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Economic Theory

ISSN 0938-2259
Volume 57
Number 2

Econ Theory (2014) 57:253-277
DOI 10.1007/s00199-014-0843-x


Volume 57 · Number 2 · October 2014

Economic Theory

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RESEARCH ARTICLE

A nonparametric analysis of multi-product oligopolies

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Received: 22 August 2013 / Accepted: 2 September 2014 / Published online: 7 October 2014

Abstract We develop revealed preference tests for models of multi-product oligopoly, building on the work in Carvajal et al. (Econometrica 81(6):2351–2379, 2013). We analyze a Cournot model with multiple goods and show that it has testable restrictions when at least one good is produced by two or more firms. We also develop a revealed preference test for Bertrand oligopoly in a setting where each firm produces a single differentiated good, and these goods are potentially substitutes for each other. Our tests require qualitative assumptions on the shape of the demand curves and (in the Bertrand

The financial support of the ESRC to this research project, through Grants RES-000-22-3771 (Andrés Carvajal) and RES-000-22-3187 (John Quah) is gratefully acknowledged. Rahul Deb and James Fenske would like to acknowledge the financial support they received from their Leylan fellowships. Part of this paper was written while John Quah was visiting the Economics Department at the National University of Singapore and while Andrés Carvajal was visiting CORE–Université catholique de Louvain; they thank these institutions for their hospitality. We are grateful to Dirk Bergemann, Don Brown, Greg Crawford, Luis Corchon, Françoise Forges, Enrico Minelli, Margaret Slade, Mike Waterson, Mike Whinston and Glen Weyl for helpful comments.

A. Carvajal
Economics Department, University of Western Ontario, London, ON, Canada
e-mail: acarvaj@uwo.ca

R. Deb
Economics Department, University of Toronto, Toronto, ON, Canada
e-mail: rahul.deb@utoronto.ca

J. Fenske · J. K.-H. Quah (✉)
Economics Department, University of Oxford, Oxford, UK
e-mail: john.quah@economics.ox.ac.uk

J. Fenske
e-mail: james.fenske@economics.ox.ac.uk

case) their evolution across observations, but they do not rely on the estimation of market demand.

Keywords Cournot equilibrium · Bertrand equilibrium · Revealed preference · Observable restrictions · Product differentiation · Supermodular games

JEL Classification C12 · C14 · C60 · C72 · D22 · D43 · L13

1 Introduction

Revealed preference gives a simple and intuitive way of checking whether a data set is consistent with a given model under minimal assumptions. Perhaps the earliest and most well-known application of this approach is to test the consumption models, where Afriat (1967) derived necessary and sufficient conditions for a finite set of price and demand observations to be consistent with utility maximization.[1] Subsequently, this approach has found applications in testing production (Varian 1984), general equilibrium (Brown and Matzkin 1996), collective household consumption (Cherchye et al. 2007) and other models.

In this paper, we derive revealed preference tests for models of multi-product oligopoly. These tests take the form of inequalities (which must have a solution for the data to be rationalizable) that are derived from the first-order conditions of best responding firms. We assume the observer is given a data set consisting of market prices and firm quantities over time and ask the following question: when does there exist a *time invariant* cost function for each firm such that the observed variation in the data can be explained by (static) Nash equilibrium play in response to a *time varying* market demand. We require the rationalizing cost functions to be increasing and convex and sometimes impose reasonable shape restrictions on demand functions, but otherwise, no parametric assumptions are made. Market demand curves are also required to be consistent with the data, in the sense that they must pass through the observed prices and output quantities. However, unlike the empirical literature in industrial organization, we do not assume that the researcher can observe variables that are known to shift or twist the demand curves, so there is no attempt at fitting the demand behavior to a particular model of demand. In this sense, our approach gives 'pure' tests of the oligopoly model that involve a minimum of ancillary restrictions.

Our revealed preference tests for oligopolies are useful for at least two reasons. Firstly, since they require very parsimonious assumptions, they can serve as a pretest before proceeding with estimation of demand or cost parameters. Such a pretest would determine whether the chosen Cournot or Bertrand model is appropriate to model the market. Secondly, such tests can be useful to antitrust authorities who consider static Nash equilibrium as a benchmark. The minimal required assumptions imply that a rejection of equilibrium by our tests provides very robust evidence that firm interaction

[1] For a more recent treatment of Afriat's Theorem see Fostel et al. (2004).

is taking a more complicated (and possibly collusive) form.[2] This could then provide a reason for authorities to conduct further analysis via more industry-specific models or by other methods.

Recently, Carvajal et al. (2013), henceforth referred to as CDFQ, derived revealed preference tests for a single-product Cournot oligopoly. Critical to those results was the assumption that firms in the market produce a single homogenous good. The aim of this paper was to extend the CDFQ results in two directions: by allowing for multi-product environments and by testing for Bertrand interaction, in addition to Cournot. These extensions are important for a number of reasons. For example, in many industries, firms compete in multiple geographical markets (third degree price discrimination), which formally correspond to each firm producing multiple goods. Under certain conditions, such multimarket contact can make collusion easier to sustain—see Bernheim and Whinston (1990) and the literature thereafter—and this makes collusion detection important in such environments. In other situations, there may be a single market, but the goods produced by each firm are differentiated rather than perfectly homogeneous. Such a market is often modeled as a Bertrand oligopoly with differentiated goods, perhaps the most widely used model for demand estimation in empirical industrial organization. Again, it is important to explore how this model could be tested from a revealed preference perspective.

In Sect. 2, we develop a test for a general multi-product Cournot oligopoly, after a quick review of the CDFQ results. Our model allows for the inverse demand function to change arbitrarily over time and the market clearing price of each product depends (potentially) on the entire vector of quantities produced. Thus, the various goods are permitted to be either complements or substitutes. Additionally, each firm can produce multiple products; its cost function is required to be convex in the output vector but we allow for there to be economies of scope in production. We show that this model has nontrivial testable restrictions on observed data, so long as there is at least one product sold by two or more firms. We also characterize the properties that a data set must satisfy to be consistent with this model.

Section 3 of the paper develops a test for a Bertrand model with differentiated products. In this case, we assume that each firm produces exactly one good, with a cost function that is increasing and convex. The demand for each firm's product has the following features: it is a log-concave function of its own price and the own-price demand elasticity is decreasing in the prices charged by other firms. Both of these shape restrictions on the demand function are standard in the literature. The first assumption is often made because it guarantees (along with a convex cost function) that each firm's profit is a quasi-concave function of its price and so the first-order conditions are also sufficient for profit maximization. The second condition is often made to help guarantee that prices are strategic complements and the Bertrand oligopoly is a supermodular game in the sense of Milgrom and Roberts (1990) or Vives (1990). With respect to the data generating process, we assume that observations are generated by shifts in the firms' demand functions, where demand changes in such a way that own-price elasticities

[2] Note that we are *not* saying that firms that fail Cournot or Bertrand rationalizability must necessarily be colluding. A discussion of the distinction between testing for Cournot rationalizability and testing for collusion can be found in Carvajal et al. (2013).

either fall or rise across *all* firms in the industry; we call this the *co-evolving property*. In particular, this excludes the possibility that data variation is primarily driven by firm-specific demand shocks or by changes to each firm's cost function.[3] We show that this model of Bertrand oligopoly has nontrivial testable restrictions on data and identify necessary and sufficient conditions on a data set for consistency with this model. Finally, we explain how it is possible to modify this basic test to incorporate idiosyncratic shocks to a firm's demand or marginal cost functions, by postulating that the observer can observe parameters that permit these shocks to be (at least partially) ordered.

1.1 Related literature

The testable implications of equilibrium behavior in abstract games have been investigated by Sprumont (2000), Ray and Zhou (2001) and Lee (2012), among others. In these papers, payoff functions remain fixed and the variability in the data arises from each player being constrained to choose from different subsets of the set of available strategies. This differs from our paper, and also from CDFQ, where the data are generated by changes to the demand—hence payoff—function. There are also results comparing Nash equilibria in abstract supermodular games as payoffs vary—see, for example, Theorem 6 in Milgrom and Roberts (1990). The demand perturbations generating the data in our analysis do not correspond to the type of parameter changes for which standard results on the monotone comparative statics of Nash equilibria are applicable. Furthermore, those results generally compare the largest (or smallest) Nash equilibrium[4] before and after the parameter change while in our analysis we do not require that an observation be rationalized as the largest (or smallest) Nash equilibrium.

2 Cournot rationalization

In this section, we derive revealed preference tests for the general Cournot oligopoly model. We consider an industry consisting of a set $\mathcal{I} = \{1, 2, \ldots, I\}$ of firms, producing the goods $\mathcal{K} = \{1, 2, \ldots, K\}$. The observer has access to a data set where each observation, t, consists of a vector of the prevailing prices of the K goods, $P_t = (P_t^k)_{k\in\mathcal{K}}$, and the output vector of each firm, $Q_{i,t} = (Q_{i,t}^k)_{k\in\mathcal{K}}$. The data set consists of T observations of this industry, which are indexed by $t \in \mathcal{T} = \{1, 2, \ldots, T\}$, and so we can write it as $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$. We denote the aggregate output of the industry at observation t by $Q_t = \sum_{i\in\mathcal{I}} Q_{i,t}$.

2.1 Cournot rationalization in a single-product oligopoly

We first recap the main result from CDFQ. This will set up the framework and provide the intuition for the multi-product generalization we present in Sect. 2.2.

[3] This mirrors the setup in CDFQ in the sense that data are generated by demand shocks rather than cost shocks and, of course, the demand shocks are never firm specific since all firms produce the same good.

[4] This notion is well defined in a supermodular game (see Milgrom and Roberts 1990).

In CDFQ, each firm produces the same homogeneous good (so $K = 1$). At each t, the market price of the good, P_t, and the output of each firm $(Q_{i,t})_{i\in\mathcal{I}}$ are observed. (In this case, both P_t and $Q_{i,t}$ are scalars.) We require $Q_{i,t} > 0$ for all (i, t). The data set $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is said to be *Cournot rationalizable* if each observation can be explained as a Cournot equilibrium arising from a different market demand function, keeping the cost function of each firm fixed across observations, and with the demand and cost functions obeying certain regularity properties. By a *cost function* of firm i, we mean a strictly increasing and convex function $\bar{C}_i : \mathbb{R}_+ \to \mathbb{R}_+$ that takes nonnegative values. The convexity assumption is standard in theoretical and econometric work as it helps to make the firm's optimization problem tractable and it is also a plausible assumption in many settings.[5] The market *inverse demand* is a function $\bar{P}_t : \mathbb{R}_+ \to \mathbb{R}$ (for each t) which is differentiable at all $q > 0$, with $\bar{P}_t'(q) < 0$. Assuming the demand curves to be downward sloping is standard and uncontroversial. Formally, the data set $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is *Cournot rationalizable* if there are cost functions $\bar{C}_i$ for each firm i and inverse demands $\bar{P}_t$ for each observation t such that

(i) $\bar{P}_t(Q_t) = P_t$, and
(ii) $Q_{i,t} \in \text{argmax}_{q_i\geq 0}\{q_i \bar{P}_t(q_i + \sum_{j\neq i} Q_{j,t}) - \bar{C}_i(q_i)\}$.

Condition (i) says that the inverse demand must agree with the observed data at each observation. Condition (ii) says that, again at each observation, each firm's observed output level maximizes its profit given the output of the other firms.

We begin by examining conditions on a data set that are necessary for it to be Cournot rationalizable. Suppose that $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is Cournot rationalizable by inverse demand functions $\{\bar{P}_t\}_{t\in\mathcal{T}}$ and convex cost functions $\{\bar{C}_i\}_{i\in\mathcal{I}}$. We denote by $\partial\bar{C}_i(Q_{i,t})$ the set of subgradients of $\bar{C}_i$ at $Q_{i,t}$; the nonemptiness of this set is guaranteed by the convexity of $\bar{C}_i$. At observation t, firm i chooses q_i to maximize its profit given the output of the other firms; at its optimal choice, $Q_{i,t}$, the first-order conditions say that there is $\delta_{i,t} \geq 0$ contained in $\partial\bar{C}_i(Q_{i,t})$ such that $Q_{i,t}\bar{P}_t'(Q_t) + \bar{P}_t(Q_t) - \delta_{i,t} = 0$ or

$$\delta_{i,t} = \bar{P}_t(Q_t) - \lambda_t Q_{i,t}, \tag{1}$$

where $\lambda_t = -\bar{P}_t'(Q_t)$ (the slope of the inverse demand curve at the observed total quantity Q_t). Additionally, since cost functions are convex, for any two observations t and t', any firm i, and any subgradient $\delta_{i,t} \in \partial\bar{C}_i(Q_{i,t})$, we have $\bar{C}_i(Q_{i,t'}) \geq \bar{C}_i(Q_{i,t}) + \delta_{i,t}(Q_{i,t'} - Q_{i,t})$ which may be written as

$$C_{i,t'} \geq C_{i,t} + \delta_{i,t}(Q_{i,t'} - Q_{i,t}), \tag{2}$$

where $C_{i,t} = \bar{C}_i(Q_{i,t})$ and $C_{i,t'} = \bar{C}_i(Q_{i,t'})$.

It follows that a simple necessary condition for a data set $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ to be Cournot rationalizable is that there exist positive constants $\delta_{i,t}$, $C_{i,t}$ and λ_t such

[5] Some restriction on the shape of the firms' cost functions is necessary for there be nontrivial testable restrictions on the data; this is formally shown in Theorem 1 of CDFQ. For a property on cost functions that is weaker than convexity but still leads to observable restrictions on data, see the working paper version of CDFQ, Carvajal et al. (2010).

that (1) and (2) are satisfied for all i, t. CDFQ prove that these conditions are also sufficient.

Theorem 1 (Carvajal et al. 2013). *The following statements on the data set* $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ *are equivalent:*

[A] *The set of observations is Cournot rationalizable.*

[B] *There are numbers* λ_t, $\delta_{i,t}$ *and* $C_{i,t}$ *such that, for all* t, $t' \in \mathcal{T}$, *and* $i \in \mathcal{I}$, *the following inequalities hold:*[6]

(i) $\lambda_t > 0$, $\delta_{i,t} > 0$, *and* $C_{i,t} > 0$

(ii) $\delta_{i,t} = P_t - \lambda_t Q_{i,t}$ *and*

(iii) $C_{i,t'} \geq C_{i,t} + \delta_{i,t}(Q_{i,t'} - Q_{i,t})$.

CDFQ showed via Theorem 1 that the Cournot model has testable implications despite there being no restrictions on how the demand curves can change over time. In the next subsection, we generalize this result and construct a test for a Cournot oligopoly with multiple goods. Importantly, we show that testable restrictions exist even in this general multi-product setting.

2.2 Cournot rationalization in a multi-product oligopoly

We now derive the tests for the general multi-product Cournot oligopoly, so we allow for $K > 1$. Once again, we consider a market consisting of I firms but now we allow each firm to produce more than one good. The production costs and demand for these goods are possibly interrelated, which allows for economies of scope in production and complementarity/substitutability in demand—see, for example, Brander and Eaton (1984) and Bulow et al. (1985). Let $\mathcal{K}_i \subseteq \mathcal{K}$, with $K_i = |\mathcal{K}_i|$, be the set of goods firm i is capable of producing. The output vector of each firm, $Q_{i,t} = (Q^k_{i,t})_{k\in\mathcal{K}}$, has $Q^k_{i,t} = 0$ if $k \notin \mathcal{K}_i$. We require that each firm has nonzero production at each observation: for each $i \in \mathcal{I}$ and $t \in \mathcal{T}$, there is $k \in \mathcal{K}_i$ (which may depend on t) such that $Q^k_{i,t} > 0$. We also assume that $Q_t = \sum_{i\in\mathcal{I}} Q_{i,t} \gg 0$ for all t; in other words, strictly positive amounts of each good are produced at all observations.

Generalizing our earlier definition, the *cost function of firm* i is a convex map $\bar{C}_i : \mathbb{R}^{K_i}_+ \to \mathbb{R}_+$ that takes nonnegative values and is strictly increasing in q^k_i for all $k \in \mathcal{K}_i$. Abusing our notation, we will often write $\bar{C}_i(q_i)$ where $q_i \in \mathbb{R}^K_+$ (for example, $\bar{C}_i(Q_{i,t})$); by this we simply mean $\bar{C}_i(\hat{q}_i)$, where $\hat{q}_i$ is the restriction of q_i to $\mathcal{K}_i$.

The market inverse demand function is a vector-valued mapping $\bar{P}_t : \mathbb{R}^K_+ \to \mathbb{R}^K$. We say that inverse demand function $\bar{P}_t$ obeys the *law of demand* if it is differentiable with a negative definite derivative matrix $\partial\bar{P}_t$.[7] This condition is the multi-product generalization of a downward sloping inverse demand curve. In particular, it implies

[6] These conditions are stated slightly differently in CDFQ. Condition (ii) is stated in CDFQ as the *common ratio property* and condition (iii), which guarantees the convexity of the firms' cost functions, is captured by the *co-monotone property*. These conditions have been restated here in a way that is more clearly related to the multi-product generalization in Theorem 2.

[7] For the use of this condition in the context of multi-product oligopolies, see Vives (1999). The micro-foundations of this property have been extensively studied; see Quah (2003) and the survey of

that the diagonal terms of $\partial \bar{P}_t$ (the own-price derivative for any good) are negative numbers, but negative definiteness is a stronger property. This generalization of the downward sloping property is not the only one possible, but it is intuitive, convenient for our purposes, and has been extensively studied.

As before, the set of observations is rationalizable if each observation can be explained as a Cournot equilibrium arising from a different market inverse demand function, keeping the cost function of each firm fixed across observations. Formally, the set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is said to be *Cournot rationalizable* if there are cost functions $\bar{C}_i$ for each firm i, and demand functions $\bar{P}_t$ obeying the law of demand at each observation t, such that:

(i) $\bar{P}_t(Q_t) = P_t$, and
(ii) $Q_{i,t} \in \text{argmax}_{q_i\in\Gamma_i} \left\{\sum_{k=1}^{K} q_i^k \bar{P}_t^k(q_i + \sum_{j\neq i} Q_{j,t}) - \bar{C}_i(q_i)\right\}$, where

$$\Gamma_i = \{q_i \in \mathbb{R}_+^K : q_i^k = 0 \;\forall\; k \notin \mathcal{K}_i\}.$$

Condition (i) requires each hypothesized inverse demand function to agree with its respective observation, while condition (ii) says that firm i's output choice is profit-maximizing, given the output of other firms, the hypothesized inverse demand function, and the output vectors the firm is capable of producing, which is the set Γ_i.

Theorem 2 below generalizes Theorem 1 by characterizing multi-product data sets that are Cournot rationalizable.

Theorem 2 *The following statements on the data set* $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ *are equivalent:*

[A] *The set of observations is Cournot rationalizable.*[8]
[B] *There are real numbers* $\lambda_t^{\ell,k}$, $\delta_{i,t}^k$ *and* $C_{i,t}$ *such that, for all* ℓ *and* $k \in \mathcal{K}$, *all* t *and* $t' \in \mathcal{T}$, *and all* $i \in \mathcal{I}$, *the following holds:*
  (i) $C_{i,t} > 0$, $\delta_{i,t}^k = 0$ *if* $k \notin \mathcal{K}_i$. $\delta_{i,t}^k > 0$ *if* $k \in \mathcal{K}_i$, *and the* $K \times K$ *matrix* $\Lambda_t = [\lambda_t^{\ell,k}]$ *is positive definite;*
  (ii) $\delta_{i,t}^k - P_t^k + \sum_{\ell=1}^{K} \lambda_t^{\ell,k} Q_{i,t}^\ell \geq 0$ *and* $(\delta_{i,t}^k - P_t^k + \sum_{\ell=1}^{K} \lambda_t^{\ell,k} Q_{i,t}^\ell) Q_{i,t}^k = 0$; *and*
  (iii) $C_{i,t'} \geq C_{i,t} + \sum_{k=1}^{K} \delta_{i,t}^k (Q_{i,t'}^k - Q_{i,t}^k)$.

Footnote 7 continued
Jerison and Quah (2008). The literature usually considers demand as a function of price, rather than the inverse demand function considered here. However, the two cases are equivalent: if $\partial \bar{P}_t(q)$ is negative definite, then $\bar{P}_t$ is locally invertible, and its inverse, the demand function $\bar{Q}_t$, has a negative definite matrix at $\bar{P}_t(q)$.

[8] Instead of requiring $\bar{P}_t$ to obey the law of demand, we could require that, for all $i \in \mathcal{I}$ and $t \in \mathcal{T}$, the derivative matrix $\partial \bar{P}_t$ is negative definite when restricted to the subset of goods $\mathcal{K}_i$, which is weaker than simply requiring $\partial \bar{P}_t$ to be negative definite. To characterize Cournot rationalizable data sets with this weaker requirement on demand, condition (i) in statement [B] must be modified. It should now require Λ_t to have the following property: for all i and t, the restriction of Λ_t to $\mathcal{K}_i$ is negative definite. This modification of Theorem 2 is possible because the weaker property on Λ_t is nonetheless sufficient to enable the construction of a rationalizing inverse demand function $\bar{P}_t$ such that each firm's profit function is concave (see Lemma 1).

Remark When there is just one good, statement [B] reduces to the following: there are numbers $\lambda_t > 0$, $\delta_{i,t} > 0$, $C_{i,t} > 0$ such that $\delta_{i,t} = P_t - \lambda_t Q_{i,t}$ and $C_{i,t'} \geq C_{i,t} + \delta_{i,t}(Q_{i,t'} - Q_{i,t})$. Thus, Theorem 1 is a special case of this result.

Theorem 2 establishes an equivalence between Cournot rationalizability and the solution to a programming problem. However, unlike Theorem 1, the program in statement [B] of Theorem 2 is not linear, because the requirement that the matrices Λ_t are positive definite imposes nonlinear conditions on the unknowns. The Tarski-Seidenberg Theorem tells us that there is an algorithm that could ascertain whether or not a solution exists and that provides a solution if it does. This theorem states that the projection of any semi-algebraic set is semi-algebraic as well, and that it can be found in finite time. A set is semi-algebraic if it can be described by a finite set of polynomial inequalities on real numbers; as the three conditions introduced in statement [B] are polynomial inequalities, the set of values of the variables that satisfy them is semi-algebraic. Thus, the projection of this set into the space of prices and quantities is also characterized by a finite set of polynomial inequalities. These inequalities, defined only on the observed prices and quantities, are necessary and sufficient conditions for Cournot rationalizability. So this gives us a way of implementing the test in principle, even if the derivation of these conditions is computationally demanding. The existing algorithms that implement this elimination of quantified variables are, in the best of cases, of exponential complexity.[9]

Note that this computational issue arises only if we insist on requiring that the rationalizing market inverse demand functions obey the law of demand. It is possible to replace this property with something a bit stronger and just as intuitive, for which the corresponding test is a (computationally less demanding) linear program; we explain this in greater detail in Sect. 2.4.

2.3 Testable restrictions of the multi-product Cournot model

While Theorem 2 provides us with a test for the Cournot model, it does not immediately follow that the test generates non-trivial restrictions on the data. Indeed, consider the important special case where each firm in the industry produces one good that is differentiated from all the others. In this case, no good is produced by more than one firm and Cournot rationalizability in the sense of Theorem 2 imposes *no* observable restrictions on the data (apart from trivial nonnegativity restrictions on prices and output). To rationalize any data set, simply assume that all cross-price elasticities equal zero, so that each firm is a single-product monopoly. It is not hard to see (and it is shown in CDFQ) that *any* sequence of price and outputs from a single-product monopoly can be rationalized with a (convex and increasing) cost function and a sequence of downward sloping demand functions that change across observations. Thus, Theorem 2 is not suitable as a test of strategic interaction in an industry where each firm is producing a differentiated good. Industries with this feature are typically

[9] The original procedure of Tarski and Seidenberg, and the alternative of Cylindrical Algebraic Decomposition are known to be doubly exponential. More recent developments in Quantifier Elimination may yield a procedure that is singly exponential: see Basu et al. (2006, Ch. 13).

modeled as Bertrand oligopolies and they feature prominently in both the theoretical and empirical IO literature. The revealed preference properties of that model will be considered in the next section.

The multi-product Cournot model *does* impose substantive restrictions on the data when, as in the case of the single-product Cournot model, there are goods that are supplied by more than one firm. The following example illustrates this phenomenon.

Example 1 Consider an industry with two goods, 1 and 2, where observations taken from the two firms in this industry are as follows:

(i) at observation t, $P^1_t = 10$, $Q^1_{i,t} = 13$, $Q^2_{i,t} = 12$, $Q^1_{j,t} = 4$, and $Q^2_{j,t} = 6$; and
(ii) at observation t', $P^1_{t'} = 1$, $P^2_{t'} = 1$, $Q^1_{j,t'} = 8$, and $Q^2_{j,t'} = 8$.

We claim that these observations are not Cournot rationalizable. Suppose, to the contrary, that they are, and invoke statement [B] in Theorem 2. Using condition (ii), we have that

$$P^1_t - Q^1_{i,t}\lambda^{1,1}_t - Q^2_{i,t}\lambda^{2,1}_t - \delta^1_{i,t} = 0,$$

and

$$P^1_t - Q^1_{j,t}\lambda^{1,1}_t - Q^2_{j,t}\lambda^{2,1}_t - \delta^1_{j,t} = 0.$$

Multiplying the first equation by $Q^2_{j,t}$ and the second equation by $Q^2_{i,t}$ and taking the difference between them, we obtain

$$\left(Q^2_{j,t} - Q^2_{i,t}\right) P^1_t - \left(Q^2_{j,t} Q^1_{i,t} - Q^2_{i,t} Q^1_{j,t}\right) \lambda^{1,1}_t - Q^2_{j,t}\delta^1_{i,t} + Q^2_{i,t}\delta^1_{j,t} = 0. \quad (3)$$

The significance of the numbers chosen for observation t is that they guarantee that $Q^2_{j,t} - Q^2_{i,t} < 0$ and $Q^2_{j,t} Q^1_{i,t} - Q^2_{i,t} Q^1_{j,t} > 0$. Recall that $\delta^1_{i,t} > 0$ and (because Λ_t is positive definite) $\lambda^{1,1}_t > 0$. Therefore, the second and third terms on the left of equation (3) are both negative. Re-arranging that equation, we obtain

$$\delta^1_{j,t} \geq \frac{Q^2_{i,t} - Q^2_{j,t}}{Q^2_{i,t}} \cdot P^1_t = \frac{6}{12} \cdot 10 = 5. \quad (4)$$

Condition (iii) in statement [B] says that

$$C_{j,t'} \geq C_{j,t} + \delta^1_{j,t}(Q^1_{j,t'} - Q^1_{j,t}) + \delta^2_{j,t}(Q^2_{j,t'} - Q^2_{j,t}) = C_{j,t} + (Q_{j,t'} - Q_{j,t}) \cdot \delta_{j,t} \quad (5)$$

and

$$C_{j,t} \geq C_{j,t'} + (Q_{j,t} - Q_{j,t'}) \cdot \delta_{j,t'}. \quad (6)$$

It follows from (5) that

$$C_{j,t'} - C_{j,t} \geq 5(8 - 4) = 20. \quad (7)$$

Condition (ii) in vector and matrix notation says that $-\Lambda_{t'} Q_{j,t'} = \delta_{j,t'} - P_{t'}$. Pre-multiplying this by $Q_{j,t'}$, we obtain (by the positive definiteness of $\Lambda_{t'}$)

$$Q_{j,t'} \cdot P_{t'} \geq Q_{j,t'} \cdot \delta_{j,t'}.$$

Furthermore, $Q_{j,t'} > Q_{j,t}$, and so

$$Q_{j,t'} \cdot P_{t'} \geq (Q_{j,t'} - Q_{j,t}) \cdot \delta_{j,t'} \geq C_{j,t'} - C_{j,t}, \tag{8}$$

where the last inequality is taken from (6). But (8) is impossible since $Q_{j,t'} \cdot P_{t'} = 16$ and we know from (7) that $C_{j,t'} - C_{j,t} = 20$.

In other words, (4) provides us with a lower bound on the marginal cost of firm j at its observed output of (4, 6):

$$\frac{\partial \bar{C}_j}{\partial q_1}(4, 6) \geq 5. \tag{9}$$

This in turn leads to a lower bound on the cost to firm j of raising its output from $Q_{j,t} = (4, 6)$ to $Q_{j,t'} = (8, 8)$ [see (7)]. On the other hand, firm j's total revenue at t' must be greater than the added cost of raising output from $Q_{j,t}$ to $Q_{j,t'}$ [see (8)], but this is impossible given the output levels and prices.

2.4 Variations on Theorem 2

Theorem 2 provides a test for data sets that are consistent with Cournot outcomes, assuming that the market inverse demand function obeys the law of demand. For various reasons, it is sometimes convenient to either strengthen or weaken this condition on the demand function, which will entail corresponding changes to the test. We now discuss a number of these cases and explain how the test could be modified.

As we have already pointed out, the test for Cournot rationalizability given in statement [B] of Theorem 2 requires us to solve a nonlinear program, where the nonlinearity arises from the positive definiteness condition imposed on Λ_t (see condition (i) in the statement). It is possible to replace the law of demand with a stronger condition that is easier to check. For example, we could require the rationalizing inverse demand function $\bar{P}_t$ to obey *diagonal dominance with uniform weights*; by this, we mean that

$$2\frac{\partial \bar{P}_t^k}{\partial q^k}(q) + \sum_{\ell \neq k} \left| \frac{\partial \bar{P}_t^k}{\partial q^\ell}(q) + \frac{\partial \bar{P}_t^\ell}{\partial q^k}(q) \right| < 0 \quad \text{for all } q \text{ and for all } k \in \mathcal{K}.$$

This intuitive condition says that own-price effects are larger than the sum of all cross-price effects. If we impose this condition on the rationalizing demand system, then the corresponding requirement on Λ_t (modifying condition (i) in [B]) is the following:[10]

[10] This property guarantees the positive definiteness of the symmetric matrix $\Lambda + \Lambda^T$, which is equivalent to the positive definiteness of Λ; see Mas-Colell et al. (1995, Appendix M.D.) for more on diagonal dominance.

$$-2\lambda_t^{k,k} + \sum_{\ell \neq k} |\lambda_t^{\ell,k} + \lambda_t^{k,\ell}| < 0 \quad \text{for all } k \in \mathcal{K};$$

note that this can condition can be equivalently stated as a set of linear conditions. The other parts of the test [as stated in conditions (i), (ii), and (iii)] remain unchanged and all of them involve only linear conditions.

In certain contexts, the modeler may have specific information on cross-price effects which he would like to impose as conditions on the rationalizing demand system, on top of those required by the law of demand or diagonal dominance. For example, it is possible to interpret the different goods in this model as the same good sold in several distinct and isolated markets; in other words, this multi-product oligopoly is an instance of third degree price discrimination, with the same firms interacting in several markets. In that case, it may be reasonable to require all cross-price effects to equal zero, i.e., $\partial \bar{P}_t^k / \partial q^\ell = 0$ for all $k \neq \ell$. Correspondingly, one would have to impose the condition $\lambda_t^{\ell,k} = 0$ for all t and whenever $\ell \neq k$, in addition to the ones listed in statement [B] of Theorem 2.[11]

Similarly, the modeler may believe that the K goods are substitutes ($\partial \bar{P}_t^k / \partial q^\ell \leq 0$ for all ℓ and k) or complements ($\partial \bar{P}_t^k / \partial q^\ell \geq 0$ for all $\ell \neq k$). The corresponding conditions are $\lambda_t^{\ell,k} \leq 0$ for all ℓ and k, and $\lambda_t^{\ell,k} \geq 0$ for all $\ell \neq k$, respectively.[12]

2.5 Proof of Theorem 2

Proof of Theorem 2: Suppose that [A] holds, so the data is rationalized by inverse demand functions $\bar{P}_t^k$, for $k \in \mathcal{K}$ and $t \in \mathcal{T}$, and cost functions $\bar{C}_i$. We set $C_{i,t} = \bar{C}_i(Q_{i,t})$ and

$$\lambda_t^{\ell,k} = -\frac{\partial \bar{P}_t^\ell}{\partial q^k}(Q_t)$$

Since $(\bar{P}_t^k)_{k \in \mathcal{K}}$ obeys the law of demand, Λ_t is positive definite as required by (i).

At observation t, firm i's revenue function, given that firm $j \neq i$ is producing $Q_{j,t}$, is

$$\bar{R}_{i,t}(q_i) = \sum_{\ell=1}^{K} q_i^\ell \bar{P}_t^\ell \left(q_i + \sum_{j \neq i} Q_{j,t} \right);$$

[11] Even though cross-price effects are zero, each firm is producing more than one good and the market for each good can be served by more than one firm, so this is *not* formally identical to the case of an industry where each firm is producing a good different from that produced by other firms. In the latter scenario, as we have already pointed out, observable restrictions do not exist if there are no cross-price effects.

[12] If we impose the condition that all the goods are substitutes, then Cournot rationalizability requires that all observed prices be nonnegative: if $P_t^{\bar{k}} < 0$ then any firm that is producing good $\bar{k}$ is strictly better off if it reduces its output of $\bar{k}$. In the case when the goods are not necessarily substitutes, the model allows for the possibility that some observed prices are negative: firms can optimally pay for a good to be consumed in order that it may raise the demand for some other good.

note that

$$\frac{\partial \bar{R}_{i,t}}{\partial q_i^k}(Q_{i,t}) = \bar{P}_t^k(Q_t) + \sum_{\ell=1}^{K} \frac{\partial \bar{P}_t^\ell}{\partial q^k}(Q_t) Q_{i,t}^\ell = P_t^k - \sum_{\ell=1}^{K} \lambda_t^{\ell,k} Q_{i,t}^\ell. \tag{10}$$

By assumption, $Q_{i,t}$ maximizes $\Pi_{i,t}(q_i) = \bar{R}_{i,t}(q_i) - \bar{C}_i(q_i)$. Therefore, there exists a vector $(\hat{\delta}_{i,t}^k)_{k\in\mathcal{K}_\rangle}$ in $\partial\bar{C}_i(Q_{i,t}) \in \mathbb{R}^{K_i}$ such that $\hat{\delta}_{i,t}^k \geq P_t^k - \sum_{\ell=1}^K \lambda_t^{\ell,k} Q_{i,t}^\ell$ for all $k \in \mathcal{K}_i$, with equality whenever $Q_{i,t}^k > 0$. Since $\bar{C}_i$ is increasing and convex, $\hat{\delta}_{i,t}^k \geq 0$ for all $k \in \mathcal{K}_i$, with $\hat{\delta}_{i,t}^k > 0$ if $Q_{i,t}^k > 0$. Let $\hat{\delta}_{i,t}^k = 0$ for $k \notin \mathcal{K}_i$. Then, $\hat{\delta}_{i,t}^k$ satisfies, with one exception, all the conditions in (i), (ii), and (iii), with (iii) holding because $\bar{C}_i$ is convex and $C_{i,t} = \bar{C}_i(Q_{i,t})$. The exception is that, if $Q_{i,t}^k = 0$ for some $k \in \mathcal{K}_i$, it is possible that $\hat{\delta}_{i,t}^k = 0$. Now define $\delta_{i,t}^k$ in the following way: for $k \in \mathcal{K}_i$, let $\delta_{i,t}^k = \hat{\delta}_{i,t}^k$ if $\hat{\delta}_{i,t}^k > 0$ and let $\delta_{i,t}^k = \epsilon > 0$ if $\hat{\delta}_{i,t}^k = 0$; for $k \notin \mathcal{K}_i$, simply let $\delta_{i,t}^k = \hat{\delta}_{i,t}^k = 0$. It is clear that $\delta_{i,t}^k$ chosen in this manner will obey conditions (i) and (ii). Furthermore, because $\bar{C}_i$ is convex and increasing, condition (iii) will also be satisfied if ϵ is sufficiently small. This completes our proof that [A] implies [B].

Lemmas 1 and 2, which we state and prove below, show immediately that [B] implies [A]. □

Lemma 1 *Suppose that, at some observation t, there are real numbers $\lambda_t^{\ell,k}$ and $\delta_{i,t}^k$ such that, for all ℓ, all $k \in \mathcal{K}$ and all $i \in \mathcal{I}$, conditions* (i) *and* (ii) *in Theorem 2 hold. Suppose also that there are cost functions $\bar{C}_i : \mathbb{R}^{\mathcal{K}_i} \to \mathbb{R}_+$ with $(\delta_{i,t}^k)_{k\in\mathcal{K}_i} \in \partial\bar{C}_i(Q_{i,t}) \subset \mathbb{R}_+^{K_i}$. Then, there exists an inverse demand function $\bar{P}_t$, obeying the law of demand, such that $\bar{P}_t(Q_t) = P_t$ and, with each firm i having the cost function $\bar{C}_i$, $\{Q_{i,t}\}_{i\in\mathcal{I}}$ constitutes a Cournot equilibrium.*

Proof We define the inverse demand function for good k by

$$\bar{P}_t^k(q) = a_t^k - \sum_{\ell=1}^{K} \lambda_t^{k,\ell} q^\ell \tag{11}$$

with a_t^k chosen such that $\bar{P}_t^k(Q_t) = P_t^k$.

Firm i's profit at observation t, given that firm $j \neq i$ is producing $Q_{j,t}$ is $\Pi_{i,t}(q_i) = \bar{R}_{i,t}(q_i) - \bar{C}_i(q_i)$, where $\bar{R}_{i,t}(q_i) = \sum_{\ell=1}^K q_i^\ell \bar{P}_t^\ell(q_i + \sum_{j\neq i} Q_{j,t})$, and marginal revenue is given by (10). Since $(\delta_{i,t}^k)_{k\in\mathcal{K}_i} \in \partial\bar{C}_i(Q_{i,t})$, condition (ii) gives the Kuhn-Tucker conditions for profit maximization. These conditions are sufficient to guarantee that firm i's choice is optimal if $\Pi_{i,t}$, is concave in q_i. Given that the cost function $\bar{C}_i$ is, by definition, convex, it suffices to check that the $\bar{R}_{i,t}$ is concave in q_i. It is straightforward to verify that, for all q_i, the Hessian $\partial^2\bar{R}_{i,t}(q_i) = -\Lambda_t^T - \Lambda_t$. Condition (i) guarantees that this matrix is negative definite, so we conclude that $\bar{R}_{i,t}$ is concave. □

Lemma 2 *Suppose that for some firm i, there are numbers $\delta_{i,t}^k$, with $\delta_{i,t}^k > 0$ for $k \in \mathcal{K}_i$ and $\delta_{i,t}^k = 0$ for $k \notin \mathcal{K}_i$, and numbers $C_{i,t} > 0$ such that, for all t and $t' \in \mathcal{T}$, condition* (iii) *in Theorem 2 is satisfied. Then, there is a cost function $\bar{C}_i : \mathbb{R}_+^{K_i} \to \mathbb{R}_+$ with $(\delta_{i,t}^k)_{k \in \mathcal{K}_i} \in \partial \bar{C}_i(Q_{i,t})$.*

Proof Define the function $\bar{C}_i$ by

$$\bar{C}_i(q) = \max_{t \in \mathcal{T}} \left\{ C_{i,t} + \sum_{k=1}^{K} \delta_{i,t}^k \left(q^k - Q_{i,t}^k \right) \right\} - \max_{t \in \mathcal{T}} \left\{ C_{i,t} - \sum_{k=1}^{K} \delta_{i,t}^k Q_{i,t}^k \right\}. \quad (12)$$

Notice that, by construction, $\bar{C}_i(0) = 0$. Since $\delta_{i,t}^k > 0$ for all $k \in \mathcal{K}_i$, function $\bar{C}_i$ is increasing and, since it is the upper envelope of linear functions, $\bar{C}_i$ is a convex function. Thus, $\bar{C}_i$ satisfies all the conditions of a cost function. Furthermore, condition (iii) implies that $\bar{C}_i(Q_{i,t}) = C_{i,t} - \max_{t \in \mathcal{T}} \{C_{i,t} - \sum_{k=1}^{K} \delta_{i,t}^k Q_{i,t}^k\} > 0$, since

$$C_{i,t} \geq C_{i,s} + \sum_{k=1}^{K} \delta_{i,s}^k \left(Q_{i,t}^k - Q_{i,s}^k \right) \text{ for all } s \in \mathcal{T}.$$

Therefore, $(\delta_{i,t}^k)_{k \in \mathcal{K}_i} \in \partial \bar{C}_i(Q_{i,t})$. □

3 Bertrand rationalization with differentiated goods

We assume that each firm $i \in \mathcal{I}$ is producing just one good, which we shall also refer to as i. Each observation in the data set consists of a vector of prevailing prices, $P_t = (P_{i,t})_{i \in \mathcal{I}}$, where $P_{i,t}$ is the price of i (produced by firm i) at observation t, and a vector of outputs, $Q_t = (Q_{i,t})_{i \in \mathcal{I}}$, where $Q_{i,t}$ is firm i's output of product i.[13] As in the previous section, we assume that $\mathcal{T}$ is finite, with $\mathcal{T} = \{1, 2, \ldots, T\}$.

We are interested in finding necessary and sufficient conditions on a data set $\{(P_t, Q_t)\}_{t \in \mathcal{T}}$ under which each observation can be explained as a Bertrand equilibrium arising from a different market demand function, keeping the cost function of each firm fixed across observations. In other words, we require cost functions $\bar{C}_i : \mathbb{R}_+ \to \mathbb{R}_+$ for each firm i (which are increasing and convex by definition), and demand functions $\bar{Q}_{i,t} : \mathbb{R}_+^I \to \mathbb{R}^I$ for each product, i, and each observation, t, with $\partial \bar{Q}_{i,t}/\partial p_i < 0$ at all $p \in \mathbb{R}_+^I$, such that:

(i) $\bar{Q}_{i,t}(P_t) = Q_{i,t}$, and
(ii) $P_{i,t} \in \text{argmax}_{p_i \geq 0} \{ p_i \bar{Q}_{i,t}(p_i, P_{-i,t}) - \bar{C}_i(\bar{Q}_{i,t}(p_i, P_{-i,t}))\}$.

[13] In the previous section, superindices were used to denote commodities, while firms were indicated by the first of two subindices. In this section, each firm produces only one product, so we shall dispense with the superindex.

3.1 A revealed preference test for Bertrand behavior

Our objective is to develop a revealed preference test that retains the spirit of Theorem 1 and its multi-product generalization, Theorem 2. The distinguishing feature of CDFQ's test of the Cournot model is that the test does not require specific information on the industry's demand curve – in contrast to empirical IO models, it does not entail the estimation of industry demand. The surprise in those results is that testable restrictions exist even when demand is not known to the observer. However, in that case, the job was made simpler by the fact that different firms are producing the *same* good, which means (by definition) that they encounter identical demand conditions, even if the demand curve is not known to the observer. In this case, we are considering firms that are producing differentiated goods. How can we formally allow for differentiation and yet capture the idea that these firms are in the same industry? And how can we develop a model where observable restrictions exist while keeping broadly to the spirit of Theorems 1 and 2, which means avoiding the imposition of industry-specific numerical bounds on price elasticities and other demand parameters (that need to be separately estimated)?

Examining Theorem 1 more closely, we notice that the single-good assumption in Theorem 1 has, in a sense, two consequences: (1) *at each observation*, when some firm i raises its output, it affects the price faced by another firm j, in the particular way given by their common demand function; (2) the price tradeoff faced by each firm (as it decides on its output) changes in the same way *across observations* since there is just one demand function. Our modeling strategy here is to model the industry's demand structure, and the demand perturbations across time, in a way that involve restrictions loosely analogous to (1) and (2), while allowing for differentiated goods. First, we impose restrictions on how, at each observation, demand for one good is affected by the price of another good produced in the same industry. Second, even though different firms produce different goods, we require that the demand conditions they face move in the same direction from one observation to the next.

Let us denote by $\bar{\varepsilon}_{i,t}(p)$ the relative decrease in the demand of good i in response to an infinitesimal increase in its price, when the vector of prices is p; that is, given the demand function for good i at observation t, $\bar{Q}_{i,t}$,

$$\bar{\varepsilon}_{i,t}(p_i, p_{-i}) = -\frac{1}{\bar{Q}_{i,t}(p_i, p_{-i})}\frac{\partial \bar{Q}_{i,t}}{\partial p_i}(p_i, p_{-i}).$$

We impose the following condition on the demand system $\{\bar{Q}_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$: at each observation t and for all goods i,

$$\bar{\varepsilon}_{i,t}(p_i, p_{-i}) \text{ is nondecreasing in } p_i \text{ and nonincreasing in } p_{-i}. \qquad (13)$$

The first of these two conditions we call the *log-concave condition* because it says that the demand for good i is a log-concave function of its own price. This is a standard assumption to make because it ensures, together with increasing marginal cost, that firm i's profit is quasi-concave in the price it charges. The second condition we call the *substitutes condition*. Clearly, this condition is equivalent to saying that the own-price elasticity of good i, $p_i \bar{\varepsilon}_{i,t}(p)$, is nondecreasing in p_{-i}, i.e., the prices of other goods. Again, this is a standard assumption and is typically made to guarantee that the Bertrand game played among firms is a game of strategic complements, i.e., firm i's optimal price is increasing in the price of other goods (see Milgrom and Shannon 1994).

The log-concave and substitutes conditions impose curvature properties on the demand function of each firm at each observation. However, it should be clear that these conditions are *not* sufficient to generate nontrivial observable restrictions on a data set, since it is possible for each firm to encounter perturbations to its demand function that are completely unrelated to that faced by other firms in the industry. We thus impose a further condition on demand that captures a temporal feature of firms that belong to the same industry: they experience common demand shocks. Formally, we assume the demand system is *co-evolving* in the following sense: for any t and t', either

$$\bar{\varepsilon}_{i,t}(p) \geq \bar{\varepsilon}_{i,t'}(p), \quad \text{for all } p \text{ and all } i \in \mathcal{I}, \tag{14}$$

or

$$\bar{\varepsilon}_{i,t}(p) \leq \bar{\varepsilon}_{i,t'}(p) \quad \text{for all } p \text{ and all } i \in \mathcal{I}. \tag{15}$$

This assumption excludes the possibility that a shock raises the demand elasticity for one firm but lowers it for another, and allows us to rank the elements in $\mathcal{T}$ according to the impact of t on demand elasticity. Of course, firms in an industry can encounter idiosyncratic shocks, so what we are doing is confining ourselves to observations over some time frame where industry-wide shocks are a reasonable assumption. While there are various ways in which this assumption can be modified or generalized (and we discuss this in Sect. 3.4), it is important to emphasize that the co-evolving property is a very natural benchmark assumption for firms that are, after all, supposed to belong to the same industry.

The data set $\{(P_t, Q_t)\}_{t \in \mathcal{T}}$ is said to be *Bertrand rationalizable* if each observation can be explained as a Bertrand equilibrium, with the demand system $\{\bar{Q}_{i,t}\}_{(i,t) \in \mathcal{I} \times \mathcal{T}}$ obeying the log-concave, substitutes and co-evolving conditions. The following result provides a test for data sets that are Bertrand rationalizable.

Theorem 3 *The following statements on* $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ *are equivalent:*

[A] *The set of observations is Bertrand rationalizable.*

[B] *There is a permutation of* $\mathcal{T}$*, denoted by the function* $\sigma : \mathcal{T} \to \mathcal{T}$*, and real numbers* $\lambda_{i,t}$*,* $\delta_{i,t}$*,* $C_{i,t}$ *for all* $t \in \mathcal{T}$ *and* $i \in \mathcal{I}$*, such that the following holds:*
  - (i) $C_{i,t} > 0$, $\delta_{i,t} > 0$, $\lambda_{i,t} > 0$*;*
  - (ii) $\delta_{i,t}\lambda_{i,t} - P_{i,t}\lambda_{i,t} + 1 = 0$*;*
  - (iii) $C_{i,t'} \geq C_{i,t} + \delta_{i,t}(Q_{i,t'} - Q_{i,t})$*; and*
  - (iv) *if* $P_{i,t'} \geq P_{i,t}$, $P_{-i,t'} \leq P_{-i,t}$ *and* $\sigma(t) < \sigma(t')$*, then* $\lambda_{i,t} \leq \lambda_{i,t'}$*.*

Statement [B] in Theorem 3 provides us with a test of Bertrand rationalizability that can be implemented in a finite number of steps. Indeed, for a *given* permutation σ of $\mathcal{T}$, the stated conditions form a linear program in $\delta_{i,t}$, $C_{i,t}$, and $1/\lambda_{i,t}$. Since $\mathcal{T}$ is finite, so is the number of permutations of $\mathcal{T}$, and thus, the entire problem can be solved in a finite number of steps.

Theorem 3 assumes that demand is co-evolving and that cost functions are unchanged over the observation period. The co-evolving assumption may not be suitable if the demand faced by different firms are subject to idiosyncratic shocks; it also leads to a computationally demanding test, since with a data set of T observations, there will be $T!$ ways in which demand could be co-evolving (even if each possible ranking leads to a simple linear test). Similarly, the cost function of each firm can change (perhaps because of a change in input prices) during the period where observations are taken.

One way of avoiding all of these issues is to use data sets consisting of a small number of observations collected over a short duration. An implementation of our test would then involve cutting up a long (possibly very long) data set of an industry's performance over many periods into much shorter segments, with each segment containing observations taken over a small number of consecutive periods. There will then be a large number of small data sets, and tests can be performed on each of them. The number of observations in each data set should be small enough so that the test is computationally feasible (which could mean fewer than 10 observations). Assuming that idiosyncratic changes to firms' demand functions or firms' cost functions occur only slowly (or infrequently) relative to industry-wide demand fluctuations, the co-evolving condition will hold and cost functions of firms will remain unchanged in each data set and so a negative test outcome will correctly be attributed to a violation of Bertrand behavior. The frequency of passing these tests then serves as an indicator of whether the industry is behaving as a Bertrand oligopoly. The empirical strategy outlined here is, in essence, the same as the one used in CDFQ's implementation of the test for Cournot rationalizability.[14]

3.2 Proof and motivation for Theorem 3

Proof (that [A] implies [B] in Theorem 3): If the data set can be rationalized, the first-order condition of profit maximization is that

$$\bar{Q}_{i,t}(P_t) + P_{i,t}\frac{\partial \bar{Q}_{i,t}}{\partial p_i}(P_t) - \delta_{i,t}\frac{\partial \bar{Q}_{i,t}}{\partial p_i}(P_t) = 0, \tag{16}$$

for some $\delta_{i,t} \in \partial \bar{C}_i'(\bar{Q}_{i,t}(P_t))$ (the set of subgradients of $\bar{C}_i$ st $Q_{i,t}$). Setting

$$C_{i,t} = \bar{C}_i(Q_{i,t}) \text{ and } \lambda_{i,t} = \bar{\varepsilon}_{i,t}(P_t),$$

[14] An issue that all revealed preference tests have to contend with is that they are binary: a data set either passes or fails the test. Thus, implementing such a test on a large data set will often be problematic, unless a way of accounting for errors is formally included. This is one reason why data sets used in revealed preference tests generally tend not to be very big. In studies of consumer demand, it is typical to implement these tests (for example, Afriat's Theorem or its variations) on a large number of subjects, with the number of observations for each subject being fairly modest (fewer than 15 or even 10 observations are not uncommon). In other words, there is a large number of tests, with each test having a small data set, and the frequency of passing the test in this collection of data sets is used as a measure of the model's success. The empirical strategy proposed here, and used in CDFQ, has a broadly similar pattern.

we obtain (i) and, through each firm's first-order condition, we obtain (ii). Condition (iii) must also hold because the cost function of each is convex. For part (iv), first observe that the co-evolving property of the demand system guarantees that there is a permutation of the observations, σ, such that if $\sigma(t) < \sigma(t')$, then

$$\bar{\varepsilon}_{i,t}(p) \leq \bar{\varepsilon}_{i,t'}(p) \text{ for all } p \geq 0 \text{ and } i \in \mathcal{I}.$$

(In other words, we are ranking the observations in $\mathcal{T}$ according to the own-price elasticity of demand; there is no ambiguity because of the co-evolving condition.) Given this, when $P_{i,t'} \geq P_{i,t}$ and $P_{-i,t'} \leq P_{-i,t}$ we obtain that

$$\bar{\varepsilon}_{i,t}(P_{i,t}, P_{-i,t}) \leq \bar{\varepsilon}_{i,t}(P_{i,t'}, P_{-i,t}) \leq \bar{\varepsilon}_{i,t}(P_{i,t'}, P_{-i,t'}) \leq \bar{\varepsilon}_{i,t'}(P_{i,t'}, P_{-i,t'}),$$

where the first and second inequalities follow from the log-concave and substitute properties of the demand system, respectively, while the third one follows from the ordering given by the co-evolving property. Hence, $\lambda_{i,t} \leq \lambda_{i,t'}$. □

It is useful to compare and contrast the conditions identified in Theorem 3 with those in Theorem 1. In both cases, condition (iii) captures the requirement that firms' cost functions are convex. Condition (ii) in both cases capture the first-order conditions that must hold at each firm's observed choice, but there is an important difference between them. Condition (ii) in Theorem 1 leads to a restriction on marginal costs *across firms*, because different firms face the same price impact from a marginal change in output.[15] This is not true of the analogous condition in Theorem 3 because the price elasticities faced by different firms are not the same at each observation (i.e., $\lambda_{i,t}$ and $\lambda_{j,t}$ need not be equal). It is not hard to see that conditions (i), (ii) and (iii) in Theorem 3 are not in themselves sufficient to impose restrictions on the data; restrictions exist only because of the inclusion of condition (iv). This condition says that, under certain circumstances, we can rank the elasticity of demand faced by firm i at observation t ($\lambda_{i,t}$) with the elasticity it faces at observation t' ($\lambda_{i,t'}$). This condition arises from the combination of the log-concave, substitutes, and co-evolving assumptions on the demand system; there is no analog to this restriction in Theorem 1 because that model does not require the inverse demand functions across observations to be globally ranked by elasticities.

Proof (that [B] implies [A] in Theorem 3): Lemma 2, condition (iii) implies that firm i has a cost function $\bar{C}_i$ with $\delta_{i,t} \in \bar{C}_i'(Q_{i,t})$. The function $\bar{C}_i$ is increasing and convex in q_i. Next, we choose, for each i and t, a continuous function $\varepsilon_{i,t} : \mathbb{R}^I_{++} \to \mathbb{R}$ that is strictly positive, nondecreasing in p_i and nonincreasing in p_{-i}, such that $\varepsilon_{i,t}(P_t) = \lambda_{i,t}$ and $\varepsilon_{i,t}(p) \leq \varepsilon_{i,t'}(p)$ for all p, whenever $\sigma(t) < \sigma(t')$. This is possible because of (iv). The log-demand function for good i is chosen to be

$$\bar{L}_{i,t}(p) = -\int_0^{p_i} \varepsilon_{i,t}(z, p_{-i})\, \mathrm{d}z + a_{i,t} \tag{17}$$

[15] Notice that (ii) is equivalent to $(P_t - \delta_{i,t})/Q_{i,t} = (P_t - \delta_{i,t})/Q_{i,t} > 0$ for any two firms i and j and for all t. In CDFQ, this is called the *common ratio property.*

where $a_{i,t}$ is chosen so that $\bar{Q}_{i,t}(P_t) = \exp(\bar{L}_{i,t}(P_t)) = Q_{i,t}$. With this choice of $\bar{Q}_{i,t}$,

$$\bar{\varepsilon}_{i,t}(P_t) = -\frac{1}{\bar{Q}_{i,t}} \frac{\partial \bar{Q}_{i,t}}{\partial p_i}(P_t) = \varepsilon_{i,t}(P_t) = \lambda_{i,t}.$$

Condition (ii) implies that, given the chosen demand curves and cost functions, the observed quantities satisfy firm i's first-order conditions for profit maximization. Furthermore, note that firm i's demand curve is log-concave (since $\varepsilon_{i,t}$ is nondecreasing in p_i) and its cost function is convex in its output; together these imply that firm i's profit function is quasiconcave in p_i and so the first-order condition is also sufficient for optimality.[16] Finally our assumptions on $\varepsilon_{i,t}$ guarantee that the demand system constructed from it with (17) also has the substitutes and co-evolving properties. □

3.3 Examples of rationalizable and nonrationalizable data sets

In order to show that our model of Bertrand oligopoly is refutable, we now present an example of data that cannot be rationalized by the model.

Example 2 Consider an industry with two, firms 1 and 2, each producing a differentiated good. Suppose that between observations t and t' the following hold: $P_{1,t} < P_{1,t'}$, $P_{2,t} > P_{2,t'}$, $Q_{1,t} > Q_{1,t'}$, $Q_{2,t} < Q_{2,t'}$. In other words, going from t to t', firm 1 is charging more and producing less, while firm 2 is doing the opposite. We claim that these observations are not Bertrand rationalizable.

Indeed, the log-concavity and substitutes properties say that

$$\bar{\varepsilon}_{1,t}(P_{1,t}, P_{2,t}) \le \bar{\varepsilon}_{1,t}(P_{1,t'}, P_{2,t}) \le \bar{\varepsilon}_{1,t}(P_{1,t'}, P_{2,t'}). \tag{18}$$

Note that the first-order condition (16) can be rewritten as

$$P_{1,s} - \delta_{1,s} = \frac{1}{\bar{\varepsilon}_{1,s}(P_{1,s}, P_{2,s})}, \tag{19}$$

[16] The profit function $\bar{\Pi}_i$ of firm i is log-concave if, and only if,

$$-\bar{\Pi}_i'(p_i) = \bar{C}_i'(\bar{Q}_i(p_i))\bar{Q}_i'(p_i) - \bar{Q}_i(p_i) - p_i\bar{Q}_i'(p_i)$$

is a single crossing function of p_i. (We are suppressing the dependence of $\bar{Q}_i$ on p_{-i} in the notation.) Since $\bar{C}_i'(\bar{Q}_i(p_i))\bar{Q}_i'(p_i) - \bar{Q}_i(p_i)$ and $-p_i\bar{Q}_i'(p_i)$ are both single crossing functions of p_i (indeed, the first is a negative-valued function and the second a positive-valued function), it suffices to show that ratio

$$\frac{C_i'(\bar{Q}_i(p_i))\bar{Q}_i'(p_i) - \bar{Q}_i(p_i)}{p_i\bar{Q}_i'(p_i)} = \frac{1 + \bar{\varepsilon}_i(p_i)C_i'(\bar{Q}_i(p_i))}{p_i\bar{\varepsilon}_i(p_i)}$$

is decreasing in p_i (see Quah and Strulovici 2012). This is true because $\bar{C}_i'$ is nondecreasing in q_i while $\bar{\varepsilon}(p_i)$ and $\bar{Q}_i$ are both nonincreasing in p_i.

(for $s = t,\ t'$) where $\delta_{1,s} \in \bar{C}_1'(Q_{1,s})$. Furthermore, $P_{1,t} < P_{1,t'}$, $Q_{1,t} > Q_{1,t'}$, and Firm 1 has a convex cost function. It follows that $P_{1,t} - \delta_{1,t} < P_{1,t'} - \delta_{1,t'}$ and thus

$$\bar{\varepsilon}_{1,t}(P_{1,t}, P_{2,t}) > \bar{\varepsilon}_{1,t'}(P_{1,t'}, P_{2,t'}). \tag{20}$$

It follows from (18) and (20) that

$$\bar{\varepsilon}_{1,t'}(P_{1,t'}, P_{2,t'}) < \bar{\varepsilon}_{1,t}(P_{1,t'}, P_{2,t'}). \tag{21}$$

In the case of Firm 2, the log-concave and substitutes properties tell us that

$$\bar{\varepsilon}_{2,t}(P_{1,t}, P_{2,t}) \geq \bar{\varepsilon}_{2,t}(P_{1,t'}, P_{2,t}) \geq \bar{\varepsilon}_{2,t}(P_{1,t'}, P_{2,t'}) \tag{22}$$

while it follows from (21) and the co-evolving property that $\bar{\varepsilon}_{2,t'}(P_{1,t'}, P_{2,t'}) \leq \bar{\varepsilon}_{2,t}(P_{1,t'}, P_{2,t'})$. Thus,

$$\bar{\varepsilon}_{2,t}(P_{1,t}, P_{2,t}) \geq \bar{\varepsilon}_{2,t'}(P_{1,t'}, P_{2,t'}).$$

Firm 2's first-order condition then guarantees (through the analogy of (19) for Firm 2) that there is $\delta_{2,s} \in \bar{C}_2'(Q_{2,s})$ for $s = t,\ t'$ such that

$$P_{2,t} - \delta_{2,t} \leq P_{2,t'} - \delta_{2,t'}, \tag{23}$$

but this cannot happen since $P_{2,t} > P_{2,t'}$ and $\delta_{2,t} \leq \delta_{2,t'}$ (the latter because $Q_{2,t} < Q_{2,t'}$).

It is also possible to check directly that the data set does not obey the conditions in [B]. Indeed, it follows from (iii) that $\delta_{1,t'} \leq \delta_{1,t}$, since $Q_{1,t'} > Q_{1,t}$. Using this and the fact that $P_{1,t} < P_{1,t'}$, condition (ii) tells us that $\lambda_{1,t} > \lambda_{1,t'}$. By (iv) and the fact that $P_{2,t} > P_{2,t'}$, this can only occur if $\sigma(t') > \sigma(t)$. An analogous argument applied to Firm 2 will tell us that $\lambda_{2,t} < \lambda_{2,t'}$ and that can only occur if $\sigma(t') < \sigma(t)$. So we obtain a contradiction.

The next example illustrates the role that cross-price effects can play in rationalizing the data. The data set in this example is Bertrand rationalizable and it can only be rationalized by a demand system with nonzero cross-price effects, which means that the firms are interacting with each other in a nontrivial way.

Example 3 Consider an industry with firms 1, 2, and 3, each producing a differentiated good and suppose that the following hold at observations t and t':

(i) for firm 1, $P_{1,t} = 10$, $P_{1,t'} = 12$, $Q_{1,t} = 10$, and $Q_{1,t'} = 8$;
(ii) for firm 2, $P_{2,t} = 10$, $P_{2,t'} = 9$, $Q_{2,t} = 5$, and $Q_{2,t'} = 6$; and
(iii) for firm 3, $P_{3,t} = 10$, $P_{3,t'} = 9$, $Q_{3,t} = 5$, and $Q_{3,t'} = 6$.

We claim that any rationalization of this data must involve nonzero cross-price effects. Note, firstly, that between t and t', the price charged by firm 1 has increased while that of firms 2 and 3 have fallen. Since firm 1's demand function has the log-concave and substitutes properties, we obtain

$$\bar{\varepsilon}_{1,t}(P_{1,t}, P_{2,t}, P_{3,t}) \leq \bar{\varepsilon}_{1,t}(P_{1,t'}, P_{2,t}, P_{3,t}) \leq \bar{\varepsilon}_{1,t}(P_{1,t'}, P_{2,t'}, P_{3,t'}). \tag{24}$$

On the other hand, since firm 1's price is higher and its output lower at t' compared with t, the first-order condition tells us that $\bar{\varepsilon}_{1,t}(P_{1,t}, P_{2,t}, P_{3,t}) > \bar{\varepsilon}_{1,t'}(P_{1,t'}, P_{2,t'} P_{3,t'})$. Thus, we obtain

$$\bar{\varepsilon}_{1,t'}(P_{1,t'}, P_{2,t'}, P_{3,t'}) < \bar{\varepsilon}_{1,t}(P_{1,t'}, P_{2,t'}, P_{3,t'}). \tag{25}$$

By the co-evolving property, firm 2's demand function must obey

$$\bar{\varepsilon}_{2,t'}(P_{1,t'}, P_{2,t'}, P_{3,t'}) \leq \bar{\varepsilon}_{2,t}(P_{1,t'}, P_{2,t'}, P_{3,t'}). \tag{26}$$

Firm 2's price is lower and its output higher at t' compared with t, the first-order condition implies that $\bar{\varepsilon}_{2,t}(P_{1,t}, P_{2,t}, P_{3,t}) < \bar{\varepsilon}_{2,t'}(P_{1,t'}, P_{2,t'}, P_{3,t'})$. Together with (26), we obtain

$$\bar{\varepsilon}_{2,t}(P_{1,t}, P_{2,t}, P_{3,t}) < \bar{\varepsilon}_{2,t}(P_{1,t'}, P_{2,t'}, P_{3,t'}). \tag{27}$$

So in *any* Bertrand rationalization of the data, (27) must hold. Since $P_{2,t'} < P_{2,t}$ and $P_{1,t'} > P_{1,t}$,

$$\bar{\varepsilon}_{2,t}(P_{1,t}, P_{2,t}, P_{3,t}) \geq \bar{\varepsilon}_{2,t}(P_{1,t'}, P_{2,t}, P_{3,t}) \geq \bar{\varepsilon}_{2,t}(P_{1,t'}, P_{2,t'}, P_{3,t}) \tag{28}$$

where the first inequality follows from the substitutes property (with respect to the price of firm 1) and the second from the log-concave property (with respect to firm 2's own price). (27) and (28) together gives us

$$\bar{\varepsilon}_{2,t}(P_{1,t'}, P_{2,t'}, P_{3,t}) < \bar{\varepsilon}_{2,t}(P_{1,t'}, P_{2,t'}, P_{3,t'}). \tag{29}$$

We conclude that any demand function for firm 2 that rationalizes the data must exhibit the following property: when the price charged by firm 3 falls from $P_{3,t}$ to $P_{3,t'}$, the own-price elasticity of firm 2's product will strictly increase.

It is straightforward to check that the data set passes the test set out in statement [B] of Theorem 3 if the unknowns are chosen as follows:

$$\begin{aligned}
&\sigma(t) = 2 > \sigma(t') = 1;\\
&\lambda_{1,t} = \frac{1}{5},\ \lambda_{1,t'} = \frac{1}{8},\ \delta_{1,t} = 5,\ \delta_{1,t'} = 4,\ C_{1,t} = 14,\ \text{ and } C_{1,t'} = 4;\\
&\lambda_{2,t} = \frac{1}{5},\ \lambda_{2,t'} = \frac{1}{4},\ \delta_{2,t} = 5,\ \delta_{2,t'} = 5,\ C_{2,t} = 25,\ \text{ and } C_{2,t'} = 30;\ \text{ and}\\
&\lambda_{3,t} = \frac{1}{5},\ \lambda_{3,t'} = \frac{1}{4},\ \delta_{3,t} = 5,\ \delta_{3,t'} = 5,\ C_{3,t} = 25,\ \text{ and } C_{3,t'} = 30.
\end{aligned}$$

3.4 Variations on Theorem 3

It is quite straightforward to adapt Theorem 3 to yield somewhat different tests of Bertrand rationalizability. In this section, we explain how we may modify the co-evolving property on demand by allowing for firm-specific demand shocks and also how we can relax the assumption that firms' cost functions do not change across observations. The idea is to incorporate observed parameters that are known to influence a firm's demand or marginal cost. We do not assume that these parameters affect demand or marginal costs via specific functional forms; instead, we assume that they convey ordinal information in the sense that they allow the observer to impose a partial order on a firm's demand elasticity or on its marginal costs. The inclusion of these parameters obviously raises the informational requirements of the test, but it enlarges the circumstances under which a test is possible.

We shall consider, in turn, the incorporation of firm-specific demand shocks and the incorporation of firm-specific cost shocks. The reader will have no difficulty seeing how a single Bertrand test that combines both these features can also be constructed but, in order to keep the exposition simple and clear, we shall not present it in that fashion.

Allowing for idiosyncratic demand shocks Suppose that an observer has access to some index of general economic conditions which he knows will affect the elasticity of demand for all firms in the industry. In addition, for each firm, he observes some index that affects only the demand facing that firm. Formally, these can be represented by an observable parameter $\alpha \in A \subseteq \mathbb{R}$ that raises demand elasticity at all firms in the same direction and, for each firm i, a parameter $\beta_i \in B_i \subseteq \mathbb{R}$ that only raises the elasticity of demand for firm i's output. The data set is then $\{(P_t, Q_t, \alpha_t, \beta_t)\}_{t\in\mathcal{T}}$, where α_t is the value of α at observation t, and $\beta_t = (\beta_{i,t})_{i\in\mathcal{I}}$ is the realization of β_i at t. This data set is Bertrand rationalizable if each observation can be explained as a Bertrand equilibrium, with the demand function for good i, $\bar{Q}_i(p, \alpha, \beta_i)$, having the following properties: it agrees with the observations, i.e., $\bar{Q}_i(P_t, \alpha_t, \beta_{i,t}) = Q_{i,t}$ (for all $t \in \mathcal{T}$), it obeys the log-concave and substitutes properties (in $p = (p_i, p_{-i})$), and the own-price elasticity of demand is nondecreasing in (α, β_i). The last property is equivalent to

$$\bar{\epsilon}_i(p, \alpha, \beta_i) \geq (=)\, \bar{\epsilon}_i(p, \alpha', \beta_i') \text{ for all } p \text{ whenever } (\alpha, \beta_i) > (=)\, (\alpha', \beta_i'), \quad (30)$$

where

$$\bar{\epsilon}_i(p, \alpha, \beta_i) = -\frac{1}{\bar{Q}_i}\frac{\partial \bar{Q}_i}{\partial p_i}(p, \alpha, \beta_i).$$

The test for this concept of rationalizability is easily obtained by modifying the one set out in Theorem 3. It would involve conditions (i), (ii), and (iii) (in [B]), with (iv) being replaced by the following new condition (iv):

[a] if $P_{i,t'} \geq P_{i,t}$, $P_{-i,t'} \leq P_{-i,t}$, and $(a_{t'}, \beta_{i,t'}) \geq (a_t, \beta_{i,t})$, then $\lambda_{i,t'} \geq \lambda_{i,t}$;
[b] if $P_{i,t'} = P_{i,t}$, $P_{-i,t'} = P_{-i,t}$, and $(a_t', \beta_{i,t'}) = (a_t, \beta_{i,t})$, then $\lambda_{i,t'} = \lambda_{i,t}$.

Notice that this test is computationally straightforward, since it involves checking for a solution to a family of linear inequalities. The necessity of these conditions is clear. As in the proof of Theorem 3, we set $C_{i,t} = \bar{C}_i(Q_{i,t})$ and $\lambda_{i,t} = \bar{\varepsilon}_{i,t}(P_t)$, from which we obtain properties (i), (ii), and (iii). It is then obvious that condition **[b]** holds. Furthermore, if $P_{i,t'} \geq P_{i,t}$ and $P_{-i,t'} \leq P_{-i,t}$, and $(a_t, \beta_{i,t}) \leq (a_{t'}, \beta_{i,t'})$, we obtain

$$\bar{\varepsilon}_{i,t}(P_{i,t}, P_{-i,t}) \leq \bar{\varepsilon}_{i,t}(P_{i,t'}, P_{-i,t}) \leq \bar{\varepsilon}_{i,t}(P_{i,t'}, P_{-i,t'}) \leq \bar{\varepsilon}_{i,t'}(P_{i,t'}, P_{-i,t'}),$$

where the first and second inequalities follow from the log-concave and substitute properties of the demand system, respectively, while the third one follows from $(a_t, \beta_{i,t}) \leq (a_{t'}, \beta_{i,t'})$. Hence, $\lambda_{i,t} \leq \lambda_{i,t'}$ as required by [**a**]. The sufficiency of conditions (i)–(iii), together with the new condition (iv), to guarantee Bertrand rationalization in the sense defined here can also be easily checked by mimicking the proof of Theorem 3.[17]

Note that to obtain substantive restrictions on data it is crucial that the observer can rank idiosyncratic shocks to each firm's demand, but it is permissible for him to be ignorant of how general shocks to demand are ranked. (In other words, the observability of α_t is not crucial.) This is clear from Example 2 in Sect. 3.3. In that case, the test for Bertrand rationalizability will require that we consider all possible rankings of the general shock to demand and then combine them with the idiosyncratic shocks to (at least partially) order each firm's demand function across observations. This is formally equivalent to introducing a fictitious parameter α_t, carrying out the test as described and, if necessary, repeating it up to $T!$ times to cover all the possible ways the general shock to demand can be ordered.

Allowing changes to firms' cost functions The test for Bertrand rationalizability provided by Theorem 3 requires firms' cost functions to stay unchanged across all observations in the data set. It is clear that if we allow cost functions to change arbitrarily across observations in a way that is unknown to the observer (in addition to the type of demand changes permitted by the test), then there can be no observable restrictions to the data. However, it is possible to modify the test to allow for changes to the cost functions, provided the observer *knows the direction* in which marginal costs have changed.

To be specific, suppose that, in addition to prices and firm-level outputs, there is also an observable parameter $w_i \in W_i \subseteq \mathbb{R}$ that has an impact on firm i's cost function, which we denote as $\bar{C}_i(\cdot; w_i)$. We assume that firm i has a differentiable cost function (so the set $\bar{C}'_i(\cdot; w_i)$ is a singleton) and its marginal cost is affected by w_i,

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[17] To construct the demand system, we choose, for each i, a function $\epsilon_i : \mathbb{R}^I_{++} \times A \times B_i \to \mathbb{R}$ such that $\epsilon(p, \alpha, \beta_i)$ is strictly positive, continuous in p, nondecreasing in p_i, nonincreasing in p_{-i} and nondecreasing in (α, β_i) such that $\epsilon_i(P_t, \alpha_t, \beta_{i,t}) = \lambda_{i,t}$. This is possible because of the new condition (iv). The log-demand function for good i is chosen to be $\bar{L}_i(p, \alpha, \beta_i) = -\int_0^{p_i} \epsilon_i(z, p_{-i}, \alpha, \beta_i)\, dz + k_i(\alpha, \beta_i)$ where the function k_i is chosen so that $\bar{Q}_i(P_t, \alpha_t, \beta_{i,t}) = \exp(\bar{L}_i(P_t, \alpha_t, \beta_{i,t}) = Q_{i,t}$ (for all $t \in \mathcal{T}$). $\bar{Q}_i(p, \alpha, \beta_i)$ has all the required properties of a demand function for product i and also satisfies

$$-\frac{1}{Q_{i,t}} \frac{\partial \bar{Q}_i}{\partial p_i}(P_t, \alpha_t, \beta_{i,t}) = \lambda_{i,t}.$$

with higher values of w_i leading to higher marginal costs, while costs are unchanged if the parameter is unchanged.[18] In other words,

$$\bar{C}_i'(q_i; \bar{w}_i) \geq (=)\, C_i'(q_i; \hat{w}_i) \text{ for all } q_i > 0 \text{ if } \bar{w}_i > (=)\, \hat{w}_i. \tag{31}$$

The data set is then $\{(P_t, Q_t, w_t)\}_{t \in \mathcal{T}}$, where $w_t = (w_{i,t})_{i \in \mathcal{I}}$ and $w_{i,t}$ is the value of w_i at observation t. We say that this data set is Bertrand rationalizable if each observation can be explained as a Bertrand equilibrium, with the demand system obeying the log-concave, substitutes and co-evolving properties and with each firm i having, at observation t, a differentiable and convex cost function $\bar{C}(\cdot; w_{i,t})$ that obeys (31).

The necessary and sufficient conditions on $\{(P_t, Q_t, w_t)\}_{t \in \mathcal{T}}$ for this hypothesis are a straightforward modification of that set out in [B] in Theorem 3: *there is a permutation of* $\mathcal{T}$, *denoted by the function* $\sigma : \mathcal{T} \to \mathcal{T}$, *and real numbers* $\lambda_{i,t}$ *and* $\delta_{i,t}$ *for all* $t \in \mathcal{T}$ *and* $i \in \mathcal{I}$, *such that the following holds:*

(i) $\delta_{i,t} > 0$ *and* $\lambda_{i,t} > 0$;
(ii) $\delta_{i,t}\lambda_{i,t} - P_{i,t}\lambda_{i,t} + 1 = 0$;
(iii) $\delta_{i,t'} \geq \delta_{i,t}$ *whenever* $(Q_{i,t'}, w_{i,t'}) > (Q_{i,t}, w_{i,t})$ *and* $\delta_{i,t'} = \delta_{i,t}$ *whenever* $(Q_{i,t'}, w_{i,t'}) = (Q_{i,t}, w_{i,t})$; *and*
(iv) *if* $P_{i,t'} \geq P_{i,t}$, $P_{-i,t'} \leq P_{-i,t}$ *and* $\sigma(t) < \sigma(t')$, *then* $\lambda_{i,t} \leq \lambda_{i,t'}$.

The proof of this claim is a straightforward modification of the arguments made in Sect. 3.2, so we shall leave the details to the reader. To check the necessity of these conditions, we set $\delta_{i,t} = \bar{C}_i'(\bar{Q}_{i,t}(P_t); w_{i,t})$ and $\lambda_{i,t} = \bar{\varepsilon}_{i,t}(P_t)$. The justification for (i), (ii), and (iv) remain the same as in Theorem 3, while condition (iii) follows from the convexity and differentiability of each firm's cost function and (31). To establish the sufficiency of these conditions, we first construct functions $\bar{m}_i(q_i; w_i)$ (for $(q_i; w_i) \in \mathbb{R}_+ \times W_i$) that are positive valued, nondecreasing in (q_i, w_i), continuous in q_i, and with $\bar{m}_i(Q_{i,t}; w_{i,t}) = \delta_{i,t}$. This is possible because of (iii). Then $\bar{C}_i(q_i; w_i) = \int_0^{q_i} \bar{m}_i(s; w_i)\, ds$ has the following properties: it is an increasing, differentiable, and convex function of q_i, it obeys (31), and $\bar{C}_i'(Q_{i,t}; w_{i,t}) = \delta_{i,t}$. The rest of the proof then follows that of Theorem 3 ([B] implies [A]) in Sect. 3.2.

Lastly, it should be quite clear that we can simultaneously incorporate idiosyncratic demand changes and changes to cost functions even though, for reasons of expositional clarity, we have treated them separately.

4 Conclusion

The purpose of this paper was to develop revealed preference tests for equilibrium in a multi-product oligopoly, akin to the classical revealed preference tests of utility maximization in consumer demand. The distinctive feature of our revealed preference

[18] We could think of w_i as the price of a factor, with a higher price leading to a higher marginal cost. Note that, instead of a one-dimensional parameter w_i, we could allow for a multi-dimensional parameter which is guaranteed to raise the marginal cost function of a firm only if it is higher in all dimensions; such a formulation is more complicated but can be handled in a similar way. The reader can consult the treatment of this issue in Carvajal et al. (2010).

tests is that they make minimal modeling assumptions on industry demand and, in particular, they do not require any information on variables that shift or twist demand curves. We have also discussed different ways in which these tests can be modified, for example, to account for additional information on the movement of firms' marginal costs. Obviously, we have not exhausted all the empirical contexts in which one may conceivably wish to carry out a test of oligopoly behavior, so it is not always possible to use our results 'off-the-shelf.' Nonetheless, the basic ideas and methods set out in this paper can inform the development of other tests of oligopoly interaction.

In cases where the revealed preference tests are not rejected, they also give information about the values of unobserved variables like firms' marginal costs. For example, we know from the proof of Theorem 2 in Section 2.5. that if a convex cost function $\bar{C}_i$ for firm i is part of a Cournot rationalization of a given data set, then there is $\left(\delta^k_{i,t}\right)_{k\in\mathcal{K}_i} \in \partial\bar{C}_i(Q_{i,t})$ (for all $t \in \mathcal{T}$) such that $\left(\delta^k_{i,t}\right)_{k\in\mathcal{K}_i}$ is part of a solution to the test [B] in Theorem 2. Conversely, if $\left(\delta^k_{i,t}\right)_{k\in\mathcal{K}_i}$ is part of the solution to [B], then there is rationalizing cost function $\bar{C}_i$ for firm i such that $(\delta^k_{i,t})_{k\in\mathcal{K}_i} \in \partial\bar{C}_i(Q_{i,t})$ (for all $t \in \mathcal{T}$). In other words, the test for Cournot rationalizability provides, as a by-product, set restrictions on each firm's marginal costs. Equation (2) is an instance of this: under the hypothesis of Cournot equilibrium, observation t in Example 1 gives us a lower bound on the marginal cost of firm j at its observed output. In that example, observation t' was chosen in such a way that the data set is not rationalizable, but it is possible for observation t to be part of a Cournot rationalizable data set; in that case, any counterfactual analysis of the industry from which this data set was obtained must respect this bound on marginal cost, so long as the Cournot hypothesis is maintained. Similarly, our test of Bertrand rationalizability yields information about firms' marginal costs. Suppose we modify the data set in Example 2 slightly and have $Q_{2,t} > Q_{2,t'}$ (while keeping the rest of the data unchanged). It is easy to see that this new data set (unlike Example 2 itself) will be consistent with Bertrand rationalizability. The arguments leading to (23) remain valid, and thus,

$$\delta_{2,t} \geq P_{2,t} - P_{2,t'} + \delta_{2,t'} \geq P_{2,t} - P_{2,t'} > 0.$$

In this way, we obtain a nontrivial lower bound on firm 2's marginal cost at all output levels $q_2 > Q_{2,t}$ (since firm 2 has a convex cost function).

Standard revealed preference tests are binary—either a data set passes the test or it fails. In empirical applications, it is often useful to introduce ways of accommodating or measuring 'small' deviations that prevent a data set from passing a test. In the context of consumer demand, where revealed preference methods are most mature, different ways have been developed to address this issue. For example, Varian (1985) has suggested a way in which Afriat's test of utility maximization could be modified to account for measurement error. In Carvajal et al. (2010), we showed how the standard test for the single-product Cournot oligopoly (stated in this paper as Theorem 1) can be modified to allow for measurement error. Besides this, there may well be other ways of modifying revealed preference tests for oligopoly models that will facilitate their empirical implementation. We leave the careful exploration of this issue for future research.

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