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On refutability of the Nash bargaining solution

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ABSTRACT

Empirical tests of the Nash bargaining solution (NBS, hereafter) are developed under different hypotheses about the behavior of disagreement utility levels.

1. Introduction

Among the most prominent solution concepts to bargaining problems is the one proposed by John Nash in 1950 and 1953. The studies that have investigated the empirical content of the NBS can be divided into two main streams: the differential approach and the revealed preference analysis. In the former, Manser and Brown (1980) studied the empirical content of household decision making (i.e., how to allocate resources and gains).[1] They define gains in a marriage to exist if the maximum utility that each individual can attain lies inside the utility possibility frontier. If this occurs, the household must decide on a distribution of resources and distribution of gains. They invoke two bargaining solution concepts, NBS and Kalai and Smorodinsky's (1975) solution concept, and also consider what they call the Pareto optimal solution, which is the dictatorial case. The comparative static properties of the individual demand functions are compared with maximizing a single utility function, using the Slutsky conditions.[2]

With respect to this approach, Chiappori (1988) has suggested that "these conditions are not restrictive, unless the agents' premarital preferences are known". Instead, Chiappori (1992) derived, from a collective setting, a set of testable restrictions on observable behavior, using labor supply: contrary to the Slutsky equations obtained from the traditional models, he suggested an alternative way of deriving structural conditions (i.e., conditions on parameters) on the functional forms for demand or labor supply functions. Chiappori and Donni (2006) extends this analysis to the case of the NBS.

The second strand of the literature follows the discussion of Samuelson (1938), Afriat (1967) and Varian (1982), which focuses on revealed preference theory of individual behavior to characterize rationalizability using a finite set of consumption data. Only recently has the literature obtained testable implications on data for game theoretical solution concepts. For example, Sprumont (2000) considers a non-cooperative game played by a finite number of players, each of whom can choose a strategy from a finite set, and identifies general necessary and sufficient conditions for Nash-rationalizability. Ray and Zhou (2001) focus on extensive form games and derive a set of necessary and sufficient conditions for sub-game Nash-rationalizability. Other related literatures include Carvajal (2010), who studied whether the Nash–Walras equilibrium imposes testable restrictions on the equilibrium prices of

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1 They step away from Becker's (1973, 1974) bargaining rule that the household maximizes one individual's utility function and explicitly allows for different utility functions within a household inhabited by two individuals.

2 A drawback of their work is the presence of a bargaining rule like the one allowed by Becker (1974) in order to guarantee the resolution of conflict. Along the same vein, McElroy and Horney (1981 and 1990), derive a Nash generalization of the Slutsky conditions for household demand function and explain behavior consistent with Nash behavior and the so-called neoclassical individual utility maximizer. For studies that refer to the differential approach, but use different solution concepts, see Lundberg and Pollack (1993), Chen and Woolley (2001). In non-strategic settings, the differential approach has allowed economists to argue that non-observed fundamentals can be unambiguously recovered from observed equilibrium outcomes; for a survey of this literature, and also of early literature on the revealed preference approach, see Carvajal et al. (2004).

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economies with externalities, and Carvajal et al. (2012), who developed the revealed preference analysis of the Cournot model of oligopolistic competition.

The purpose of this paper is to investigate the testable restrictions that the NBS has on the allocations of an aggregate resource (i.e., sharing of a pie) from a revealed preference analysis. We develop polynomial tests for the NBS, under different hypotheses about the behavior of disagreement levels, and use the Tarski–Seidenberg algorithm to characterize rationalizable data as those that satisfy a finite system of polynomial inequalities. Following the revealed preference approach, we do not impose any parametric assumptions; thus we do not test the consistency of the observed behavior conditional on a specific functional form. For instance, our results do not rely on empirical work that uses parametric specifications of preferences, or other restrictive assumptions of the parameters of the model.

In this respect, Chambers and Echenique (forthcoming) is closely related in the sense that they too focus on the allocation of a single-dimensional resource. They assume that the analyst has available data on how some amounts of money are divided amongst a group of agents, but has no information on the individuals' preferences and the protocol that leads to the division. They select three theories that could possibly explain the division of money, those of the utilitarian, Nash and egalitarian max–min models (assuming that the observed disagreement utility levels are fixed) and show that all three models are observationally equivalent. A main difference is that we characterize rationalizable data as those that satisfy a system of quadratic inequalities and apply the Tarski–Seidenberg algorithm to obtain a test for the NBS (under various hypotheses about the behavior of the default utility levels), unlike them who use a dual characterization of the problem that satisfies a system of polynomial inequalities. Also, Chambers and Echenique only consider a subset of the scenarios that we study here.

Cherchye et al. (2013) also investigate the empirical content of the NBS, but they focus on a different setting from ours: they consider a model where a pair of agents bargain over income and allow agents to make individual consumption purchases on their own. This implies therefore that they assume disagreement points that vary endogenously, whereas here the default utility levels are exogenously determined. More importantly, their test is developed for the joint hypothesis that the income is distributed according to the NBS, while the individual consumption bundles are chosen by rational, price-taking consumers. If their test is rejected, it is thus not clear which of the two hypotheses failed. In this sense, the joint application of our test and theirs would allow an analyst to determine the failure of the NBS hypothesis independently.[3]

We find that the NBS hypothesis is refutable – upon observation of the allocation of the total resources – if the analyst has information on the behavior of the default utility levels. This result is of practical relevance, as are our other results, because it allows us to show that not every set of allocations that exhausts the total amount of the resource can be explained by the symmetric solution. For example, within the literature on household behavior, our results are useful to derive falsifiable conditions upon household labor supplies from a non-parametric viewpoint.

The outline of the paper is as follows. In Section 2, we define the NBS under the assumption that the outside econometrician can observe both the allocation of an aggregate endowment and the utility levels individuals can obtain, provided that they do not reach an agreement. We derive the necessary and sufficient conditions for a data set to be rationalizable by the NBS model. Then, in Section 3, we drop the strong assumption that default utility levels are observable and find that the hypothesis has no scientific meaning, from a Popperian viewpoint: it is not falsifiable. We restore falsifiability when we impose some conditions on the unobservable default utility levels. After showing that the solution concept can be refuted, under conditions on the unobserved default levels, in Sections 4 and 5 we assume that the econometrician has information about the behavior of the income levels that the individuals can attain outside the cooperative agreement and test if a data set like this is rationalizable.

Due to the constricted nature of our tests (i.e. the data either satisfies the NBS hypothesis, or it does not) if a data set is not rationalizable, we do not know if by applying a small perturbation to the data the test of rationalizability may pass.[4] In Section 6, and in the tradition of Varian (1985), we measure the magnitude of departure from the NBS hypothesis, i.e. we construct a statistical version of the test of rationalizability. In Section 7 we generalize the setting of Section 2 for an arbitrary number of players and asymmetric bargaining powers. Finally, in Section 8 we conclude.

2. Nash bargaining model

Suppose that we observe, for a finite number of situations, the way in which two people split a common endowment: for each t in the set $\{1, \ldots, T\}$, we observe the allocation $(x_t^1, x_t^2) \in \mathbb{R}^2$, of an aggregate resource $X_t \in \mathbb{R}$. When can we guarantee that this information can be modeled by the NBS, given that we cannot observe the individuals' utility functions?

Let us denote by $u^i(x)$ the utility level that individual i attains if she consumes x units of the resource. If the collective decision of the two people is consistent with the Nash solution, each observation must solve the program

$$\max_{x^1,x^2}\{[u^1(x^1) - v_t^1][u^2(x^2) - v_t^2] : x^1 + x^2 = X_t \text{ and } u^i(x^i) > v_t^i\}, \quad (1)$$

where v_t^1 and v_t^2 are exogenously determined utility levels that the players can obtain by themselves if they break up the negotiations. Note that we assume, as in the original work of Nash, that there is a feasible allocation that leaves both individuals strictly better off than if they do withdraw from the negotiations. Now, suppose that all an outside econometrician can observe is the data

$$\left\{(x_t^1, x_t^2, X_t) : t = 1, \ldots, T\right\}, \quad (2)$$

but he does not know the utility functions u^1 and u^2. What restrictions does the structure of Program (1) impose on data set (2)?[5]

2.1. Under observation of default utility levels

Surely, utility levels are not observable by an analyst. But for the sake of argument, suppose for the moment that the analyst also has information about the utility levels that both individuals would

[3] Cherchye, Demuynck and De Rock also design and conduct an experimental application of their tests.

[4] In the words of Varian (1985), "If some data fail the tests, but only by a small amount, we might be tempted to attribute this failure to measurement error, left out variables, or other sorts of stochastic influences rather than to reject the hypothesis outright".

[5] It is important to note that we test the hypothesis that the observed outcome is derived from the NBS under a restricted domain of problems: those corresponding to a given pair of individuals, for a finite set of aggregate resources and default payoffs. This is in contrast to the axiomatic analysis of the solution, where the entire universe of bargaining problems is considered. In this sense, our results do not offer an axiomatization of the solution, but just the set of all the necessary conditions that the data have to satisfy if they are derived from the solution in the finite set of observations of the individuals.

attain if they did not agree on how to share the resource.[6] In this case, the data set would be of the form

$$\left\{(x_t^1, x_t^2, X_t, v_t^1, v_t^2) : t = 1, \ldots, T\right\}, \tag{3}$$

where we assume that $x_t^1 + x_t^2 = X_t$ at all observations.[7]

We say that data set (3) *is rationalizable* if there exist utility functions $u^1 : \mathbb{R} \to \mathbb{R}$ and $u^2 : \mathbb{R} \to \mathbb{R}$, both of which are strictly increasing and strictly concave, such that at each observation t, the pair (x_t^1, x_t^2) solves Program (1).[8]

2.2. A test of rationalizability: necessary conditions

Proposition 1. *If a data set of form (3) is rationalizable, then there exists an array of numbers*

$$\left\{(\mu_t^1, \mu_t^2, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the following system: for all t and t',

$$\mu_{t'}^i \le \mu_t^i + \lambda_t(\mu_t^i - v_t^i)(x_{t'}^i - x_t^i), \tag{4}$$

with strict inequality if $x_t^i \neq x_{t'}^i$; for all i and all t,

$$\mu_t^i > v_t^i; \tag{5}$$

and for all t,

$$\lambda_t > 0. \tag{6}$$

Proof. Under strict monotonicity, we can re-write Program (1) as

$$\min_x \left\{f_t(x) : u^1(x) > v_t^1 \text{ and } u^2(X_t - x) > v_t^2\right\}, \tag{7}$$

where

$$f_t(x) := [v_t^1 - u^1(x)][u^2(X - x) - v_t^2].$$

Since both utility functions are concave, they are Lipschitz continuous and, hence, a necessary condition for x_t^1 to solve Program (7) is that $0 \in \partial f_t(x_t^1)$.[9] By construction, $0 \in \partial f_t(x_t^1)$ only if there exist $\delta_t^1 \in \partial u^1(x_t^1)$ and $\delta_t^2 \in \partial u^2(x_t^2)$ such that

$$\delta_t^1[u^2(X_t - x_t^1) - v_t^2] = \delta_t^2[u^1(x_t^1) - v_t^1].$$

Since function u^1 is strictly increasing, if we define the number

$$\lambda_t := \frac{\delta_t^1}{u^1(x_t^1) - v_t^1} > 0,$$

we get that, for both individuals, necessarily, $\delta_t^i = \lambda_t[u^i(x_t^i) - v_t^i]$. Since both u^1 and u^2 are strictly concave and $\delta_t^i \in \partial u^i(x_t^i)$, the latter implies that, for all $x \in \mathbb{R}$, $x \neq x_t^i$,

$$u^i(x) < u^i(x_t^i) + \lambda_t[u^i(x_t^i) - v_t^i](x - x_t^i).$$

Thus, we can define the numbers $\mu_t^i := u^i(x_t^i) > v_t^i$. □

A comment on the implications of this proposition is in order.[10] It is well known that the NBS treats the preferences of the individuals as cardinal objects. It may then seem paradoxical that our revealed preference approach is applicable in this context, since this approach is normally of ordinal nature. But it is then important to note that the necessary condition (4) imposes a common factor to the term that appears on the right-hand side of the usual Afriat expansion: λ_t is common to both individuals. This commonality gives the whole system cardinal content: the λ_t that solves Eq. (4) is *not* robust to arbitrary transformations of the preferences of the individuals, even if they preserve their ordinal content.

While this result provides a necessary condition for rationalizability, by itself it does not constitute a test of the hypothesis, for in principle it could be that the necessary condition is tautological, in which case any data set would be rationalizable and the hypothesis would not be refutable.

2.3. Power of the test: sufficiency and non-tautology of the necessary condition

Our next claim is that the existence of a solution to the system of inequalities defined by Eqs. (4)–(6) exhausts the necessary conditions of the hypothesis that a data set is rationalizable, as this condition is also sufficient for the hypothesis.

Proposition 2. *Given a data set of form (3), suppose that there exists an array of numbers*

$$\left\{(\mu_t^1, \mu_t^2, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the system of inequalities (4)–(6), defined in Proposition 1. Then, the data set is rationalizable, and the utility functions that rationalize it can be constructed in the class $\mathbf{C}^2$.

Proof. Given a solution $\left\{(\mu_t^1, \mu_t^2, \lambda_t) : t = 1, \ldots, T\right\}$ to the system of inequalities, construct the following utility functions: for each player i,

$$u_0^i(x) := \min\left\{\mu_t^i + \lambda_t(\mu_t^i - v_t^i)(x - x_t^i) : t = 1, \ldots, T\right\},$$

mapping $\mathbb{R}$ into $\mathbb{R}$. These functions are continuous, concave and strictly increasing and are $\mathbf{C}^\infty$ at all but a finite number of points in $\mathbb{R}$.

With these constructions, and for any observation t, note that $u_0^i(x_t^i) \le \mu_t^i$. If this inequality was strict, then for some other observation t' we would have that

$$\mu_{t'}^i + \lambda_{t'}(\mu_{t'}^i - v_{t'}^i)(x_t^i - x_{t'}^i) < \mu_t^i,$$

which contradicts Eq. (4),[11] so we must conclude that $u_0^i(x_t^i) = \mu_t^i$.

Now, consider any pair $(x^1, x^2) \neq (x_t^1, x_t^2)$ that is feasible in the program

$$\max_{x^1,x^2}\{[u_0^1(x^1) - v_t^1][u_0^2(x^2) - v_t^2] : x^1 + x^2 \le X_t \text{ and } u_0^i(x^i) > v_t^i\}. \tag{8}$$

By definition and construction,

$$0 < u_0^i(x^i) - v_t^i \le \mu_t^i + \lambda_t(\mu_t^i - v_t^i)(x^i - x_t^i) - v_t^i$$

for both $i = 1, 2$. Multiplying, we thus get that

$$\begin{aligned}
&[u_0^1(x^1) - v_t^1][u_0^2(x^2) - v_t^2] \\
&\le [\mu_t^1 + \lambda_t(\mu_t^1 - v_t^1)(x^1 - x_t^1) - v_t^1] \\
&\quad \times [\mu_t^2 + \lambda_t(\mu_t^2 - v_t^2)(x^2 - x_t^2) - v_t^2] \\
&= (\mu_t^1 - v_t^1)(\mu_t^2 - v_t^2)[1 + \lambda_t(x^1 - x_t^1)][1 + \lambda_t(x^2 - x_t^2)] \\
&= (\mu_t^1 - v_t^1)(\mu_t^2 - v_t^2)[1 + \lambda_t(x^1 + x^2 - X_t) \\
&\quad + (\lambda_t)^2(x^1 - x_t^1)(x^2 - x_t^2)],
\end{aligned}$$

6 Certainly, we will later drop this assumption.

7 We comment below why we do not allow for waste of the aggregate resource.

8 Since we impose strict monotonicity of preferences, it is an immediate testable implication of the rationalizability definition that the individuals do not want to waste resources. Of course, there are other solution concepts in which the latter is not true, and the assumption that $x_t^1 + x_t^2 = X_t$ would be restrictive if we were studying those solutions. For the focus of our paper, the question is what restrictions, other than the fact that there is no waste, are implied by the solution hypothesis.

9 We use ∂g to denote the subgradient of any function g.

10 We thank Alejandro Saporiti for this observation.

11 By Eqs. (4) the linear approximation must be the tangent line to the general function, i.e. $u_0^i(x_t^i) = u^i(x_t^i)$.

where, in the last line, we have used the fact that $x_t^1 + x_t^2 = X_t$. Now, consider each of the terms on the right-hand side of the latter expression: first, by feasibility of (x^1, x^2), we have that

$$\mu_t^1 - v_t^1 > 0 \quad \text{and} \quad \mu_t^2 - v_t^2 > 0;$$

also, by Eq. (6) and feasibility, we have that

$$\lambda_t(x^1 + x^2 - X_t) \leq 0;$$

and, finally, we have that

$$(x^1 - x_t^1)(x^2 - x_t^2) \leq 0,$$

since again, by feasibility, $x^1 + x^2 \leq X_t = x_t^1 + x_t^2$. Since $(x^1, x^2) \neq (x_t^1, x_t^2)$, the previous two inequalities cannot hold with equality at the same time. This implies that

$$1 + \lambda_t(x^1 + x^2 - X_t) + (\lambda_t)^2(x^1 - x_t^1)(x^2 - x_t^2) < 1,$$

and hence that

$$\begin{aligned}(\mu_t^1 - v_t^1)(\mu_t^2 - v_t^2)[1 + \lambda_t(x^1 + x^2 - X_t) \\ + (\lambda_t)^2(x^1 - x_t^1)(x^2 - x_t^2)] \\ < (\mu_t^1 - v_t^1)(\mu_t^2 - v_t^2).\end{aligned}$$

We conclude, hence, that

$$\begin{aligned}[u_0^1(x^1) - v_t^1][u_0^2(x^2) - v_t^2] &< (\mu_t^1 - v_t^1)(\mu_t^2 - v_t^2) \\ &= [u_0^1(x_t^1) - v_t^1][u_0^2(x_t^2) - v_t^2], \end{aligned} \tag{9}$$

and since, by Eq. (5), $u_0^i(x_t^i) > v_t^i$ for both $i = 1, 2$, and $x_t^1 + x_t^2 = X_t$, we conclude that (x_t^1, x_t^2) solves Program (8).

To complete the proof, we ought to show that u_0^1 and u_0^2 can be deformed into functions u^1 and u^2 that are strictly concave and smooth. This can be done, using a convolution, since the inequalities of Eqs. (4) and (9) are all strict whenever $x_t^i \neq x_{t'}^i$, and since the number of observations is finite. The details of this construction are deferred to Appendix. □

An implication of the proposition is that, as is commonly the case in the revealed preference literature, an analyst would need at least two observations in order to be able to reject the hypothesis of the NBS: if $T = 1$, then condition (4) is vacuous and the whole system is always satisfied.[12]

More importantly, together with Proposition 1 this result allows us to claim that the hypothesis of rationalizability is testable and to state the type of test an analyst can develop.

Proposition 3. *There exists a non-tautological condition that a data set of form (3) satisfies if, and only if, it is rationalizable. Moreover, this condition is a finite set of polynomial inequalities on* $\{(x_t^1, x_t^2, X_t, v_t^1, v_t^2) : t = 1, \dots, T\}$.

Proof. The set of values of

$$\{(x_t^1, x_t^2, X_t, v_t^1, v_t^2, \mu_t^1, \mu_t^2, \lambda_t) : t = 1, \dots, T\}$$

that satisfy the system of inequalities defined by Eqs. (4)–(6) is, by definition, a semi-algebraic set. By the Tarski–Seidenberg algorithm, the projection of this set into the space of data (that is, of values of $\{(x_t^1, x_t^2, X_t, v_t^1, v_t^2) : t = 1, \dots, T\}$) is semi-algebraic as well, which means that it can be characterized by a finite set of polynomial inequalities. Then, by Propositions 1 and 2, a data set is rationalizable if, and only if, it satisfies these polynomial inequalities (or, put another way, if it lies in the latter projected set).

To see that such a system of polynomial inequalities is not tautological, it suffices to find a non-rationalizable data set. To see this, consider a set of the form (3) where, for a pair of observations t and t' one has that $v_t^1 = v_{t'}^1$, $v_t^2 = v_{t'}^2$, $X_t < X_{t'}$ and $x_t^1 > x_{t'}^1$. If such a set were rationalizable, there would exist $\delta_t^i \in \partial u^i(x_t^i)$ and $\delta_{t'}^i \in \partial u^i(x_{t'}^i)$, for $i = 1, 2$, such that

$$\delta_t^1[u^2(x_t^2) - v_t^2] = \delta_t^2[u^1(x_t^1) - v_t^1] \tag{10}$$

and

$$\delta_{t'}^1[u^2(x_{t'}^2) - v_{t'}^2] = \delta_{t'}^2[u^1(x_{t'}^1) - v_{t'}^1]. \tag{11}$$

By the concavity of the utility functions, $\delta_t^1 < \delta_{t'}^1$ and $\delta_t^2 > \delta_{t'}^2$, given that $x_t^2 < x_{t'}^2$ since $X_t < X_{t'}$. By their monotonicity, similarly, $u^1(x_t^1) > u^1(x_{t'}^1)$ and $u^2(x_t^2) < u^2(x_{t'}^2)$. It then follows that

$$\begin{aligned}\delta_t^1[u^2(x_t^2) - v_t^2] &< \delta_{t'}^1[u^2(x_{t'}^2) - v_t^2] \\ &= \delta_{t'}^1[u^2(x_{t'}^2) - v_{t'}^2] \\ &= \delta_{t'}^2[u^1(x_{t'}^1) - v_{t'}^1] \\ &= \delta_{t'}^2[u^1(x_{t'}^1) - v_t^1] \\ &< \delta_t^2[u^1(x_t^1) - v_t^1] \\ &= \delta_t^1[u^2(x_t^2) - v_t^2],\end{aligned}$$

where the second equality follows from Eq. (11), and the last one from (10). This is obviously impossible, so we conclude that the data set cannot be rationalized. □

In fact, it is useful to complement the proof of this proposition with an analysis of the comparative statics of Program (1), for the case when the utility functions are differentiable twice. In such a case, note that we can write the first-order condition of the program

$$\max_{x^1, x^2}\{[u^1(x^1) - v^1][u^2(X - x^1) - v^2] : u^1(x^1) > v^1 \text{ and } u^2(X - x^1) > v^2\},$$

as

$$\partial u^1(x^1)[u^2(X - x^1) - v^2] = \partial u^2(X - x^1)[u^1(x^1) - v^1].$$

If we totally differentiate this equation, we get that, over the manifold of solutions to the program, dx^1 equals

$$\frac{\{\partial^2 u^2(x^2)[u^1(x^1) - v^1] - \partial u^1(x^1)\partial u^2(x^2)\}\, dX - \partial u^2(x^2)dv^1 + \partial u^1(x^1)dv^2}{\partial^2 u^1(x^1)[u^2(x^2) - v^2] - 2\partial u^1(x^1)\partial u^2(x^2) + \partial^2 u^2(x^2)[u^1(x^1) - v^1]}. \tag{12}$$

Since both utility functions are strictly increasing and strictly concave, and since $u^1(x^1) > v^1$ and $u^2(X - x^1) > v^2$, it follows that the denominator in this equation is negative. For the same reasons, the term that multiplies dX in the numerator is negative too, while the terms that multiply dv^1 and dv^2 are both positive. As a consequence, it follows that if $dX > 0$, $dv^1 > 0$ and $dv^2 < 0$, then, unambiguously, $dx^1 > 0$. It then follows that a data set of form (3) is *not* rationalizable if it contains a pair of distinct observations, t and t', such that

$$X_t > X_{t'}, \qquad v_t^1 > v_{t'}^1, \qquad v_t^2 < v_{t'}^2 \quad \text{and} \quad x_t^1 < x_{t'}^1.$$

Subsequently, this observation will allow us to claim that, under differentiable utility functions, the hypothesis of rationalizability is refutable on the basis of a test.

[12] An explicit construction that rationalizes any single observation (x_1, x_2, X, v_1, v_2) is as follows: assume, with no loss of generality, that $x_2 > x_1 > 0$, and let $u^1(x) = \ln x + \alpha$ and $u^2(x) = \ln x + \beta$, with $\beta > v_2 + \max\{0, -\ln x_2\}$ and

$$\alpha = v^1 - \ln x_1 + \frac{x_2}{x_1}(\ln x_2 + \beta - v_2).$$

3. Without observation of default utility levels

Suppose now that the observer has no information about the default utility levels of the two individuals so that the data set reduces to (2).[13]

3.1. No assumptions on the default utility levels: unfalsifiability

Suppose that the analyst is not willing to impose any conditions on the (unobserved) utility levels that the individuals could have obtained by withdrawing from the negotiation. In this setting, we shall say that the data set of form (2) *is rationalizable* if there exist default utility levels v_t^1 and v_t^2, for $t = 1, \ldots, T$, and individual utility functions $u^1 : \mathbb{R} \to \mathbb{R}$ and $u^2 : \mathbb{R} \to \mathbb{R}$, both of which are strictly increasing and strictly concave, and are such that at each observation t, the pair (x_t^1, x_t^2) solves Program (1).

Our next result shows that in this case the hypothesis of rationalizability becomes irrefutable.

Proposition 4. *Any data set of the form (2) is rationalizable. Moreover, in the rationalization both utility functions can be constructed in the class* $\mathbf{C}^2$.

Before proving the proposition, we introduce a lemma that will be useful for the result.

Lemma 1. *For any finite set of numbers* $\{x_s : s = 1, \ldots, S\}$, *there exists an array of pairs of numbers,* $\{(\mu_s, \delta_s) : s = 1, \ldots, S\}$, *such that* $\delta_s > 0$ *for all s, and*

$$\mu_{s'} < \mu_s + \delta_s(x_{s'} - x_s)$$

for all s and all $s' \neq s$.

Proof. With no loss of generality, let us assume that $x_1 > x_2 > \cdots > x_S$. We can re-write the desired conditions as

$$\mu_{s'} - \mu_s + \delta_s(x_s - x_{s'}) < 0,$$

for all s and all $s' \neq s$, with $-\delta_s < 0$ for all s.

If such a system has no solution, then, by the Theorem of the Alternative,[14] we can find a double array of non-negative numbers $\{\alpha_{s,s'} : s, s' = 1, \ldots, S, s' \neq s\}$ and an array of non-negative numbers $\{\beta_s : s = 1, \ldots, S\}$ such that, for all s,

$$\sum_{s' \neq s} \alpha_{s,s'} = \sum_{s' \neq s} \alpha_{s',s} \qquad (13)$$

and

$$\sum_{s' \neq s} \alpha_{s,s'}(x_s - x_{s'}) = \beta_s, \qquad (14)$$

with at least one of the numbers in these two arrays being different from zero.

Aggregating Eq. (14) across observations, $\sum_s \sum_{s' \neq s} \alpha_{s,s'}(x_s - x_{s'}) = \sum_s \beta_s$. By Eq. (13), the left-hand side of the latter expression is null, which in turn implies that $\beta_s = 0$ for all s. Using this, Eq. (14) implies that $\sum_{s' \neq 1} \alpha_{1,s'}(x_1 - x_{s'}) = 0$. But note that, since $x_1 > x_s$ for all $s \neq 1$, this equality is possible only if $\alpha_{1,s'} = 0$ for all $s' \neq 1$. Then, (13) implies that, moreover, $\alpha_{s',1} = 0$ for all $s' \neq 1$. But then, for $s = 2$, (14) implies that

$$\sum_{s \geq 3} \alpha_{2,s'}(x_2 - x_{s'}) = \sum_{s' \neq 2} \alpha_{2,s'}(x_2 - x_{s'}) = 0.$$

Again, since $x_2 > x_s$ for all $s \geq 3$, the latter implies that $\alpha_{2,s'} = 0$ for all $s' \neq 2$, and (13) again implies that $\alpha_{s',2} = 0$ for all $s' \neq 2$. Continuing in this fashion, we obtain that $\alpha_{s,s'} = 0$ for all s and all $s' \neq s$, which contradicts the fact that at least one of the numbers in the two arrays is different from zero. □

This lemma is the key step in the Proof of Proposition 4.

Proof of Proposition 4. Given the data set, by Lemma 1 we have that for each individual $i = 1, 2$, we can find numbers μ_t^i and $\delta_t^i > 0$ for all t, such that

$$\mu_{t'}^i \leq \mu_t^i + \delta_t^i(x_{t'}^i - x_t^i),$$

with strict inequality if $x_{t'}^i \neq x_t^i$.

Now, fix an arbitrary array of numbers $\left\{v_t^1 : t = 1, \ldots, T\right\}$ such that $v_t^1 < \mu_t^1$ at all t. Using these numbers, define, for each t,

$$\lambda_t := \frac{\delta_t^1}{\mu_t^1 - v_t^1}$$

and

$$v_t^2 := \mu_t^2 - \frac{\delta_t^2}{\lambda_t}.$$

By construction, $\lambda_t > 0$ and $v_t^2 < \mu_t^2$, while it is immediate that

$$\mu_t^i + \lambda_t(\mu_t^i - v_t^i)(x_{t'}^i - x_t^i) = \mu_t^i + \delta_t^i(x_{t'}^i - x_t^i) \geq \mu_{t'}^i$$

with strict inequality if $x_{t'}^i - x_t^i$. By Proposition 2, it follows that there are individual utility functions u^1 and u^2 that satisfy the desired properties and are such that at each observation t, the pair (x_t^1, x_t^2) solves Program (1). □

Proposition 4 tells the analyst that if he is not willing to make any assumptions on the levels of utility that the two individuals could secure for themselves by breaking up the negotiations, or on the evolution of these levels, the hypothesis that the data can be explained using the NBS is not refutable and, hence, the hypothesis itself is unscientific from a Popperian perspective. In fact, a careful look at the proof of that proposition shows that even if the analyst had observations of the default utility levels of one of the individuals, but not on the levels of the other one, then the hypothesis of NBS would be unfalsifiable: this is, indeed, what the proof actually shows. Moreover, with a small modification of the proof one can show that the player for whom the analyst has observed the default utility levels need not be the same at all observations: if, for each observation, one observes v_t^i but not $v_t^{\neg i}$, the argument continues to hold by defining

$$\lambda_t := \frac{\delta_t^i}{\mu_t^i - v_t^i} > 0$$

and

$$v_t^{\neg i} := \mu_t^{\neg i} - \frac{\delta_t^{\neg i}}{\lambda_t} < \mu_t^{\neg i};$$

even in this case, the hypothesis continues to be irrefutable.

3.2. Some assumptions on the default utility levels: falsifiability restored

We now know that in order to have a testable theory, the analyst has to impose conditions on the evolution of default utility levels. These conditions will now form part of the hypothesis being tested,[15] and rejection of this joint test does not inform which

[13] We maintain the assumption that $x_t^1 + x_t^2 = X_t$ at all observations.

[14] See, for example, Rockafellar (1970).

[15] As much as, for instance, the conditions that define the class of utility functions that are being allowed by our definitions.

of the hypotheses drove the rejection. Still, if in a given exercise the analyst can reasonably impose structure on these unobserved variables, if this structure is 'enough' it may restore the refutability of the hypothesis.

For instance, say that a data set of the form (2) *is rationalizable with invariant default utility levels* if there exist two numbers v^1 and v^2, and individual utility functions $u^1 : \mathbb{R} \to \mathbb{R}$ and $u^2 : \mathbb{R} \to \mathbb{R}$, both of which are strictly increasing and strictly concave, such that at each observation t, the pair (x_t^1, x_t^2) solves the program

$$\max_{x^1,x^2} \left\{[u^1(x^1) - v^1][u^2(x^2) - v^2] : x^1 + x^2 \leq X_t \text{ and } u^i(x^i) > v^i\right\}.$$

Our next result is that this extra assumption on the unobserved utility levels strengthens the hypothesis and renders it refutable again.

Proposition 5. *There exists a non-tautological condition that a data set of form (2) satisfies if, and only if, it is rationalizable with invariant default utility levels. Moreover, this condition is a finite set of polynomial inequalities on*

$$\left\{(x_t^1, x_t^2, X_t) : t = 1, \ldots, T\right\}.$$

Proof. Since the details of the argument are similar to previous ones, we shall omit them. First, as in the Proof of Proposition 1 if the data set is rationalizable with invariant default utility levels, there must exist a pair of numbers (v^1, v^2) and an array of numbers

$$\left\{(\mu_t^1, \mu_t^2, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the following system:

$$\mu_{t'}^i \leq \mu_t^i + \lambda_t(\mu_t^i - v^i)(x_{t'}^i - x_t^i), \tag{15}$$

with strict inequality if $x_{t'}^i \neq x_t^i$,

$$\mu_t^i > v^i, \tag{16}$$

and

$$\lambda_t > 0. \tag{17}$$

On the other hand, it follows from Proposition 2 that the existence of this solution to the system is sufficient for the data set to be rationalizable, and since the pair (v^1, v^2) remains constant across all observations, the data set is, *a fortiori*, rationalized with invariant default utility levels.

Now, as in the Proof of Proposition 3, the system of inequalities (15)–(17) defines a semi-algebraic set, so its projection into the space of data is semi-algebraic as well and is characterized by a finite set of polynomial inequalities. In order to see that such a condition is non-tautological, it suffices to observe that, as in Proposition 3, any data set in which x_t^i and X_t are not co-monotone cannot be rationalized with invariant default utilities.[16] □

The invariance of the unobserved utility levels is not the only case in which the analyst can recover refutability: our argument can be extended to argue that the hypothesis is refutable if it imposes that the unobserved utility levels $\left\{(v_t^1, v_t^2) : t = 1, \ldots, T\right\}$, while not being observed, satisfy that v_t^1 is weakly co-monotone with X_t and v_t^2 is weakly anti-co-monotone with X_t. Indeed, in this case the necessary and sufficient system of inequalities is defined on the array

$$\left\{(\mu_t^1, \mu_t^2, v_t^1, v_t^2, \lambda_t) : t = 1, \ldots, T\right\},$$

and it includes Eqs. (4)–(6), as well as the following two requirements, which are immediate from the new assumptions: for every t and every t'

$$(v_t^1 - v_{t'}^1)(X_t - X_{t'}) \geq 0,$$

while

$$(v_t^2 - v_{t'}^2)(X_t - X_{t'}) \leq 0.$$

Since these extra inequalities are polynomial, the set of solutions remains semi-algebraic and our analysis of its projection is still valid. On the other hand, it follows immediately from Eq. (12) that in this case x_t^1 and X_t must be co-monotonic, so the system that characterizes the projection is again non-tautological.[17]

4. Default income levels

The model under consideration, and the way in which we have dealt with it so far, specifies utility levels that the individuals can attain by withdrawing from the negotiation without reaching an agreement. We have seen that even when utility levels are observed, or restricted by the analyst when they are unobserved, the empirical implications are significant. However, it is more realistic to assume that the analyst has some information, or hypothesis, about the behavior of the *income* levels that the individuals can obtain outside the cooperative agreement as opposed to the *utility* they derive from these income levels. So, suppose now that the data set is of the form

$$\left\{(x_t^1, x_t^2, X_t, y_t^1, y_t^2) : t = 1, \ldots, T\right\}, \tag{18}$$

where y_t^i represents the default income level that individual i could have obtained at observation t if an agreement had not been reached. We continue to assume that $x_t^1 + x_t^2 = X_t$, and additionally, suppose that $x_t^i > y_t^i$ at all observations.

We will say that a data set of the form (18) *is rationalizable* if there exist utility functions $u^1 : \mathbb{R} \to \mathbb{R}$ and $u^2 : \mathbb{R} \to \mathbb{R}$, both of which are strictly increasing and strictly concave, such that at each observation t, the pair (x_t^1, x_t^2) solves the program

$$\max_{x^1,x^2}\{[u^1(x^1) - u^1(y_t^1)][u^2(x^2) - u^2(y_t^2)] : x^1 + x^2 \leq X_t \text{ and } u^i(x^i) > u^i(y_t^i)\}. \tag{19}$$

The next result, which is analogous to Proposition 1, says that a necessary condition for rationalizability is the existence of a solution to a system of polynomial inequalities. The system required for this new setting, however, is slightly more complicated as we now need to account for more points in the domain of the utility functions.

Proposition 6. *If a data set of the form (18) is rationalizable, then there exists an array of numbers*

$$\left\{(\mu_t^1, \mu_t^2, v_t^1, v_t^2, \delta_t^1, \delta_t^2, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the following system:

$$\mu_{t'}^i \leq \mu_t^i + \lambda_t(\mu_t^i - v_t^i)(x_{t'}^i - x_t^i), \tag{20}$$

with strict inequality if $x_{t'}^i \neq x_t^i$;

$$v_{t'}^i \leq v_t^i + \delta_t^i(y_{t'}^i - y_t^i), \tag{21}$$

with strict inequality if $y_{t'}^i \neq y_t^i$;

$$\mu_{t'}^i \leq v_t^i + \delta_t^i(x_{t'}^i - y_t^i), \tag{22}$$

[16] Variables y_t and z_t are said to be co-monotone if $y_t \geq y_{t'}$ occurs when, and only when, $z_t \geq z_{t'}$. They are anti-co-monotone if $y_t \geq y_{t'}$ occurs when, and only when, $z_t \leq z_{t'}$.

[17] Now, though, we cannot say anything about the co-variation of x_t^2 and X_t.

with strict inequality if $x^i_{t'} \neq y^i_t$;

$$v^i_{t'} \leq \mu^i_t + \lambda_t(\mu^i_t - v^i_t)(y^i_{t'} - x^i_t) \tag{23}$$

with strict inequality if $y^i_{t'} \neq x^i_t$;

$$\mu^i_t > v^i_t \tag{24}$$

and

$$\delta^i_t > 0 \tag{25}$$

for all i, all t and all $t' \neq t$; *and*

$$\lambda_t > 0 \tag{26}$$

for all t.

Proof. As in the Proof of Proposition 1, the first-order necessary condition of Program (19) requires that for some $\delta^1_t \in \partial u^1(x^1_t)$ and $\delta^2_t \in \partial u^2(x^2_t)$, we have that

$$\delta^1_t[u^2(x^2_t) - u^2(y^2_t)] = \delta^2_t[u^1(x^1_t) - u^1(y^1_t)].$$

Since u^1 is strictly increasing, defining the number

$$\lambda_t = \frac{\delta^1_t}{u^1(x^1_t) - u^1(y^1_t)} > 0,$$

we get that

$$\delta^i_t = \lambda_t[u^i(x^i_t) - v^i_t].$$

Now, if we also pick $\delta^i_t \in \partial u^i(y^i_t) > 0$, $\mu^i_t = u^i(x^i_t)$ and $v^i_t = u^i(y^i_t)$, we get Eqs. (20)–(23) from the fact that u^1 and u^2 are strictly concave, while we get Eq. (24) from their strict monotonicity, given that $x^i_t > y^i_t$ by assumption. The other two conditions, Eqs. (25) and (26), also follow from monotonicity. □

Since we now need to account for more points in the domains of the utility functions, the system of inequalities in this setting is more complex. The first condition in the system, Eq. (20), compares the utility level at the attained income of observation t' with its first-order approximation around the attained income of observation t. This was the same comparison that we had in the system of Proposition 1. Now, we also need to compare the utility level at the default income of observation t' with its linear approximation around the default income of observation t, which is done by Eq. (21); the utility level at the attained income of observation t' with its linear approximation around the default income of observation t, which is Eq. (22); and the utility at the default income of observation t' with its approximation around the attained income of observation t, namely Eq. (23). Importantly, while the hypothesis of rationalizability imposes a condition on the derivatives of the utility functions at the attained income levels (i.e., the equality given by the first-order condition of Program (19)), their derivatives at the default income levels are only constrained to be positive.

These extra conditions have to be imposed as part of the system, for otherwise we cannot guarantee its sufficiency. We thus obtain the following result.

Proposition 7. *Given a data set of the form (18), suppose that there exists an array of numbers*

$$\left\{(\mu^1_t, \mu^2_t, v^1_t, v^2_t, \delta^1_t, \delta^2_t, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the system of inequalities defined in Proposition 6. Then, the data set is rationalizable and, moreover, the utility functions that rationalize it can be constructed in the class $\mathbf{C}^2$.

Proof. The argument resembles the Proof of Proposition 2. Given a solution

$$\left\{(\mu^1_t, \mu^2_t, v^1_t, v^2_t, \delta^1_t, \delta^2_t, \lambda_t) : t = 1, \ldots, T\right\}$$

to the system of inequalities, construct the functions

$$u^i_0(x) := \min\left\{\min\left\{\mu^i_t + \lambda_t(\mu^i_t - v^i_t)(x - x^i_t), v^i_t + \delta^i_t(x - y^i_t)\right\} : t = 1, \ldots, T\right\}$$

which satisfy that $u^i_0(x^i_t) = \mu^i_t$ and $u^i_0(y^i_t) = v^i_t$.[18]

Taking, as before, any pair $(x^1, x^2) \neq (x^1_t, x^2_t)$ that is feasible in the program

$$\max_{x^1,x^2}\left\{[u^1_0(x^1) - v^1_t][u^2_0(x^2) - v^2_t] : x^1 + x^2 \leq X_t \text{ and } u^i_0(x^i) > v^i_t\right\},$$

we still get that

$$0 < u^i_0(x^i) - v^i_t \leq \mu^i_t + \lambda_t(\mu^i_t - v^i_t)(x - x^i_t) - v^i_t.$$

Since this is what was required for the argument of Proposition 2, we can omit the remaining details. □

Importantly, the conditions that are added to the system are still polynomial inequalities, so the set of arrays

$$\left\{(x^1_t, x^2_t, X_t, y^1_t, y^2_t, v^1_t, v^2_t, \mu^1_t, \mu^2_t, \delta^1_t, \delta^2_t, \lambda_t) : t = 1, \ldots, T\right\}$$

that satisfy the system is semi-algebraic, and so is, therefore, its projection into the space of data. As before, this set is, thus, characterized by a finite set of polynomial inequalities, which constitute the strongest possible test of the hypothesis of rationalizability in this setting. Assuming that the utility functions that rationalize the data are $\mathbf{C}^2$, we can see that the test is not a tautology by replacing $\mathrm{d}v^i = \partial u^i(y^i)\mathrm{d}y^i$ in Eq. (12), in order to obtain comparative statics for the case that we are considering: $\mathrm{d}x^1$ now equals the ratio of

$$\left\{\partial^2 u^2(x^2)[u^1(x^1) - u^1(y^1)] - \partial u^1(x^1)\partial u^2(x^2)\right\}\mathrm{d}X - \partial u^2(x^2)\partial u^1(y^1)\mathrm{d}y^1 + \partial u^1(x^1)\partial u^2(y^2)\mathrm{d}y^2$$

and

$$\partial^2 u^1(x^1)[u^2(x^2) - u^2(y^2)] - 2\partial u^1(x^1)\partial u^2(x^2) + \partial^2 u^2(x^2)[u^1(x^1) - u^1(y^1)].$$

Under monotonicity, the latter implies that no set that contains a pair of observations for which

$$X_t > X_{t'}, \qquad y^1_t > y^1_{t'}, \qquad y^2_t < y^2_{t'} \quad \text{and} \quad x^1_t < x^1_{t'}$$

can be rationalized. Thus, the following proposition summarizes these observations.

Proposition 8. *There exists a non-tautological condition that a data set of form (18) satisfies if, and only if, it is rationalizable by utility functions in the class* $\mathbf{C}^2$. *Moreover, this condition is a finite set of polynomial inequalities on*

$$\left\{(x^1_t, x^2_t, X_t, y^1_t, y^2_t) : t = 1, \ldots, T\right\}.$$

18 And which are continuous, concave and strictly increasing, and are $\mathbf{C}^\infty$ at all but a finite number of points in $\mathbb{R}$.

5. Bounds on unobserved default income levels

The analysis of the case when default income levels are observed allows us to address a weakness that our previous results display. Consider again the setting of Section 2.1, where we have data of the form (3). Our definition of rationalizability in that section did not require the existence of an unobserved default income level at which the individuals would obtain the observed default utility levels under our construction of the utility functions: indeed, the definition of rationalizability does not require that there exist numbers

$$\{(y_t^1, y_t^2) : t = 1, \ldots, T\}$$

such that $u^1(y_t^1) = v_t^1$ and $u^2(y_t^2) = v_t^2$ at all observations. And while our construction in Proposition 2 delivers for us this extra requirement, for the functions u^i constructed in the proof are unbounded below, it may still be the case that in such construction the implicit y_t^i for which $u^i(y_t^i) = v_t^i$ is not plausible from the point of view of the analyst, for some i and some t—for instance, it could be the case that such an income level has to be allowed to take negative values.

This problem can be addressed by extending the system of Proposition 1 in the same way as Eqs. (20)–(26). For instance, let $Y_t^i \subseteq \mathbb{R}$ be a nonempty set of default income levels that the analyst considers possible for individual i at observation t, and denote by Y the collection of all these constraints:

$$Y := \left\{(Y_t^1, Y_t^2) : t = 1, \ldots, T\right\}.$$

Say that a data set of the form (3) is *rationalizable with respect to Y* if there exist utility functions $u^1 : \mathbb{R} \to \mathbb{R}$ and $u^2 : \mathbb{R} \to \mathbb{R}$, both of which are $\mathbf{C}^2$, strictly increasing and strictly concave, and default income levels

$$\left\{(y_t^1, y_t^2) : t = 1, \ldots, T\right\}$$

such that (i) at each observation t, the pair (x_t^1, x_t^2) solves Program (1); (ii) at each observation t, the default income levels are feasible, in the sense that $y_t^1 \in Y_t^1$ and $y_t^2 \in Y_t^2$; and (iii) the default income levels deliver the observed default utilities: for all t, $u^1(y_t^1) = v_t^1$ and $u^2(y_t^2) = v_t^2$.

The following analysis strengthens the results of Section 2.1 for this definition of rationalizability. In addition to the assumptions introduced there, we here assume that all the sets Y_t^i contain at least one $y < x_t^i$. As the results can be proven by arguments similar to the ones given in Section 4, we state them without a detailed proof.

Proposition 9. *Fix the collection Y of constraints. A data set of the form (18) is rationalizable with respect to Y if, and only if, there exists an array of numbers*

$$\left\{(\mu_t^1, \mu_t^2, y_t^1, y_t^2, \delta_t^1, \delta_t^1, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the system of Eqs. (20)–(26), with $y_t^i \in Y_t^i$ and $y_t^i < x_t^i$ for all i and all t. If, moreover, all the constraints in Y are semi-algebraic, then there exists a non-tautological condition that a data set satisfies if, and only if, it is rationalizable with respect to Y. This condition is a finite set of polynomial inequalities on

$$\left\{(x_t^1, x_t^2, X_t, v_t^1, v_t^2) : t = 1, \ldots, T\right\}.$$

Proof. Only the fact that the condition imposed by the hypothesis of rationalizability with respect to Y is non-tautological requires a comment. The reason why this is the case is, simply, that if a set is rationalizable with respect to Y then it is rationalizable (in the sense of Section 2.1). But since there are data sets that cannot be rationalized, then there are sets that cannot be rationalizable with respect to Y and it follows that the projection of the set of solutions to the system defined in the proposition into the space of data has to be a proper subset of that space. Since we are assuming that all the sets in Y are semi-algebraic, the fact that the condition is non-tautological is implied by the Tarski–Seidenberg Theorem. □

6. A statistical test

A possible weakness of the results we have obtained so far is the lack of a measure of how strong a rejection of the hypothesis of rationalizability is: that is, the application of our tests is dichotomic in the sense that a data set is rationalizable or not, but when a test is not rationalizable we still do not know how big or small a perturbation to the data would make it consistent with the hypothesis. If a very 'small' perturbation to one of the observations sufficed for the data set to pass the test of rationalizability, the analyst may want to consider the data consistent with the hypothesis, attributing the 'small error' to causes like, for instance, an error in the collection of the data.

This criticism is common to all the basic literature on revealed preferences, but can be addressed by extending the analysis in the direction of the construction of statistical versions of the test (as is done in that literature as well) following, Varian (1985).

For the most plausible framework, suppose that the observed data consists of

$$\left\{(x_t^1, x_t^2, y_t^1, y_t^2) : t = 1, \ldots, T\right\},$$

as in Section 4, only with the caveat that we now let X_t be constructed as $x_t^1 + x_t^2$, instead of it being observed.[19] The analyst may believe that the real income of individual i at observation t was $x_t^i + \varepsilon_{i,t}^x$, and that her default income was $y_t^i + \varepsilon_{i,t}^y$, accounting for measurement error by the (unobserved) perturbations

$$\varepsilon := \left\{(\varepsilon_{1,t}^x, \varepsilon_{2,t}^x, \varepsilon_{1,t}^y, \varepsilon_{2,t}^y) : t = 1, \ldots, T\right\}.$$

In this setting, as in Varian (1985), we can construct a statistical version of the test of rationalizability. First, let $\mathcal{D}$ be the set of all values of

$$\left\{(\breve{x}_t^1, \breve{x}_t^2, \breve{y}_t^1, \breve{y}_t^2) : t = 1, \ldots, T\right\}$$

that satisfy the condition given by Proposition 3, using $\breve{X}_t = \breve{x}_t^1 + \breve{x}_t^2$ (or, equivalently for which there exists a solution to the system of conditions defined in Eqs. (20)–(26)). Second, for the array

$$e := \left\{(e_{1,t}^x, e_{2,t}^x, e_{1,t}^y, e_{2,t}^y) : t = 1, \ldots, T\right\},$$

define χ as the minimum value of the sum of squared errors, $e \cdot e$, subject to the constraint that

$$(x_t^1 + e_{1,t}^x, x_t^2 + e_{2,t}^x, y_t^1 + e_{1,t}^y, y_t^2 + e_{2,t}^y)_{t=1}^T \in \mathcal{D}.$$

Under the assumption that the perturbation vector ε follows a normal distribution with mean $(0, \ldots, 0)$ and variance–covariance matrix $\sigma\mathbb{I}$, it is immediate that χ/σ follows the χ^2 distribution with $4T$ degrees of freedom. This statistic can be used to test the hypothesis of rationalizability, using the null hypothesis that $\chi = 0$.

7. Generalization: asymmetric bargaining and arbitrary number of players

The results we have obtained so far can be generalized to an arbitrary numbers of players whose bargaining powers may differ (but are assumed to be constant). For the simplicity of presentation,

[19] The reason for this will be apparent momentarily.

we concentrate on the setting where default utility levels are observed.

That is, suppose that the analyst has observed data of the form

$$\left\{(x_t^i, v_t^i) : i = 1, \ldots, I \text{ and } t = 1, \ldots, T\right\}, \tag{27}$$

and let $X_t := \sum_i x_t^i$. Fix a vector $\alpha = (\alpha_1, \ldots, \alpha_I) \gg 0$, and say that one such data set *is α-rationalizable* if there exist utility functions $u^i : \mathbb{R} \to \mathbb{R}$, for $i = 1, \ldots, I$, all of which are $\mathbf{C}^2$, strictly increasing and strictly concave, such that each vector $(x_t^1, \ldots, x_t^I)$ solves the program

$$\max_{(x^1,\ldots,x^I)} \left\{\prod_i [u^i(x^i) - v_t^i]^{\alpha_i} : \sum_i x^i \leq X_t \text{ and } u^i(x^i) > v_t^i\right\}, \tag{28}$$

where the coefficient $\alpha_i > 0$ represents individual i's bargaining power.

As in Section 2.1, the first-order conditions of this problem are that for all individuals we have that

$$\frac{\alpha_i \partial u^i(x_t^i)}{u^i(x_t^i) - v_t^i} = \lambda_t, \tag{29}$$

for some $\lambda_t > 0$.[20] For all individuals, then,

$$\partial u^i(x_t^i) = \frac{1}{\alpha_i}\lambda_t[u^i(x_t^i) - v_t^i].$$

Since u^i is strictly concave, the latter implies that, for all $x \in \mathbb{R}$, $x \neq x_t^i$,

$$u^i(x) < u^i(x_t^i) + \frac{1}{\alpha_i}\lambda_t[u^i(x_t^i) - v_t^i](x - x_t^i),$$

and, therefore, defining the numbers $\mu_t^i := u^i(x_t^i)$, we have proved the following proposition.

Proposition 10. *Let $\alpha \gg 0$. If a data set of form (27) is α-rationalizable, then there exists an array of numbers*

$$\left\{(\mu_t^1, \ldots, \mu_t^I, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the following system:

$$\mu_{t'}^i \leq \mu_t^i + \frac{1}{\alpha_i}\lambda_t(\mu_t^i - v_t^i)(x_{t'}^i - x_t^i), \tag{30}$$

with strict inequality if $x_{t'}^i \neq x_t^i$,

$$\mu_t^i > v_t^i, \tag{31}$$

and

$$\lambda_t > 0. \tag{32}$$

Perhaps it is not surprising, but it is important that the necessary condition we just proposed is sufficient as well.

Proposition 11. *Given a data set of form (27) and a vector $\alpha \gg 0$, suppose that there exists an array of numbers*

$$\left\{(\mu_t^1, \ldots, \mu_t^I, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the system of inequalities defined in Proposition 10. Then, the data set is α-rationalizable.

Proof. Given the array $\left\{(\mu_t^1, \ldots, \mu_t^I, \lambda_t) : t = 1, \ldots, T\right\}$ that solves the system of inequalities, construct, for each player i,

$$u_0^i(x) := \min\left\{\mu_t^i + \frac{1}{\alpha_i}\lambda_t(\mu_t^i - v_t^i)(x - x_t^i) : t = 1, \ldots, T\right\},$$

mapping $\mathbb{R}$ into $\mathbb{R}$. This function gives $u_0^i(x_t^i) = \mu_t^i$ at all t, is continuous, concave and strictly increasing, and is $\mathbf{C}^\infty$ at all but a finite number of points in $\mathbb{R}$. Since the inequalities are strict, using the argument presented in the Appendix, the function can be transformed into a $\mathbf{C}^2$, strictly concave and strictly increasing function u^i such that $u^i(x_t^i) = u_0^i(x_t^i)$ and $\partial u^i(x_t^i) = \partial u_0^i(x_t^i)$.

With these constructions, since Program (28) is log-concave, it suffices to show that the vector $(x_t^1, \ldots, x_t^I)$ satisfies its first-order conditions, namely Eq. (29), for it to be its solution. But this is immediate, by construction, for

$$\frac{\alpha_i \partial u^i(x_t^i)}{u^i(x_t^i) - v_t^i} = \frac{\alpha_i \partial u_0^i(x_t^i)}{u_0^i(x_t^i) - v_t^i} = \lambda_t$$

for all i. □

Also, the comparative statics of Eq. (12) generalize to the current setting. For instance, keeping for simplicity the assumption that there are only two players but letting their bargaining powers be α_1 and α_2, we get that given in Box I, which suffices to imply that no data set can be α-rationalized if it contains a pair of distinct observations, t and t', such that

$$X_t > X_{t'}, \qquad v_t^1 > v_{t'}^1, \qquad v_t^2 < v_{t'}^2 \quad \text{and} \quad x_t^1 < x_{t'}^1.$$

(In the case of I players, if there are t and t', such that

$$X_t > X_{t'}, \qquad v_t^1 > v_{t'}^1,$$
$$v_t^i < v_{t'}^i \quad \text{for all } i \neq 1 \quad \text{and} \quad x_t^1 < x_{t'}^1.)$$

In consequence, we have proven the following result.

Proposition 12. *Let $\alpha \gg 0$. There exists a non-tautological condition that a data set of form (27) satisfies if, and only if, it is α-rationalizable. Moreover, this condition is a finite set of polynomial inequalities on*

$$\left\{(x_t^i, v_t^i) : i = 1, \ldots, I \text{ and } t = 1, \ldots, T\right\}.$$

8. Conclusions

We study the empirical implications of the NBS, assuming that the outside analyst observes a finite set of bargaining outcomes, under different hypotheses about the behavior of the disagreement levels. We first consider a model where two agents bargain over the division of an aggregate endowment, and follow a revealed preference approach to verify the empirical validity of the NBS. In addition, we extend our study for the case of an arbitrary number of agents, whose bargaining power may be asymmetric (but constant).

We study the cases where default levels are observed, and when the analyst has some information on the behavior of the disagreement levels, for instance, the invariance of unobserved default utility levels. With no information on the evolution of default utility levels, the NBS hypothesis is not refutable. We recover refutability under the hypothesis that the disagreement point, while not being observed or invariant, is weakly co-monotone with the aggregate resource, in the case of one agent, and weakly anti-co-monotone with the aggregate resource for the other.

We use the Tarski–Seidenberg algorithm to construct the type of test an observer can develop to claim the hypothesis that

[20] With a slight abuse of notation, we now use ∂u^i to denote the derivative of function u^i, given that we have assumed that these functions are all differentiable.

$$dx^1 = \frac{\left\{\alpha_2 \partial^2 u^2(x^2)[u^1(x^1) - v^1] - \alpha_1 \partial u^1(x^1)\partial u^2(x^2)\right\} dX - \alpha_2 \partial u^2(x^2) dv^1 + \alpha_1 \partial u^1(x^1) dv^2}{\alpha_1 \partial^2 u^1(x^1)[u^2(x^2) - v^2] - (\alpha_1 + \alpha_2)\partial u^1(x^1)\partial u^2(x^2) + \alpha_2 \partial^2 u^2(x^2)[u^1(x^1) - v^1]}$$

Box I.

the data can be explained by using the NBS. We also construct a statistical version of the rationalizability test, to give specific content as to when we can attribute the failure of the hypothesis of rationalizability to measurement error.

Appendix. Strict concavity and differentiability of preferences

The purpose of this appendix is to illustrate how the piecewise linear preferences that were constructed in the results of the paper can be transformed into strictly concave $\mathbf{C}^2$ functions. For the sake of simplicity in our presentation, we concentrate on the system of inequalities obtained in Section 1 of the paper, but the reader can readily check that the analysis extends to the rest of the cases considered. Also, since the construction is made individual by individual, here we simplify our notation by ignoring the super-index that identifies the agents that are considered in the paper.

That is, suppose that an analyst is given an array of numbers

$$\{(x_t, v_t, \mu_t, \lambda_t) : t = 1, \dots, T\}$$

that satisfies the following system:

$$\mu_{t'} \leq \mu_t + \lambda_t(\mu_t - v_t)(x_{t'} - x_t), \tag{33}$$

with strict inequality if $x_{t'} \neq x_t$, $\mu_t > v_t$, and $\lambda_t > 0$.

As in Theorem 2 in Matzkin and Richter (1991), take a strictly convex function $h(x) \geq 0$ such that $h(x) = 0$ only at $x = 0$, and whose derivative is always less than 1. Given Eq. (30) and the fact that the number of observations is finite, one can construct functions

$$v_t(x) = \mu_t + \lambda_t(\mu_t - v_t)(x - x_t) - \epsilon_t h(x_t - x),$$

where ϵ_t is small enough that

$$\mu_{t'} \leq \mu_t + \lambda_t(\mu_t - v_t)(x_{t'} - x_t) - \epsilon_t h(x_t - x_{t'}),$$

with strict inequality whenever $x_t \neq x_{t'}$.

Now, define

$$u_0(x) = \min\{v_t(x) : t = 1, \dots, T\}.$$

This function is continuous, strictly concave and strictly monotone. By construction, it also satisfies that $u_0(x_t) = \mu_t$ and that

$$u_0(x) - v_t \leq \mu_t + \lambda_t(\mu_t - v_t)(x - x_t) - v_t,$$

which is the inequality that is critical for the argument that the functions constructed in Section 2.1 rationalize the observed data.

It only remains to show that the function can be further perturbed to smooth out its (finitely many) kinks. For this one can use a deformation like the one proposed by Chiappori and Rochet (1987). By construction, one can fix $\varepsilon > 0$ such that, for all t, $u(x) = v_t(x)$ whenever $|x - x_t| < \varepsilon$. Then, define the function $\rho : \mathbb{R} \to \mathbb{R}$ as

$$\rho(\psi) = \begin{cases} \exp\left(-\frac{1}{\psi^2 - 1}\right)\left[\int_{\mathbb{R}} \exp\left(-\frac{1}{\mu^2 - 1}\right) d\mu\right]^{-1}, \\ \quad \text{if } |\psi| \leq 1; \\ 0, \quad \text{otherwise.} \end{cases}$$

Now, consider the following mapping, which takes the convolution of u_0 and ρ:

$$u(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}} u_0(x - \psi)\rho\left(\frac{\psi}{\epsilon}\right) d\psi.$$

Chiappori and Rochet show that this function is $\mathbf{C}^\infty$, strictly concave and strictly increasing, and rationalizes the same maxima as u_0.

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