},$$

so that $u^i(x^i, y^i, y^{\neg i})$ represents person i's utility level if her consumption bundle is (x^i, y^i) and the profile of consumption of the last commodity by the other agents in the society is $y^{\neg i} = (y^j)_{j \in \mathcal{I} \setminus \{i\}}$.

Denote by u the profile, $(u^i)_{i=1}^I$, of utility functions, and, similarly, by (e, k) the profile of individual endowments. In this setting, an *economy* is completely described by the profile of individual preferences and endowments; for simplicity of notation, we may sometimes write the economy as $\mathcal{E} = (u, e, k)$.

All commodities are traded in competitive markets, and their prices are denoted by (p, q), where p is the vector of prices of the first L commodities and q is the price of the last commodity. Given an economy, an allocation is a profile, $(x, y) = (x^i, y^i)_{i=1}^I$, of consumption bundles, and *a Nash–Walras equilibrium* consists of a vector of prices and an allocation of commodities, (p, q, x, y), such that:

1. each consumer's demand is rational, given the prices and the choices of other individuals: each (x^i, y^i) solves the problem

$$\max_{\tilde{x}, \tilde{y}} u^i\left(\tilde{x}, \tilde{y}, y^{\neg i}\right) : p\left(\tilde{x} - e^i\right) + q\left(\tilde{y} - k^i\right) \leq 0;$$

2. all markets clear: $\sum_i (x^i - e^i, y^i - k^i) = 0$.

Let $\mathcal{W}(\mathcal{E})$ denote the set of Nash–Walras equilibria of $\mathcal{E}$, and let $W(\mathcal{E})$ be the projection of this set into the space of prices.

1.2 Rationalizability of data

A *data set* is a finite sequence of pairs consisting of a vector of prices and a profile of individual endowments, $(p_t, q_t, e_t, k_t)_{t=1}^T$. For simplicity of the notation and of later arguments, assume that all observed prices and endowments are strictly positive. Profile of preferences u *rationalizes* a data set if each observation of prices is consistent with the corresponding (observed) profile of endowments, in the sense of being a vector of Nash–Walras equilibrium prices: for every t, $(p_t, q_t) \in W(u, e_t, k_t)$.

The main question with which this paper is concerned is the determination of the conditions that a data set must satisfy if it is rationalizable by a profile of preferences that lie in some given class. Observation of data violating these 'testable implications' would contradict the hypothesis that all the observed values of the endogenous variables (prices) can be determined by the mechanism of Nash–Walras equilibrium, given the observed values of the exogenous variables (endowments) and under fixed unobservable fundamentals (invariant preferences in the relevant class).

If the economy does not exhibit externalities, in equilibrium each individual's consumption bundle must maximize her preferences, and, hence, when data on individual demands is available, each person's consumption must obey the axioms of revealed preference. When individual consumption has not been observed, as in the data sets considered here, still there must exist allocations of commodities, one for each observation, that satisfy market-clearing and nonnegativity constraints, and such that each individual's consumption obeys, again, revealed preferences axioms. The remarkable contribution of Brown and Matzkin (1996) is to show that these allocations need not always exist: the data may be such that the first two requirements, namely nonnegativity and market-clearing, make the satisfaction of the weak axiom of revealed preference by all individuals impossible.

In the case of externalities, the first property of the definition of equilibrium, individual rationality, loses some of its predictive power: in this case, each person only maximizes the cross-section of her utility function with respect to her own consumption, and, as a consequence, if consumption were observed, the axioms of revealed preference only apply across observations where the profile of demands of the commodity that causes the externality is invariant. When the data provides no direct information on individual consumption, the possibility that this invariance never took place cannot be ruled out: for *any* data set, one can find a sequence $(x_t, y_t)_{t=1}^T$ such that consumption is positive,

$$(x_t^i, y_t^i) \gg 0; \tag{1}$$

satisfies Walras's law,

$$p_t x_t^i + q_t y_t^i = p_t e_t^i + q_t k_t^i; \tag{2}$$

and clears markets,

$$\sum_i \left(x_t^i, y_t^i\right) = \sum_i \left(e_t^i, k_t^i\right); \tag{3}$$

and where demand for the externality commodity is never invariant,

$$y_{t'}^i \neq y_t^i, \quad \text{whenever } t' \neq t. \tag{4}$$

(For instance, the output of the algorithm presented in Appendix A1 will satisfy these properties.) The implication of this impossibility is that an analyst can never rule out the possibility that the individuals in the economy never chose according to the same (cross-section of their) preferences, and cannot, therefore, apply a test based on revealed-preference analysis. The following result says that, in fact, rationalizability by a profile of preferences defined in the basic class of conditions used to guarantee existence of Nash–Walras equilibrium imposes no testable implications.

Proposition *Any data set is rationalizable by a profile of continuous preferences that are strongly concave and strictly monotone in own consumption* (x^i, y^i).[4]

This result, the most basic one in the paper, is in fact a corollary of Proposition 1 below, so a formal proof for it can be omitted. Simply put, the result highlights the fact that the hypothesis of individual rationality imposes no empirical restrictions in the presence of a commodity that causes external effects, if its demand is not observed at the individual level. Then, the proposition exploits the fact that this observation continues to be true even if one has observed individual consumption of every other commodity, and if market clearing is imposed.

2 Further assumptions

2.1 Concavity and monotonicity

The basic proposition obtained above does not imply that rationalizability with respect to proper subclasses of preferences is not testable. For instance, the possibility that concavity or monotonicity may fail with respect to the externality is allowed by that proposition. The next result says that even this stronger requirement fails to impose any empirical regularity. The reason is again that the axioms of revealed preferences need not apply in any meaningful sense: given $(x_t, y_t)_{t=1}^T$ that obeys Eqs. (1)–(4) one can always find numbers u_t^i and μ_t^i, and strictly positive numbers $\lambda_t^i > 0$, for all individuals and observations, such that[5]

$$u_{t'}^i < u_t^i + \lambda_t^i \left(p_t \left(x_{t'}^i - x_t^i \right) + q_t \left(y_{t'}^i - y_t^i \right) \right) + \mu_t^i \left(y_{t'}^j - y_t^j \right). \tag{5}$$

This system, which resembles the analysis of Afriat, can be used to construct preferences that rationalize the data set, as in the following proposition.

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[4] That is, for every $y^{\neg i}$, function $u^i(\cdot, \cdot, y^{\neg i})$ is strongly concave and strictly monotone.

[5] Fix i, let $j \neq i$, and, for simplicity of notation and using Eq. (4), suppose that $y_1^j > y_2^j > \cdots > y_T^j$, but the system of Eq. (5) below has no solution with all $\lambda_t^i > 0$. Then, it follows from the Theorem of the Alternative for strict inequalities, Rockafellar (1970, §22.2), that there exist $\alpha_{t,t'} \geq 0$, for all t and all $t' \neq t$, and $\beta_t \geq 0$, for all t, with at least one of these numbers strictly positive, such that for each t,

(i) $\sum_{t' \neq t} \alpha_{t,t'} = \sum_{t' \neq t} \alpha_{t',t}$;

(ii) $\sum_{t' \neq t} \alpha_{t,t'} (p_t (x_t^i - x_{t'}^i) + q_t (y_t^i - y_{t'}^i)) = \beta_t$; and

(iii) $\sum_{t' \neq t} \alpha_{t,t'} (y_t^j - y_{t'}^j) = 0$.

Since $y_1^j > y_t^j$ for all $t \geq 2$, it follows from (iii) that $\alpha_{1,t'} = 0$ for every $t' \neq 1$. Then, from (i), it must be that also $\alpha_{t',1} = 0$ for every $t' \neq 1$, and, hence, (ii) implies that $\alpha_{2,t'} = 0$ for every $t' \neq 2$. From (i) again, one has that also $\alpha_{t',2} = 0$ for every $t' \neq 2$. Continuing recursively, it follows that all $\alpha_{t,t'} = 0$, which implies, by (ii), that every $\beta_t = 0$, which is impossible.

Proposition 1 *Any data set is rationalizable by a profile of continuous, strongly concave and strictly monotone preferences.*

Proof As in Theorem 2 in Matzkin and Richter (1991), one can find a strongly convex function $h(x, y, y^{\neg})$ that satisfies the properties that $h(x, y, y^{\neg}) = 0$ only at $(x, y, y^{\neg}) = 0$, and that all its partial derivatives are less than 1. Using a solution to system (5) for individual i, one can construct functions

$$v_t^i\left(x^i, y^i, y^{\neg i}\right) = u_t^i + \lambda_t^i\left(p_t\left(x^i - x_t^i\right) + q_t\left(y^i - y_t^i\right)\right) + \mu_t^i\left(y^j - y_t^j\right)$$
$$-\epsilon_t^i h\left(\left(x_t^i, y_t^i, y_t^{\neg i}\right) - \left(x^i, y^i, y^{\neg i}\right)\right),$$

where $\epsilon_t^i \in (0, \lambda_t^i \min\{\min_l\{p_{t,l}\}, q_t\})$ is small enough, so that for all $t' \neq t$,

$$u_{t'}^i < u_t^i + \lambda_t^i\left(p_t\left(x_{t'}^i - x_t^i\right) + q_t\left(y_{t'}^i - y_t^i\right)\right) + \mu_t^i\left(y_{t'}^j - y_t^j\right)$$
$$-\epsilon_t^i h\left(\left(x_t^i, y_t^i, y_t^{\neg i}\right) - \left(x_{t'}^i, y_{t'}^i, y_{t'}^{\neg i}\right)\right).$$

Now, we can define

$$u^i\left(x^i, y^i, y^{\neg i}\right) = \min_t\left\{v_t^i\left(x^i, y^i, y^{\neg i}\right)\right\} + \left(\max\left\{1, -\min_t\{\mu_t^i - \epsilon_t^i\}\right\} + 1\right)\sum_{j' \neq i} y^{j'}.$$

The utility functions constructed in this way are continuous, strongly concave and strictly monotone. By construction, and using Eq. (4), they also rationalize the data. □

The proposition means that the assumption that convexity applies only to the variables that each individual chooses is innocuous from the perspective of testability: the Nash–Walras hypothesis is nontestable with and without that assumption.

2.2 Strategic complementarities

The literature on monotone comparative statics (see, for example, Milgrom and Shannon 1994; Topkis 1998; Quah 2007), and in particular its application to abstract games, Carvajal (2004b), have shown that the assumption of strategic complementarity strengthens, significantly, the tests of hypotheses of noncooperative behavior: in general, the revealed preference axioms imply very weak restrictions for Nash equilibrium in continuous games, since, as here, each player maximizes a different preference relation when her opponents have changed their play; but if the game exhibits strategic complementarities, the restrictions strengthen, as the players' best responses, absent constraints, should be co-monotone. The following theorem shows that such result does not extend to the present setting.

A utility function u^i will be said to exhibit *strategic complementarity* if

$$u^i\left(x^i, y^i, y^{\neg i}\right) - u^i\left(x^i, \hat{y}^i, y^{\neg i}\right) \geq u^i\left(x^i, y^i, \hat{y}^{\neg i}\right) - u^i\left(x^i, \hat{y}^i, \hat{y}^{\neg i}\right),$$

whenever $y^i \geq \hat{y}^i$ and $y^{\neg i} \geq \hat{y}^{\neg i}$. If individual demands are observed, this property implies strong properties of monotone comparative statics that cannot be satisfied simply by the observation that each individual may be maximizing a different cross-section of her utility function. It turns out, however, that one can always construct Afriat systems that will lead to rationalizations of the data that exhibit strategic complementarity. To see this, fix an individual i, let $j \neq i$, and assume that $y_1^j < y_2^j < \cdots < y_T^j$. As before, one can always find numbers $u_t^i, \lambda_t^i > 0$ and μ_t^i that satisfy the inequality

$$-u_t^i + u_{t'}^i + \lambda_t^i \left(p_t \left(x_t^i - x_{t'}^i \right) + q_t \left(y_t^i - y_{t'}^i \right) \right) + \mu_t^i \left(y_t^j - y_{t'}^j \right) < 0, \quad (6)$$

whenever $t \neq t'$, and are such that

$$\lambda_t^i q_t < \lambda_{t+1}^i q_{t+1}, \quad \text{whenever } t \leq T-1, \quad (7)$$

and

$$\mu_t^i > \mu_{t+1}^i, \quad \text{whenever } t \leq T-1. \quad (8)$$

With these numbers, one can define individual i's preferences by the function

$$u^i \left(x^i, y^i, y^{\neg i} \right) = \min_t \left\{ u_t^i + \lambda_t^i \left(p_t \left(x^i - x_t^i \right) + q_t \left(y^i - y_t^i \right) \right) + \mu_t^i \left(y^j - y_t^j \right) \right\},$$

which is continuous and concave in all arguments, is strictly monotone on (x^i, y^i), and satisfies strategic complementarity.[6] Strict monotonicity in $y^{\neg i}$ can be obtained as in the proof of Proposition 1, while Eq. (6) guarantees that the profile of preferences constructed in this way rationalizes the data set. Hence, we have proved the following proposition.[7]

[6] It suffices to observe that its right-hand partial derivative with respect to y^i is nondecreasing in $y^{\neg i}$. To see that this is the case, let t and $\tilde{t}$ be such that the right-hand partial at $(x^i, y^i, y^{\neg i})$ is $\lambda_t^i q_t$, and the right-hand partial at $(x^i, y^i, \tilde{y}^{\neg i})$ is $\lambda_{\tilde{t}}^i q_{\tilde{t}}$ By construction,

$$u_t^i + \lambda_t^i \left(p_t \left(x^i - x_t^i \right) + q_t \left(y^i - y_t^i \right) \right) + \mu_t^i \left(y^j - y_t^j \right) \leq u_{\tilde{t}}^i + \lambda_{\tilde{t}}^i \left(p_{\tilde{t}} \left(x^i - x_{\tilde{t}}^i \right) + q_{\tilde{t}} \left(y^i - y_{\tilde{t}}^i \right) \right) + \mu_{\tilde{t}}^i \left(y^j - y_{\tilde{t}}^j \right),$$

and

$$u_{\tilde{t}}^i + \lambda_{\tilde{t}}^i \left(p_{\tilde{t}} \left(x^i - x_{\tilde{t}}^i \right) + q_{\tilde{t}} \left(y^i - y_{\tilde{t}}^i \right) \right) + \mu_{\tilde{t}}^i \left(\tilde{y}^j - y_{\tilde{t}}^j \right) \leq u_t^i + \lambda_t^i \left(p_t \left(x^i - x_t^i \right) + q_t \left(y^i - y_t^i \right) \right) + \mu_t^i \left(\tilde{y}^j - y_t^j \right).$$

Adding these two inequalities gives $(\mu_t^i - \mu_{\tilde{t}}^i)(y^j - \tilde{y}^j) \leq 0$, while from Eqs. (7) and (8), it follows that $(\lambda_t^i q_t - \lambda_{\tilde{t}}^i q_{\tilde{t}})(y^j - \tilde{y}^j) \geq 0$.

[7] Details of the arguments, including the fact that Eqs. (6)–(8) can *always* be satisfied, are available from the author upon request.

Proposition 2 *Any data set is rationalizable by a profile of continuous, concave, strictly monotone preferences that satisfy strategic complementarity.*

This proposition may seem to contradict the results of Chipman (1977), Quah (2007) and Chambers and Echenique (2009) that imply that the rational demand of an individual whose preferences are representable by a supermodular and concave utility function must be normal. It does not, as strategic complementarity here only refers to the effect of consumption of other people on the utility that the individual derives from her consumption of one commodity, given the consumption of every other commodity; that is, there is no effect associated to supermodularity between two commodities that the individual chooses.

It follows from the proposition that even the combination of overall concavity, strategic complementarity and monotonicity imposes no testable implications. The following result exploits the fact that the results above do not assume that endowments are different across observations; it is immediate from the Propositions 1 and 2, by letting the data set be $(p_s, q_s, e, k)_{s=1}^S$. It resembles the results of Mas-Colell (1977), exploiting the presence of externalities to allow for further properties on the unobserved individual preferences.

Corollary *Any finite set of prices is a subset of Nash–Walras equilibrium prices for some economy with a given profile of endowments and a profile of continuous, concave, strictly monotone preferences that satisfy strategic complementarity. Formally, for any finite set of strictly positive prices,* $\{(p_s, q_s)\}_{s=1}^S$, *and any profile of strictly positive endowments,* (e, k), *there exists a profile of preferences,* u, *satisfying the conditions above, such that* $\{(p_s, q_s)\}_{s=1}^S \subseteq W(u, e, k)$. *Alternatively, all preferences can be taken to be strongly concave and strictly monotone.*

2.3 Separability

Obviously, the assumption that each individual's utility function is additive separable in the consumption of the rest of agents immediately restores the testable implications of the model without externalities: the presence of externalities has welfare effects, but it does not affect the behavior of any agent. The purpose of the following result is to determine whether under weaker separability assumptions the equilibrium models does impose some restrictions as well. For the purposes of this result, I assume that there are at least two commodities other than the one that causes the externality, namely that $L \geq 2$.

A utility function u^i will be said to be *weakly separable* (in x^i) if it can be written as

$$u^i\left(x^i, y^i, y^{\neg i}\right) = U^i\left(V^i(x), y^i, y^{\neg i}\right),$$

for a strictly monotone function $V^i : \mathbb{R}_+^L \to \mathbb{R}$ and a function $U^i : V[\mathbb{R}_+^L] \times \mathbb{R} \times \mathbb{R}_+^{I-1} \to \mathbb{R}$ that is strictly monotone in its first two arguments (see Blackorby et al. 1978, §3). We will say that u^i is *weakly, smoothly separable* if it is differentiable and has interior contour sets, and if the functions U^i and V^i above are differentiable.

Given a data set, define the following system over vectors $x_t^i \geq 0$ and ρ_t^i, and scalars $y_t^i \geq 0$, $u_t^i, v_t^i, \theta_t^i, \lambda_t^i > 0$ and $\mu_t^i > 0$:[8] for every individual i and every observation t,

$$\rho_t^i = \lambda_t^i p_t \quad \text{and} \quad \theta_t^i = \mu_t^i q_t, \tag{9}$$

$$p_t x_t^i + q_t y_t^i = p_t e_t^i + q_t k_t^i, \tag{10}$$

$$u_{t'}^i \leq u_t^i + \frac{\mu_t^i}{\lambda_t^i}\left(V_{t'}^i - V_t^i\right) + \theta_t^i \left(y_{t'}^i - y_t^i\right), \quad \text{whenever} \;\; y_{t'}^{\neg i} = y_t^{\neg i}, \tag{11}$$

$$v_{t'}^i \leq v_t^i + \rho_t^i \left(x_{t'}^i - x_t^i\right), \tag{12}$$

and

$$\sum_i \left(x_t^i, y_t^i\right) = \sum_i \left(e_t^i, k_t^i\right). \tag{13}$$

Lemma 1 *Fix a data set* $(p_t, q_t, e_t, k_t)_{t=1}^T$.

1. *If the set is rationalizable by a profile of weakly, smoothly separable preferences that satisfy concavity and strong monotonicity in own consumption,* (x^i, y^i)*, then the system of Eqs.* (9)–(13) *has a solution with all* $(x_t^i, y_t^i) \gg 0$.
2. *If the system of equations has a solution, then the data set is rationalizable by a profile of weakly separable, concave and strongly monotone preferences.*

Lemma 1 does not give a test of rationalizability under the assumption of weakly separable preferences, because of the existential quantifiers it invokes, but is instrumental for the proposition that follows: in order to obtain a proper test, one simply needs to argue that all existential quantifiers can be eliminated from the characterization given by the lemma, without obtaining a tautology.

Proposition 3 *Rationalizability by a profile of weakly separable preferences is testable using a finite set of inequalities on data. Formally, for any finite sequence of profiles of strictly positive individual endowments,* $d = (e_t, k_t)_{t=1}^T$*, there exists a semialgebraic set of sequences of strictly positive prices for all commodities,* Δ_d*, such that*:

1. *if* $(p_t, q_t, e_t, k_t)_{t=1}^T$ *is rationalizable by a profile of weakly, smoothly separable preferences that satisfy concavity and strict monotonicity in own demands, then* $(p_t, q_t)_{t=1}^T \in \Delta_d$;
2. *if* $(p_t, q_t)_{t=1}^T \in \Delta_d$*, then* $(p_t, q_t, e_t, k_t)_{t=1}^T$ *is rationalizable by a profile of weakly separable, concave and strictly monotone preferences.*

Also, there exist d for which Δ_d *is a proper subset of* $(\mathbb{R}_{++}^L \times \mathbb{R}_{++})^T$.

Proof Given d consider the set of arrays $(p_t, q_t, (x_t^i, y_t^i, \rho_t^i, \nu_t^i, \lambda_t^i, \mu_t^i, u_t^i, v_t^i)_{i=1}^I)_{t=1}^T$ that solve the system of Eqs. (9)–(13). This is a finite set of polynomial inequalities,

[8] The system is based on Varian (1983) and Brown and Matzkin (1996).

so it follows from the Tarski–Seidenberg theorem (see Mishra 1993, Theorem 8.6.6) that its projection into the space of prices only, Δ_d, is semialgebraic. Parts 1 and 2 of the Proposition then follow, respectively, from parts 1 and 2 of Lemma 1, while Example 1 below illustrates d for which $\Delta_d \subsetneq (\mathbb{R}^L_{++} \times \mathbb{R}_{++})^T$. □

In the analysis of Brown and Matzkin (1996), the data identify individual budget constraints at each observation, and the tests exploit an existing tension between the principles of market-clearing (in the form of the nonnegativity constraints on consumption) and individual rationality (in the form of the weak axiom of revealed preferences). Here, the overall budget constraint of each individual is observed, but not the 'reduced' budget that constrains the individual's choice of the commodities in which her preferences are separable; nonnegativity constraints on all other commodities, however, do impose bounds on the position of these reduced budgets, and, then, within these bounds, the tension between market clearing and individual rationality yields the testable implications: in order to complete the argument of Proposition 3, we must show that there exist data sets which cannot be rationalized under the separability hypothesis; the following example is one such data set.

Example 1 Suppose that $I = L = 2$. Suppose that a data set includes the following two observations:

$$
\begin{array}{llll}
e^1_1 = (1, 4), & e^1_2 = (4, 1), & e^2_1 = (2, 1), & e^2_2 = (1, 2), \\
k^1_1 = 0.01, & k^1_2 = 0.005, & k^2_1 = 0.01, & k^2_2 = 0.005, \\
p_1 = (1, 10), & p_2 = (10, 1), & q_1 = 0.1, & q_2 = 0.2.
\end{array}
$$

Suppose that the data set is rationalized by the profile of weakly separable preferences (u^1, u^2). Let (x^1_t, y^1_t) solve

$$\max_{x^1, y^1} u^1 \left(x^1, y^1, y^2_t\right) : p_t \cdot \left(x^1 - e^1_t\right) + q_t \left(y^1 - k^1_t\right) \leq 0,$$

and suppose that $u^1(x^1, y^1, y^2) = U^1(V^1(x^1), y^1, y^2)$. Then, it must be that each x^1_t solves the problem

$$\max_{x} V^1(x) : p_t x \leq T^1_t,$$

with $T^1_t = p_t e^1_t - q_t(y^1_t - k^1_t)$. By aggregate feasibility,

$$40.999 = p_t e^1_t - q_t k^2_t \leq T^1_t \leq p_t e^1_t + q_t k^1_t] = 41.001.$$

Also, since $e^1_1 + e^2_1 = (3, 5)$ and $e^1_2 + e^2_2 = (5, 3)$, feasible values of x^1_1 and x^1_2 can only be, respectively, in

$$X_1 = \left\{ (x_1, x_2) | x_1 \in [0, 3], x_2 = \frac{T^1_1}{10} - 0.1 x_1 \right\}$$

and

$$X_2 = \left\{ (x_1, x_2) | x_2 \in [0, 3], x_1 = \frac{T_2^1}{10} - 0.1x_2 \right\}$$
$$\subset \left\{ (x_1, x_2) | x_1 \in [3.7999, 4.2], x_2 = T_2^1 - 10x_1 \right\}.$$

Since $X_1 \cap X_2 = \emptyset$, necessarily $x_1^1 \neq x_2^1$. Since $x_1^1 \in X_1$, then

$$p_2 x_1^1 = 10x_{1,1}^1 + \frac{T_1^1}{10} - 0.1x_{1,1}^1 \leq 9.9(3) + 4.1001 < T_2^1 = p_2 x_2^1,$$

whereas, since $x_2^1 \in X_2$, then

$$p_1 x_2^1 = x_{2,1}^1 + 10\left(T_2^1 - 10x_{2,1}^1\right) \leq 410.01 - 99(3.7999) < T_1^1 = p_1 e_1^1,$$

which is impossible, as it violates the weak axiom of revealed preferences.

It follows that under weak separability the hypothesis of competitive equilibrium does impose testable restrictions that take the form of a finite set of inequalities on data. The results will still hold as long as for each individual there are two of the $L+1$ commodities for which the person's preferences are weakly separable: what is critical in the example is the fact that, under nonnegativity constraints, the observation of the graduate endowment of all commodity imposes bounds on the levels of expenditure that each individual must have had on any subset of commodities; if there are two or more commodities on which the individual's preferences are weakly separable, then she must satisfy the axioms of revealed preferences with respect to her demand for that set of commodities; what the example does is to exploit the fact that, even if the exact expenditure in those commodities is not observable, the bounds imposed on it are sufficiently tight that, under the observed prices, it is impossible to satisfy the weak axiom of revealed preferences over the commodities on which preferences are hypothesized to be separable.

3 Public goods

Another canonical case is when the externality is actually a public commodity: its consumption is nonrival and nonexclusive. In this case, preferences are $u^i : \mathbb{R}_+^L \times \mathbb{R}_+ \to \mathbb{R}$, and the utility level of individual i given a consumption allocation (x, y) is $u^i(x^i, \sum_j y^j)$.

For this situation to be of interest, I introduce production in the economy: I assume that there exists an aggregate technology $\mathcal{F} \subseteq \mathbb{R}^L \times \mathbb{R}$, so that a production plan of netputs (X, Y) is technologically feasible if, and only if, $(X, Y) \in \mathcal{F}$. The ownership structure of the economy is $\theta = (\theta^i)_{i=1}^I$, a vector of nonnegative numbers such that $\sum_i \theta^i = 1$.

An *economy* is now a profile of individual preferences and endowments, a technology and an ownership structure: $\mathcal{E} = (u, e, k, \theta, \mathcal{F})$. For a given economy, *a Nash–Walras equilibrium* is a vector of prices, a profile of demands and a production plan, (p, q, x, y, X, Y), such that:

1. the firm maximizes profits: plan (X, Y) solves the problem

$$\max_{\tilde{X},\tilde{Y}} p\tilde{X} + q\tilde{Y} : (\tilde{X}, \tilde{Y}) \in \mathcal{F};$$

2. every consumer is rational, given prices and the demands of other consumers: each (x^i, y^i) solves the problem

$$\max_{\tilde{x},\tilde{y}} u^i \left(\tilde{x}, \tilde{y} + \sum_{j \neq i} y^j \right) : p\tilde{x} + q\tilde{y} \leq pe^i + qk^i + \theta^i (pX + qY); \quad (14)$$

3. all markets clear: $\sum_i (x^i - e^i, y^i - k^i) = (X, Y)$.

Under nonnegativity of consumption, it is immediate that the second condition in the definition of Nash–Walras equilibrium can be replaced by the requirement that, for each individual i, the pair $(x^i, \sum_j y^j)$ solve the problem

$$\max_{\tilde{x},\tilde{y}} u^i (\tilde{x}, \tilde{y}) : \begin{cases} p\tilde{x} + q\tilde{y} \leq pe^i + q(k^i + \sum_{j\neq i} y^j) + \theta^i (pX + qY), \\ \tilde{y} \geq \sum_{j \neq i} y^j. \end{cases} \quad (15)$$

As before, let $\mathcal{W}(\mathcal{E})$ denote the set of Nash–Walras equilibria of $\mathcal{E}$, and let $W(\mathcal{E})$ be the projection of this set into the space of prices. A *data set* is now $(p_t, q_t, e_t, k_t, \theta_t)_{t=1}^T$, a finite sequence of prices, endowments and ownership structures; as before, all prices and endowments are taken to be strictly positive. A data set is *rationalizable* if there exist a profile of preferences, u, and a technology, $\mathcal{F}$, such that $(p_t, q_t) \in W(u, e_t, k_t, \theta_t, \mathcal{F})$, for every observation t.

I follow the same strategy as in the case of separable preferences, in order to show that the hypothesis of Nash–Walras equilibrium in this setting does impose testable implications: first, I obtain a (partial) characterization of rationalizability mediated by existential quantifiers; then, I argue that quantified variables can be eliminated to obtain an equivalent set of conditions on observable variables only, and use an example to show that these conditions are not tautological.

3.1 Revealed preference with public goods

First, I obtain revealed-preference conditions, in the form of Afriat inequalities, for a rational consumer facing a choice problem with a public good. For the purposes of this analysis, consider a single consumer and ignore its index. Suppose that one has observed a sequence of prices, nominal incomes, m_t, demands for all commodities, (x_t, y_t), and aggregate demand for the public good by the rest of the consumers, $\bar{y}_t$; denoted this sequence by $(p_t, q_t, m_t, x_t, y_t, \bar{y}_t)_{t=1}^T$ These data are consistent with individual rationality, in the sense of Eq. (15),[9] if there exists a utility function u such that each (x_t, y_t) solves the problem

$$\max_{\tilde{x},\tilde{y}} u(\tilde{x}, \tilde{y}) : p_t\tilde{x} + q_t\tilde{y} \leq m_t + q_t\bar{y}_t \quad \text{and} \quad \tilde{y} \geq \bar{y}_t;$$

if that is the case, we say that function u *rationalizes* the data sequence.

Lemma 2 *Fix a sequence of data* $(p_t, q_t, m_t, x_t, y_t, \bar{y}_t)_{t=1}^T$.

1. *If there exists a continuous, strictly monotone utility function u that rationalizes it, then* $p_t x_t + q_t y_t = m_t + q_t \bar{y}_t$ *and there exist numbers* u_t *and* $\lambda_t > 0$ *such that*

$$u_{t'} \leq u_t + \lambda_t(p_t(x_{t'} - x_t) + q_t(\max\{y_{t'}, \bar{y}_t\} - y_t)),$$

for all t and t′.

2. *If, for all t,* $p_t x_t + q_t y_t = m_t + q_t \bar{y}_t$ *and* $y_t \geq \bar{y}_t$*, and there exist numbers* u_t *and* $\lambda_t > 0$ *such that*

$$u_{t'} \leq u_t + \lambda_t(p_t(x_{t'} - x_t) + q_t(\max\{y_{t'}, \bar{y}_t\} - y_t)),$$

for all t and t′, then there exists a continuous, monotone utility function u that rationalizes the data sequence. If, furthermore,

$$u_{t'} \leq u_t + \lambda_t(p_t(x_{t'} - x_t) + q_t(y_{t'} - y_t)),$$

for all t and t′, then u can be taken to be concave and strictly monotone.

The result is based on the generalization of classical revealed preference theory presented by Forges and Minelli (2009). As with the other lemmas, its proof is deferred to Appendix A3.

3.2 Revealed profit maximization

Consider now an analogous exercise for the case of a firm: suppose that a sequence of prices and production plans of the firm, $(p_t, q_t, X_t, Y_t)_{t=1}^T$, has been observed. Again, the data are consistent with the profit maximization of some firm, if there exists a technology, $\mathcal{F}$, such that each production plan (X_t, Y_t) solves the program

$$\max_{\tilde{X},\tilde{Y}} p_t\tilde{X} + q_t\tilde{Y} : \left(\tilde{X}, \tilde{Y}\right) \in \mathcal{F},$$

in which case we say that technology $\mathcal{F}$ rationalizes the data sequence.

Lemma 3 *Fix a sequence of data* $(p_t, q_t, X_t, Y_t)_{t=1}^T$. *There exists a nonempty, closed, convex, negative monotonic no-free-lunch technology* $\mathcal{F}$ *that rationalizes it if, and only if,*

[9] And hence of Eq. (14).

1. *for all* t *and* t', $p_t X_{t'} + q_t Y_{t'} \leq p_t X_t + q_t Y_t$; *and*
2. *for some* $\rho \in \mathbb{R}^L_{++}$ *and some* $\varphi \in \mathbb{R}_{++}$, *it is true that* $\rho X_t + \varphi Y_t \leq 0$ *for all* t.

The following lemma extends the analysis of Varian (1984) to derive testable implications of individual profit maximization to the case of no-free-lunch technologies.

3.3 A characterization of rationalizability of a data set

Consider again a data set at the aggregate level, including prices and individual demands for all commodities, and the ownership structure of the firm, $(p_t, q_t, e_t, k_t, \theta_t)_{t=1}^T$. Define the following system of equations, over vectors x_t^i and X_t, and scalars y_t^i, u_t^i, Y_t and λ_t^i: for every individual i and every observation t,

$$x_t^i \geq 0, \quad y_t^i \geq 0 \quad \text{and} \quad \lambda_t^i > 0, \tag{16}$$

$$p_t x_t^i + q_t y_t^i = p_t e_t^i + q_t k_t^i + \theta_t^i (p_t X_t + q_t Y_t), \tag{17}$$

$$u_{t'}^i \leq u_t^i + \lambda_t^i \left(p_t \left(x_{t'}^i - x_t^i \right) \right) + q_t \left(\max \left\{ \sum_j y_{t'}^j - \sum_j y_t^j, -y_t^i \right\} \right), \tag{18}$$

$$p_t X_{t'} + q_t Y_{t'} \leq p_t X_t + q_t Y_t, \tag{19}$$

and

$$\sum_i \left(x_t^i, y_t^i \right) = \sum_i \left(e_t^i, k_t^i \right) + (X_t, Y_t). \tag{20}$$

Furthermore, we may strengthen Eq. (18) to

$$u_{t'}^i \leq u_t^i + \lambda_t^i \left(p_t \left(x_{t'}^i - x_t^i \right) + q_t \left(\sum_j y_{t'}^j - \sum_j y_t^j \right) \right); \tag{21}$$

and we may also introduce to the system two more variables, the scalars $\rho \gg 0$ and $\varphi > 0$, and the equation

$$\rho X_t + \varphi Y_t \leq 0. \tag{22}$$

Lemma 4 *Fix a data set* $(p_t, q_t, e_t, k_t, \theta_t)_{t=1}^T$.

1. *If this set is rationalizable by a nonempty technology and a profile of continuous, strictly monotone preferences, then there exists a solution to the system of Eqs.* (16)–(20).
2. *If the system of Eqs.* (16)–(20) *has a solution, then the data ser is rationalizable by a nonempty, closed, convex and negative monotonic technology, and a profile of continuous monotone preferences.*

3. *If the system of Eqs. (16)–(22) has a solution,*[10] *then the data set is rationalizable by a nonempty, closed, convex, negative monotonic, no-free-lunch technology, and a profile of continuous, strictly monotone and concave preferences.*

The system is inspired by Brown and Matzkin (1996) and Forges and Minelli (2009), and similar in spirit to Lemma 1. It provides a (weak) characterization of rationalizable data sets, which is mediated by existential quantifiers, so it does not, by itself, constitute a test of the hypothesis of rationalizability.

3.4 The test

The derivation of the test now follows, again, by eliminations of the quantifiers of the characterization.

Proposition 4 *In the case of public goods, rationalizability is testable using a finite set of inequalities on data. Formally, for any finite sequence consisting of profiles of strictly positive individual endowments and an ownership structure,* $d = (e_t, k_t, \theta_t)_{t=1}^T$, *there exist semialgebraic sets of sequences of strictly positive prices for all commodities,* Δ_d^C *and* Δ_d, *with* $\Delta_d^C \subseteq \Delta_d$, *such that*:

1. *if* $(p_t, q_t, e_t, k_t, \theta_t)_{t=1}^T$ *is rationalizable by a nonempty technology and a profile of continuous, strictly monotone preferences, then* $(p_t, q_t)_{t=1}^T \in \Delta_d$;
2. *if* $(p_t, q_t)_{t=1}^T \in \Delta_d$, *then* $(p_t, q_t, e_t, k_t, \theta_t)_{t=1}^T$ *is rationalizable by a nonempty, closed, convex and negative monotonic technology, and a profile of continuous, monotone preferences*;
3. *if* $(p_t, q_t)_{t=1}^T \in \Delta_d^C$, *then* $(p_t, q_t, e_t, k_t, \theta_t)_{t=1}^T$ *is rationalizable by a nonempty, closed, convex, negative monotonic no-free-lunch technology, and a profile of strictly monotone and concave preferences.*

Also, there exist d *for which* Δ_d *is a proper subset of* $(\mathbb{R}_{++}^L \times \mathbb{R}_{++})^T$.

Proof As before, notice that, given $(e_t, k_t, \theta_t)_{t=1}^T$, the systems defined by Eqs. (16)–(20) and by Eqs. (16)–(22) are finite sets of polynomial inequalities, so the sets of their solutions are semialgebraic. Let Δ_d be Δ_d^C the projections into the space of sequences of prices of the set of solutions to these systems, respectively. By the Tarski–Seidenberg Theorem, these sets are semialgebraic, and the result then follows from Lemma 4 and Example 2 below. □

3.5 Nonrationalizable data

In a production economy, one might expect refutability to fail because: (i) if profits are not observed, individual incomes are undetermined and hence the restrictions imposed by individual rationality may be weakened; and (ii) in the presence of production, nonnegativity constraints on consumption are less informative: production may transform

[10] Of course, in this case Eq. (18) is redundant.

endowments so as to allow consumption allocations *outside* the original Edgeworth boxes. Introducing production, however, adds profit maximization as an element of the model and this additional structure may as well be a source of refutability for the hypothesis.

On the other hand, the assumption of a public good yields sufficient structure to impose empirical implications on individually-rational behavior. While it is true that the decisions of other agents in the economy induce changes in the utility function of each individual in terms of the variables that she chooses, the fact that these effects have the public-good structure implies that one can see the problem as the one of an individual who always maximizes a constant utility function, only subject to not-necessarily-linear budget constraints: the intuition of the construction of Lemma 2 and Proposition 4 is the same as in Lindahl's canonical analysis, where one models individual behavior as if each person were to choose the aggregate level of provision of the public good; here, though, instead of introducing personalized prices and allowing for unconstrained trade of the good, I maintain the anonymous price, but restrict each person's aggregate demand for the public good to be at least what the data reveals to be the aggregate private demand of the rest of the economy. The case in which personalized, Lindahl prices are introduced (although they are not assumed to be observed) was studied by Snyder (1999) and constitutes a test of an alternative hypothesis: that the economy has provided the public good at a Pareto efficient level.

The argument that the test derived in Proposition 4 is nontautological is given, once again, by an example of data that cannot be rationalized under the hypothesis of public goods. Consider the following case.

Example 2 Suppose that $I = 2$ and $L = 1$. Suppose that a data set includes the following two observations:

$$\begin{aligned}
&e_1^1 = 9, && e_2^1 = 1, && e_1^2 = 1, && e_2^2 = 1,\\
&k_1^1 = 1, && k_2^1 = 9, && k_1^2 = 1, && k_2^2 = 1,\\
&\theta_1^1 = 1, && \theta_2^1 = 1, && \theta_1^2 = 0, && \theta_2^2 = 0,\\
&p_1 = 100, && p_2 = 1, && q_1 = 1, && q_2 = 100.
\end{aligned}$$

Suppose that a data set is rationalized by a nonempty technology and a profile of continuous, strictly monotone preferences. Let $x_t^i \geq 0$ and $y_t^i \geq 0$, for $i, t = 1, 2$, be the associated consumption levels, and let (X_t, Y_t), for $t = 1, 2$, be the associated production plans. Profit maximization, monotonicity of preferences and market clearing immediately imply the following conditions:

$$p_1 X_1 + q_1 Y_1 \geq p_1 X_2 + q_1 Y_2, \tag{23}$$

$$p_2 X_2 + q_2 Y_2 \geq p_2 X_1 + q_2 Y_1, \tag{24}$$

$$p_1 x_1^1 + q_1 y_1^1 = p_1 \left(e_1^1 + X_1\right) + q_1 \left(k_1^1 + Y_1\right), \tag{25}$$

$$p_2 x_2^1 + q_2 y_2^1 = p_2 \left(e_2^1 + X_2\right) + q_2 \left(k_2^1 + Y_2\right), \tag{26}$$

$$p_1 x_1^2 + q_1 y_1^2 = p_1 e_1^2 + q_1 k_1^2, \tag{27}$$

$$p_2 x_2^2 + q_2 y_2^2 = p_2 e_2^2 + q_2 k_2^2, \tag{28}$$

$$x_1^1 + x_1^2 = e_1^1 + e_1^2 + X_1, \tag{29}$$

$${}^1_2 + x_2^2 = e_2^1 + e_2^2 + X_2, \tag{30}$$

$$y_1^1 + y_1^2 = k_1^1 + k_1^2 + Y_1, \tag{31}$$

$$y_2^1 + y_2^2 = k_2^1 + k_2^2 + Y_2. \tag{32}$$

By direct computation (see Appendix A2), these conditions imply, furthermore, that

$$p_1 x_2^1 + q_1 \left(y_2^1 + y_2^2\right) < p_1 \left(e_1^1 + X_1\right) + q_1 \left(k_1^1 + Y_1\right) + q_1 y_1^2, \tag{33}$$

$$p_2 x_1^1 + q_2 \left(y_1^1 + y_1^2\right) < p_2 \left(e_2^1 + X_2\right) + q_2 \left(k_2^1 + Y_2\right) + q_2 y_2^2, \tag{34}$$

$$p_1 x_2^1 < p_1 e_1^1 + q_1 k_1^1 + p_1 X_1 + q_1 Y_1, \tag{35}$$

$$p_2 x_1^1 < p_2 e_2^1 + q_2 k_2^1 + p_2 X_2 + q_2 Y_2. \tag{36}$$

But this is impossible, as it contradicts individual rationality for individual 1. Indeed, since each $(x_t^1, y_t^1 + y_t^2)$ solves the problem

$$\max_{\tilde{x},\tilde{y}} u^1(\tilde{x}, \tilde{y}) : \begin{cases} p_t \tilde{x} + q_t \tilde{y} \leq p_t (e_t^1 + X_t) + q_t (k_t^i + Y_t) + q_t y_t^2, \\ \tilde{y} \geq y_t^2, \end{cases}$$

it must also solve the problem

$$\max_{\tilde{x},\tilde{y}} u^1(\tilde{x}, \tilde{y}) : \begin{cases} p_t \tilde{x} + q_t \tilde{y} \leq p_t (e_t^1 + X_t) + q_t (k_t^i + Y_t) + q_t y_t^2, \\ p_t \tilde{x} \leq p_t (e_t^1 + X_t) + q_t (k_t^i + Y_t). \end{cases}$$

But then, it follows from Proposition 1 in Forges and Minelli (2009) that the following acyclicity condition must be observed: for $t \neq t'$, if

$$p_t x_{t'}^1 + q_t \left(y_{t'}^1 + y_{t'}^2\right) \leq p_t \left(e_t^1 + X_t\right) + q_t \left(k_t^1 + Y_t\right) + q_t y_t^2$$

and $p_t x_{t'}^1 \leq p_t \left(e_t^1 + X_t\right) + q_t \left(k_t^1 + Y_t\right)$, then, either

$$p_{t'} x_t^1 + q_{t'} \left(y_t^1 + y_t^2\right) \geq p_{t'} \left(e_{t'}^1 + X_{t'}\right) + q_{t'} \left(k_{t'}^1 + Y_{t'}\right) + q_{t'} y_{t'}^2,$$

or $p_{t'} x_t^1 \geq p_{t'} \left(e_{t'}^1 + X_{t'}\right) + q_{t'} \left(k_{t'}^1 + Y_{t'}\right)$.

It follows, once again, that the assumption that the externality is in the form of a produced public good gives testable restrictions on prices, given the observation of real endowments and of the ownership structure of the economy.

4 Further information

The previous results show that further restrictions on the class of fundamentals allowed in a rationalization may (but need not) restore testability for the competitive hypothesis under externalities. Alternatively, I now show that if one observes individual demands for the externality commodity, some restrictions arise even under the more general classes of preferences. I consider again the simpler case of an exchange economy with general types of externalities.

Some concepts need to be redefined, although I keep the existing notation, for simplicity. Now, a data set is $(p_t, q_t, e_t, k_t, y_t)_{t=1}^T$, a finite sequence that includes information on y_t^i for every individual and observation. I assume, again, that all observed prices, endowments and consumptions are strictly positive, and that $\sum_i (y_t^i - k_t^i) = 0$, for all t, and $p_t e_t^i + q_t(k_t^i - y_t^i) \geq 0$, for all i and all t.

Consistently, $W(\mathcal{E})$ now denotes the projection of the set of Nash–Walras equilibria of economy $\mathcal{E}$ into the space of prices and individual consumptions of the externality, and a data set is *rationalized* by a profile of preferences u, if $(p_t, q_t, y_t) \in W(u, e_t, k_t)$, for every observation t.

4.1 A characterization and the test

Once again, given a data set we can define a system over vectors x_t^i and ρ_t^i, and numbers u_t^i, θ_t^i and λ_t^i, as follows: for each individual i and each observation t,

$$x_t^i \geq 0, \quad \rho_t^i \gg 0, \quad \theta_t^i > 0 \quad \text{and} \quad \lambda_t^i > 0, \tag{37}$$

$$\rho_t^i = \lambda_t^i p_t \quad \text{and} \quad \theta_t^i = \lambda_t^i q_t, \tag{38}$$

$$p_t x_t^i = p_t e_t^i + q_t \left(k_t^i - y_t^i\right), \tag{39}$$

$$u_{t'}^i \leq u_t^i + \rho_t^i \left(x_{t'}^i - x_t^i\right) + \theta_t^i \left(y_{t'}^i - y_t^i\right), \quad \text{whenever } y_{t'}^{\neg i} = y_t^{\neg i}, \tag{40}$$

and

$$\sum_i x_t^i = \sum_i e_t^i. \tag{41}$$

Lemma 5 *Fix a data set is* $(p_t, q_t, e_t, k_t, y_t)_{t=1}^T$.

1. *If the set is rationalizable by a profile of continuous preferences that are locally nonsatiated in own consumption, then there exists a solution to the system defined by Eqs.* (37)–(41).
2. *If the system has a solution, then the set is rationalizable by a profile of continuous, concave and strongly monotone preferences.*

With this characterization, the testability result is the following proposition. It follows that, upon observation of prices, individual endowments and individual demands for the externality commodity, the hypothesis of Nash–Walras equilibrium is testable.

However, it is important to notice how mild these (exhaustive) restrictions are: any feasible data set generated randomly with nonatomic measures is rationalizable with full probability.

Proposition 5 *Rationalizability under observation of demands for the externality is testable using a finite set of inequalities on data. Formally, for any finite sequence of profiles of strictly positive individual endowments and demands for the externality, $d = (e_t, k_t, y_t)_{t=1}^{T}$, there exists a semialgebraic set of sequences of strictly positive prices for all commodities, Δ_d, such that:*

1. *if $(p_t, q_t, e_t, k_t, y_t)_{t=1}^{T}$ is rationalizable by a profile of continuous preferences that satisfy concavity and strong monotonicity on own demands, then $(p_t, q_t)_{t=1}^{T} \in \Delta_d$;*
2. *if $(p_t, q_t)_{t=1}^{T} \in \Delta_d$, then $(p_t, q_t, e_t, k_t, y_t)_{t=1}^{T}$ is rationalizable by a profile of continuous, concave and strongly monotone preferences.*

Also, there exist d for which Δ_d is a proper subset of $(\mathbb{R}^L_{++} \times \mathbb{R}_{++})^T$.

Proof The argument is similar to the proof of Propositions 3 and 4: the system defines a semialgebraic set, so its projection into the space of prices is semialgebraic as well; Lemma 5 then yields the two statements, while the fact that there are d for which $\Delta_d \subsetneq (\mathbb{R}^L_{++} \times \mathbb{R}_{++})^T$ follows from Example 3 below. □

4.2 Nonrationalizable data

The following example completes the proof of Proposition 5:

Example 3 Suppose that $I = L = 2$. The information of the data set with partial observation includes the following two observations:

$$\begin{array}{llll} e_1^1 = (1, 4), & e_2^1 = (4, 1), & e_1^2 = (2, 1), & e_2^2 = (1, 2), \\ k_1^1 = 1, & k_2^1 = 0.5, & k_1^2 = 1, & k_2^2 = 1.5, \\ p_1 = (1, 10), & p_2 = (10, 1), & q_1 = 1, & q_2 = 2, \\ y_1^1 = 0, & y_2^1 = 0, & y_1^2 = 2, & y_2^2 = 2. \end{array}$$

Suppose that the data set is rationalized by preference profile (u^1, u^2). Since $y_1^2 = y_2^2$, it follows that consumer 1 maximizes the same utility function, $u^1(\cdot, \cdot, y_1^2)$, at both observations. Let x_t^1, for $t = 1, 2$, be such that each (x_t^1, y_t^1) solves the problem

$$\max_{x^1, y^1} u^1\left(x^1, y^1, y_t^2\right) : p_t\left(x^1 - e_t^1\right) + q_t\left(y^1 - k_t^1\right) \leq 0.$$

Define

$$X_1 = \{x | x_1 \in [0, 3], x_2 = 4.2 - 0.1 x_1\}$$

and

$$X_2 = \{x | x_1 \in [3.9, 4.2], x_2 = 42 - 10 x_1\}.$$

Since

$$p_1 e_1^1 + q_1 \left(k_1^1 - y_1^1\right) = 42 = p_2 e_2^1 + q_2 \left(k_2^1 - y_2^1\right),$$

$e_1^1 + e_1^2 = (3, 5)$ and $e_2^1 + e_2^2 = (5, 3)$, it follows that $x_1^1 \in X_1$ and $x_2^1 \in X_2$. Since $X_1 \cap X_2 = \varnothing$, then $x_1^1 \neq x_2^1$. Since $x_1^1 \in X_1$, then

$$p_2 x_1^1 + q_2 y_1^1 = 9.9 x_{1,1}^1 + 4.2 \leq 9.9(3) + 4.2 < 42 = p_2 e_2^1 + q_2 k_2^1,$$

whereas, since $x_2^1 \in X_2$, by a similar argument, $p_1 x_2^1 + q_1 y_2^1 < p_1 e_1^1 + q_1 k_1^1$. But this is impossible, since it constitutes a violation of the weak axiom of revealed preferences, which applies.

5 Concluding remarks

The results obtained here show that the assumption that there exist consumption externalities significantly weakens the testable implications that the hypothesis of competitive equilibrium imposes on the response of commodity prices to observed perturbations in individual wealth levels. Under basic assumptions on individual preferences, the Nash–Walras equilibrium hypothesis imposes no restrictions on finite subsets of the equilibrium manifold of the economy. Intuitively, this occurs because the presence of externalities allows for the construction of utility functions whose cross-sections with respect to own consumption are maximized by each agent at only one of the observations in the data sample; immediately, revealed preference arguments are vacuous, and the tensions existing between individual rationality and market clearing, which yields the results of Brown and Matzkin (1996), disappears. Importantly, this is the case even when one imposes the hypothesis that all individual preferences exhibit strategic complementarity in the consumption of the commodity that causes the externality commodity, which contrasts with the literature on monotone comparative statics, and with the application of this literature to abstract games.

Stronger restrictions on unobservables, or the observation of individual consumptions of the externality, are required to restore testability. Separability of preferences in own consumption would yield the results trivially: if consumption of the commodity that has external effects by other agents of the economy is assumed to not affect each individual's behavior, the restrictions obtained for the case when there are no externalities immediately hold. What is interesting, though, is how much less separability needs to be assumed if some restrictions are to be maintained: it suffices for preferences to be weakly separable in two of the commodities traded in competitive markets.

Another important case corresponds to the hypothesis that the externality takes the form a public good. Here, I have considered the case in which production takes place, and have, for simplicity, assumed the existence of only one firm. In terms of data, I have assumed that the ownership structure over this aggregate technology is observed as well. The results show that the structure imposed by the principles of profit maximization and individual rationality, along with the structure imposed by the hypothesis of the public good, suffice to yield testable implications on prices, endowments and stock distribution.

Finally, I have argued that for general classes of preferences, under observation of individual demands for the externality, some nontautological restrictions do exist. Inspection of these conditions, however, shows that the exhaustive set of restrictions imposed is extremely mild.

Some of the results obtained here are easy to generalize. For example, at the cost of some technical complication, one can substitute the assumption that the externalities come from consumption of some commodity, for one in which they stem from some abstract action on continuous individual sets. In this case, of course, there are no prices attached to the externality, but arbitrary bounds may be imposed instead. The results obtained here extend to that setting, under the proviso that preferences be restricted to be locally Lipschitzian. The case considered here, consumption externalities, is then a particular case of that abstract setting, although some of the results obtained here can only be derived from the general case for compact arbitrary subsets of the domain.

Appendix A1: An algorithm that gives Eqs. (1)–(4)

Rationalizations for any data set can be constructed on the basis the following algorithm. Here, the notation $\mathbf{e} = (1, 0, \ldots, 0) \in \mathbb{R}^L$ is used.

Algorithm *Input: a data set* $(p_t, q_t, e_t, k_t)_{t=1}^T$

1. $t = 1$.
2. *If* $y_{t'}^i \neq k_t^i$, *for all* i *and all* $t' < t$, *then let* $x_t^i = e_t^i$ *and* $y_t^i = k_t^i$, *for all* i, *and go to step 7.*
3. *Let* $\mathcal{J} = \{i | \exists t' < t : y_{t'}^i \neq k_t^i\}$.
4. *If* $\mathcal{J} = \varnothing$, *let* $\epsilon = 1$ *and go to step 6.*
5. *Let* $\epsilon = \min_{i \in \mathcal{J}}\{\min_{t' < t : y_{t'}^i \neq k_t^i}\{|y_{t'}^i - k_t^i|\}\}$.
6. *Let* $\gamma = \min_{i \neq 1}\{\frac{(I-1)p_{t,1}e_{t,1}^i}{q_t}\}$, $\delta = \frac{1}{2}\min\{\epsilon, k_t^1, \gamma\}$, $y_t^1 = k_t^1 - \delta$, $x_t^1 = e_t^1 + \frac{q_t\delta}{p_{t,1}}\mathbf{e}$, *and, for every* $i \neq 1$, $y_t^i = k_t^i + \frac{\delta}{I-1}$ *and* $x_t^i = e_t^i - \frac{q_t\delta}{p_{t,1}(I-1)}\mathbf{e}$.
7. *If* $t = T$, *stop. Else,* $t = t + 1$ *and go to step 2.*

Output: $(x_t, y_t)_{t=1}^T$.

Note first that this algorithm runs in finite time. Also, note that, since T is finite, if δ is defined at some pass through the algorithm, then $\delta > 0$. Equations (1)–(3) are hence immediate.

For Eq. (4), it suffices to show that if at the tth pass through the algorithm, it is true that $y_{t'}^i \neq y_{t''}^i$ for every distinct $t', t'' < t$, then $y_{t'}^i \neq y_t^i$ for all $t' < t$. This is tautological if $t = 1$, and follows from step 2 if $(x_t^i, y_t^i)_{i=1}^I = (e_t^i, k_t^i)_{i=1}^I$. Now, consider $t \geq 2$ and assume that $(x_t^i, y_t^i)_{i=1}^I$ is given by steps 3–6. Consider three different cases:

Case 1 If $t = 2$ and $\mathcal{J} = \varnothing$. Then, $k_2^i = y_1^i$ for all i, and since $\delta > 0$, follows that $y_2^i \neq k_2^i = y_1^i$.

Case 2 If $t = 2$ and $\mathcal{J} \neq \varnothing$. Then, if $1 \notin \mathcal{J}$, it is true that $y_1^1 = k_2^1$, and, since $\delta > 0$, it follows that $y_2^1 = k_2^1 - \delta = y_1^1 - \delta \neq y_1^1$. Else, $1 \in \mathcal{J}$ and one has that if

$y_1^1 = y_2^1$, then, since $y_2^1 = k_2^1 - \delta$, so it would follow that

$$\left|y_1^1 - k_2^1\right| = \delta \leq \frac{\epsilon}{2} < \epsilon \leq \left|y_1^1 - k_2^1\right|,$$

an obvious contradiction. On the other hand, for each $i \in \mathcal{I} \backslash (\mathcal{J} \cup \{1\})$, it is true that $y_1^i = k_2^i$, and, since $\delta > 0$, it follows that

$$y_2^i = k_2^i + \frac{\delta}{I-1} = y_1^i + \frac{\delta}{I-1} \neq y_1^i.$$

Finally, for each $i \in \mathcal{J} \backslash \{1\}$, if one had that $y_1^i = y_2^i$, then, since $y_2^i = k_2^i + \frac{\delta}{I-1}$ and $\delta > 0$, it would follow that

$$\left|y_1^i - k_2^i\right| = \frac{\delta}{I-1} \leq \delta \leq \frac{\epsilon}{2} < \epsilon \leq \left|y_1^i - k_2^i\right|,$$

again, a contradiction.

Case 3 If $t \geq 3$. In this case, by the induction assumption, $\mathcal{J} = \mathcal{I}$, from where, if there exists $t' \leq t-1$ with $y_{t'}^1 = y_t^1$, then $y_t^1 = k_t^1 - \delta$ and $\delta > 0$ would imply that $y_{t'}^1 \neq k_t^1$ and

$$\left|y_{t'}^1 - k_t^1\right| = \delta \leq \frac{\epsilon}{2} < \epsilon \leq \min_{t'' \leq t-1 : y_{t''}^1 \neq k_t^1} \left\{\left|y_{t''}^1 - k_t^1\right|\right\},$$

an obvious contradiction. Similarly, if for some $i \neq 1$, one had that for some $t' \leq t-1$ it is true that $y_{t'}^i = y_t^i$, then, $y_t^i = k_t^i + \frac{\delta}{I-1}$ and $\delta > 0$ imply that $y_{t'}^i \neq k_t^i$ and

$$\left|y_{t'}^i - k_t^i\right| = \frac{\delta}{I-1} \leq \delta \leq \frac{\epsilon}{2} < \epsilon \leq \min_{t'' \leq t-1 : y_{t''}^i \neq k_t^i} \left\{\left|y_{t''}^i - k_t^i\right|\right\},$$

again a contradiction.

Appendix A2: Proofs of claims

The following claims apply to the data set defined in Example 2.

Claim 1 $X_1 \geq X_2 \geq -2$ *and* $Y_2 \geq Y_1 \geq -2$ *Moreover,* $y_2^1 + y_2^2 > k_1^1 + y_1^2$.

Proof From Eqs. (23) and (24), by direct substitution,

$$100X_1 \geq 100X_2 + (Y_2 - Y_1) \geq 100X_2 + \frac{1}{100}(X_1 - X_2),$$

so $X_1 \geq X_2$. Also, by Eq. (30), given nonnegativity, $X_2 = x_2^1 + x_2^2 - e_2^1 - e_2^2 \geq -e_2^1 - e_2^2 = -2$. That $Y_2 \geq Y_1 \geq -2$ is proven similarly. Now, Eq. (32) and the previous inequality imply that

$$y_2^1 + y_2^2 = 10 + Y_2 \geq 10 + Y_1 > 3 + Y_1 \geq 3 + Y_1 - y_1^1.$$

But, then,

$$y_2^1 + y_2^2 > \left(2k_1^1 + k_1^2\right) + Y_1 - y_1^1 = k_1^1 + y_1^2.$$

by direct substitution and Eq. (31). □

Claim 2

$$p_1 x_2^1 + +q_1 \left(y_2^1 + y_2^2\right) < p_1 \left(e_1^1 + X_1\right) + q_1 \left(k_1^1 + Y_1\right) + q_1 y_1^2.$$

Proof By direct substitution and Eq. (26),

$$\begin{aligned} p_1 x_2^1 + q_1 \left(y_2^1 + y_2^2\right) &= 100 x_2^1 + y_2^1 + y_2^2 \\ &= 9.01 + \frac{1}{100}(X_2 + 100 Y_2) \\ &\quad + \left(100 - \frac{1}{100}\right)\left(e_2^1 + e_2^2 + X_2 - x_2^2\right) + y_2^2, \end{aligned}$$

where the last equality comes from Eq. (30). By direct substitution, nonnegativity and Eq. (28), then,

$$\begin{aligned} p_1 x_2^1 + q_1 \left(y_2^1 + y_2^2\right) &= 208.99 + 100 X_2 + Y_2 - \left(100 - \frac{1}{100}\right) x_2^2 + y_2^2 \\ &\leq 208.99 + 100 X_2 + Y_2 + y_2^2 \\ &= 208.99 + p_1 X_2 + q_1 Y_2 + \frac{p_2 e_2^2 + q_2 k_2^2 - p_2 x_2^2}{q_2} \\ &\leq 208.99 + p_1 X_2 + q_1 Y_2 + \frac{p_2 e_2^2 + q_2 k_2^2}{q_2} \\ &= 211 + p_1 X_2 + q_1 Y_2. \end{aligned}$$

By Eq. (23), then,

$$\begin{aligned} p_1 x_2^1 + q_1 \left(y_2^1 + y_2^2\right) &\leq 211 + p_1 X_1 + q_1 Y_1 \\ &< 901 + p_1 X_1 + q_1 Y_1 \\ &= p_1 e_1^1 + q_1 k_1^1 + p_1 X_1 + q_1 Y_1 \\ &\leq p_1 e_1^1 + q_1 k_1^1 + p_1 X_1 + q_1 Y_1 + q_1 y_1^2, \end{aligned}$$

where the last inequality comes from nonnegativity. □

Claim 3

$$p_2x_1^1 + +q_2\left(y_1^1 + y_1^2\right) < p_2\left(e_2^1 + X_2\right) + q_2\left(k_2^1 + Y_2\right) + q_2y_2^2.$$

Proof By direct substitution, using Eqs. (25) and (31),

$$\begin{aligned} p_2x_1^1 + q_2\left(y_1^1 + y_1^2\right) &= x_1^1 + 100\left(y_1^1 + y_1^2\right) \\ &= 208.99 + X_1 + 100Y_1 + \frac{1}{100}y_1^2 \\ &= 208.99 + p_2X_1 + q_2Y_1 + \frac{1}{100}\frac{p_1\left(e_1^2 - x_1^2\right) + q_1k_1^2}{q_1}, \end{aligned}$$

where the last equality comes from Eq. (27). Then,

$$\begin{aligned} p_2x_1^1 + q_2\left(y_1^1 + y_1^2\right) &\leq 208.99 + p_2X_2 + q_2Y_2 + \frac{1}{100}\frac{p_1e_1^2 + q_1k_1^2}{q_1} \\ &= 211 + p_2X_2 + q_2Y_2 \\ &< 901 + p_2X_2 + q_2Y_2 \\ &\leq 901 + p_2X_2 + q_2Y_2 + q_2y_2^2 \\ &= p_2e_2^1 + q_2k_2^1 + p_2X_2 + q_2Y_2 + q_2y_2^2, \end{aligned}$$

by Eq. (24) and nonnegativity. □

Claim 4

$$p_1x_2^1 < p_1e_1^1 + q_1k_1^1 + p_1X_1 + q_1Y_1.$$

Proof By Claim 2,

$$p_1x_2^1 + q_1\left(y_2^1 + y_2^2\right) < p_1\left(e_1^1 + X_1\right) + q_1\left(k_1^1 + Y_1\right) + q_1y_1^2.$$

It must then be that

$$p_1x_2^1 < p_1\left(e_1^1 + X_1\right) + q_1Y_1 < p_1\left(e_1^1 + X_1\right) + q_1\left(k_1^1 + Y_1\right),$$

since, by Claim 1, $y_2^1 + y_2^2 > k_1^1 + y_1^2$. □

Claim 5 $p_2x_1^1 < p_2e_2^1 + q_2k_2^1 + p_2X_2 + q_2Y_2$.

Proof By direct substitution and Eqs. (25), (27) and (31),

$$p_2x_1^1 = 8.99 + X_1 + \frac{1}{100}\left(p_1\left(e_1^2 - x_1^2\right) + q_1k_1^2\right),$$

By nonnegativity and direct substitution, then,

$$\begin{aligned} p_2 x_1^1 &\leq 8.99 + X_1 + \frac{1}{100}\left(p_1 e_1^2 + q_1 k_1^2\right) \\ &= 11 + X_1 \\ &< 701 + X_1. \end{aligned}$$

Since, by Claim 1, $Y_1 \geq -2$, it follows that

$$\begin{aligned} p_2 x_1^1 &< 901 + X_1 + 100 Y_1 \\ &= 901 + p_2 X_1 + q_2 Y_1 \\ &\leq 901 + p_2 X_2 + q_2 Y_2 \\ &= p_2 e_2^1 + q_2 k_2^1 + p_2 X_2 + q_2 Y_2, \end{aligned}$$

by Eq. (24). □

Appendix A3: Proofs the lemmata

Proof of Lemma 1 For the first part, by definition of equilibrium there exist individual consumption plans $(x_t^i, y_t^i) \gg 0$ that solve the problem

$$\max_{\tilde{x}, \tilde{y}} u^i\left(\tilde{x}, \tilde{y}, y_t^{\neg i}\right) : p_t \tilde{x} + q_t \tilde{y} \leq p_t e_t^i + q_t k_t^i.$$

and satisfy Eq. (13). Equation (10) follows immediately from monotonicity of u^i. By separability and strict monotonicity of U^i in its first argument it must be that x_t^i solves

$$\max_{\tilde{x}} V^i(\tilde{x}) : p_t \tilde{x} \leq p_t e_t^i + q_t\left(k_t^i - y_t^i\right),$$

so, by the Kühn–Tucker theorem, there exist $\lambda_t^i > 0$ and $\mu_t^i > 0$ such that $\partial_{(x^i, y^i)} u^i \left(x_t^i, y_t^i, y_t^{\neg i}\right) = \mu_t^i (p_t, q_t)$ and $\partial V^i(x_t^i) = \lambda_t^i q_t$. Equation (9) follows if we define $\rho_t^i = \lambda_t^i p_t$ and $\theta_t^i = \mu_t^i q_t$. By concavity of V^i, it follows that $V^i(x^i) \leq V^i(x_t^i) + \theta_t^i(x^i - x_t^i)$. By the chain rule,

$$\partial_{V^i} U^i\left(V^i\left(x_t^i\right), y_t^i, y_t^{\neg i}\right) = \left(\partial V^i\left(x_t^i\right)\right)^{-1} \partial_{x^i} u^i\left(x_t^i, y_t^i, y_t^{\neg i}\right) = \frac{\mu_t^i p_{t,1}}{\lambda_t^i p_{t,1}} = \frac{\mu_t^i}{\lambda_t^i},$$

so it follows, by convexity of U^i, that

$$U^i\left(V^i, y^i, y_t^{\neg i}\right) \leq U^i\left(V^i\left(x_t^i\right), y_t^i, y_t^{\neg i}\right) + \theta_t^i\left(y^i - y_t^i\right) + \frac{\mu_t^i}{\lambda_t^i}\left(V^i - V^i\left(x_t^i\right)\right).$$

Defining $v_t^i = V^i(x_t^i)$ and $u_t^i = U^i(V_t^i, y_t^i, y_t^{\neg i})$ yields Eqs. (11) and (12).

For the second part, fix an individual i and notice that, again by Theorem of the Alternative for systems of weak inequalities, Rockafellar (1970, §22.1), the exist real numbers w_t^i and vectors ν_t^i such that

$$w_{t'}^i \leq w_t^i + \frac{\mu_t^i}{\lambda_t^i}\left(v_{t'}^i - v_t^i\right) + \theta_t^i\left(y_{t'}^i - y_t^i\right) + \nu_t^i\left(y_{t'}^{\neg i} - y_t^{\neg i}\right).$$

Then, we can define functions

$$V^i(x^i) = \min_t \left\{V_t^i + \rho_t^i\left(x^i - x_t^i\right)\right\},$$

and

$$U^i(V, y^i, y^{\neg i}) = \min_t \left\{w_t^i + \frac{\mu_t^i}{\lambda_t^i}\left(V - v_t^i\right) + \theta_t^i\left(y^i - y_t^i\right) + \nu_t^i\left(y^{\neg i} - y_t^{\neg i}\right)\right\},$$

to obtain a profile $(u^i)_{i=1}^I$ of continuous, concave and weakly separable preferences that are strictly monotone in (x^i, y^i), while strict monotonicity of each u^i in $y^{\neg i}$ can be obtained as in the proof of Proposition 2. This profile rationalizes the data set. □

Proof of Lemma 2 For part 1, define, for each t, the continuous and increasing function

$$g_t(x, y) = \max\{p_t x + q_t y - m_t - q_t \bar{y}_t, p_t x - m_t\}.$$

By construction, since u is strictly monotone, $p_t x + q_t y = m_t + q_t \bar{y}_t$ and it is true that each (x_t, y_t) solves the program

$$\max_{x,y} u(x, y) : p_t x + q_t y \leq m_t + q_t \bar{y}_t \quad \text{and} \quad p_t x \leq m_t.$$

This means that each (x_t, y_t) solves the program

$$\max_{x,y} u(x, y) : g_t(x, y) \leq 0,$$

and that $g_t(x_t, y_t) = 0$. It then follows from Proposition 1 in Forges and Minelli (2009) that there exist numbers u_t and $\lambda_t > 0$, for all t, such that $u_{t'} \leq u_t + \lambda_t g_t(x_{t'}, y_{t'})$, for all t and t'. For part 2, define the continuous and monotone function

$$u(x, y) = \min_t\{u_t + \lambda_t(p_t(x - x_t) + q_t(\max\{y, \bar{y}_t\} - y_t))\}.$$

Fix t and suppose that $p_t x + q_t y \leq m_t + q_t \bar{y}_t$ and $y \geq \bar{y}_t$. It is immediate that $u(x, y) \leq u_t$, while, by the conditions on u_t and λ_t, $u(x_t, y_t) = u_t$. Under the extra hypotheses, letting

$$u(x, y) = \min_t\{u_t + \lambda_t(p_t(x - x_t) + q_t(y - y_t))\}$$

gives concavity and monotonicity. □

Proof of Lemma 3 For necessity, the first item follows directly from profit maximization. Now, suppose that for no $(\rho, \varphi) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ is it true that $\rho X_t + \varphi Y_t \leq 0$ for all t. It follows from Theorem 22.2 in Rockafellar (1970), again, that there exist $\alpha \in \mathbb{R}^T_+$ and $\beta \in \mathbb{R}^{L+1}_+ \backslash \{0\}$, such that

$$\sum_t \alpha_t (X_t, Y_t) = \beta > 0.$$

It follows that $\alpha > 0$ and, by convexity of $\mathcal{F}$, that

$$0 < (X, Y) = \frac{1}{\sum_t \alpha_t} \sum_t \alpha_t (X_t, Y_t) \in \mathcal{F},$$

contradicting no-free-lunch.

For sufficiency, as in Varian (1984), let $\mathcal{F}$ be the convex hull of the set $\cup_t ((X_t, Y_t) - \mathbb{R}^L_+ \times \mathbb{R}_+)$, which is nonempty, closed and convex. That each (X_t, Y_t) solves

$$\max_{(X,Y) \in \mathcal{F}} p_t X + q_t Y$$

follows from Theorem 2 in Varian (1984). To see that Y satisfies no free lunch, suppose that $(X, Y) \in \mathcal{F}$ and $(X, Y) > 0$; by construction, we can find a sequence $(\tilde{X}_t, \tilde{Y}_t, \alpha_t)_{t=1}^T$ such that $(\tilde{X}_t, \tilde{Y}_t) \leq (X_t, Y_t)$ and $0 \leq \alpha_t \leq 1$, for all t, and $\sum_t \alpha_t (\tilde{X}_t, \tilde{Y}_t) = (X, Y)$. Then,

$$0 < \rho X + \varphi Y = \rho \sum_t \alpha_t \tilde{X}_t + \varphi^t \tilde{y}^t \leq \rho \cdot \sum_{t=1}^T \alpha^t y^t = \sum_{t=1}^T \alpha^t \rho \cdot y^t \leq 0,$$

an obvious impossibility. □

Proof of Lemma 4 The first statement follows from Lemma 2, part 1, and is routine.

For part 2, fix the vectors x^i_t and X_t, and the numbers y^i_t, u^i_t, Y_t and λ^i_t that solve the system of Eqs. (16)–(20). Notice that, by the second part of Lemma 2, there exists continuous, monotone utility functions u^i such that each $(x^i_t, \sum_j y^j_t)$ solves the problem

$$\max_{\tilde{x}, \tilde{y}} u^i(\tilde{x}, \tilde{y}) : \begin{cases} p_t \tilde{x} + q_t \tilde{y} \leq p_t e^i_t + q_t k^i_t + \theta^i_t (p_t X_t + q_t Y_t) + q_t \sum_{j \neq i} y^j_t, \\ \tilde{y} \geq \sum_{j \neq i} y^j_t. \end{cases}$$

Also, by Theorem 3 in Varian (1984), there exists a nonempty, closed, convex and negative monotonic technology, $\mathcal{F}$, such that each (X_t, Y_t) solves the program

$$\max_{\tilde{X}, \tilde{Y}} p_t \tilde{X} + q_t \tilde{Y} : \left(\tilde{X}, \tilde{Y}\right) \in \mathcal{F}.$$

The conclusion follows, since markets clear by Eq. (20).

Given a solution to system (16)–(22), the proof of the third statement is similar, using Lemmas 2 and 3. □

Proof of Lemma 5 The first part follows from Walras's law and Afriat's theorem (see Varian 1982), by definition of rationalizability.

The proof of the second part is similar to the proof of the second part of Lemma 1, so details can be omitted. System (37)–(41) implies, again by the Theorem of the Alternative, that there exist numbers w_t^i and vectors ν_t^i, for all individual and observation, such that

$$w_{t'}^i - w_t^i + \nu_t^i \left(y_t^{\neg i} - y_{t'}^{\neg i}\right) \leq \lambda_t^i \left(p_t \left(x_{t'}^i - x_t^i\right) + q_t \left(y_{t'}^i - y_t^i\right)\right).$$

Individual preferences can be defined by

$$u^i \left(x^i, y^i, y^{\neg i}\right) = \min_t \left\{w_t^i + \lambda_t^i \left(p_t \left(x^i - x_t^i\right) + q_t \left(y^i - y_t^i\right)\right) + \nu_t^i \left(y^{\neg i} - y_t^{\neg i}\right)\right\},$$

and an additive term can be used to guarantee strong monotonicity in all arguments. □

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