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Testable restrictions on the equilibrium manifold under random preferences

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Abstract

General equilibrium theory was criticized for its apparent irrefutability, as seemingly implied by the Sonnenschein–Mantel–Debreu theorem. This view was challenged by Brown and Matzkin [Econometrica, 64 (1996) 1249], who showed the existence of testable restrictions on the equilibrium manifold. Brown and Matzkin, however, maintain the assumption that individual preferences are invariant (against psychological evidence). I consider the Brown–Matzkin problem under random preferences: for each profile of endowments one observes a distribution of prices; does there exist a probability distribution of preferences that explains the observed distributions of prices via Walrasian equilibria? I argue that even under random utilities general equilibrium theory is falsifiable.

JEL classification: D12; D70; D51

Keywords: General equilibrium; Revealed preferences; Random utility; Testable restrictions

1. Introduction

The work of Sonnenschein (1973), Mantel (1974) and Debreu (1974) led economists to believe that general equilibrium did not impose any (strong) regularity that could be refuted with data, unless one observed individual behavior, which is unlikely.

Such pessimistic view was challenged in 1996 by Brown and Matzkin. They exploited a tension existing between the two fundamental concepts of the theory, individual rationality and market clearing, to show that whenever individual endowments are observed, the theory imposes non-trivial restrictions on prices.

The argument of Brown and Matzkin crucially assumes that individual preferences are invariant and uses revealed-preference theory in order to argue the existence of data which

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are inconsistent with general equilibrium. While this assumption may seem natural in economics, it receives strong criticism from other disciplines.[1] In particular, research in human behavior seems to have convinced psychologists that human preferences are probabilistic in nature and that, therefore, so are human behavior and choices.[2]

Does randomness in preferences imply that general equilibrium theory is unfalsifiable? This paper incorporates the theory of random utility to the problem and shows that whenever individual endowments are observed, the theory imposes non-trivial restrictions on probabilistic distributions of prices.

The paper is organized as follows. Section 2 lays down the problem. In Section 3, a characterization of data that are consistent with general equilibrium under random preferences is obtained. Given that this characterization is mediated by existential quantifiers, it fails to provide for a direct test and it is unclear whether the null hypothesis of consistency can ever be refuted. Section 4, which introduces two examples of non-rationalizable datasets, makes the case for refutability, while Section 5 provides another characterization of rationalizability and uses standard results in quantifier elimination to determine the abstract form that restrictions on data alone have.

Kubler (2003) also deals with the problem of falsifiability of equilibrium theory in a non-deterministic environment. Kubler studies whether intertemporal general equilibrium imposes restrictions on prices of commodities and assets, given a stochastic process of dividends and aggregate endowments. He finds that if one restricts individual preferences to be additively-separable, expected-utility preferences, then there do exist testable restrictions. There are two differences between Kubler's problem and the one dealt with here. Kubler's setting is intertemporal and agents have to make their decisions under uncertainty. The problem studied here, on the other hand, *has no intertemporal features* and, although preferences are assumed to be random, *agents decide under certainty*. Besides, Kubler assumes the observation of a joint stochastic process of prices, aggregate endowments and dividends, while here one assumes very different data: for each profile of endowments, one observes a probability measure on the space of prices—there is no sequentiality in either the set of endowments or the way in which prices are observed.

For a comprehensive survey of related literature, see Carvajal et al. (in press).

2. The problem and definitions

There are a finite set, $\mathcal{I} = \{1, \ldots, I\}$, of consumers and a finite number, $L \in \mathbb{N}$, of commodities. For each consumer, the consumption set is $\mathbb{R}^L_+$.

[1] From a philosophical standpoint, Kroeber-Riel (1971) has written that "revealed-preference theory proves to have ... little foundations in reality ... The alleged advantages of increased empirical relevance ... of this theory prove to be linguistic declarations without factual meaning ..." See also Hausman (2000).

[2] According to Luce (1959) "[a] basic presumption ... is that choice is best described as a probabilistic, not an algebraic phenomenon ..." Similarly, Block and Marshack (1960) wrote that "[i]n interpreting human behavior there is a need to substitute 'stochastic consistency of choices' for 'absolute consistency of choices'. The latter is usually assumed in economic theory, but is not well supported by experience." Also, Luce and Supes (1965) justify the choice of a probabilistic understanding of human behavior by saying that "[h]istorically, the algebraic theories ... have been used in economics and statistics almost exclusively. The probabilistic ones are largely the product of psychological thought, forced upon [psychologists] by the data [they] collect in the laboratory."

Prices are normalized to lie in $\mathcal{S}$, the $(L-1)$-dimensional unit simplex. $\mathcal{S}$ is endowed with a σ-algebra Ξ.

Suppose that one observes a non-empty set of profiles of endowments, $E \subseteq (\mathbb{R}^L_{++})^I$, and that for each observed profile, $e = (e^i)_{i\in\mathcal{I}} \in E$, one observes a probability measure on the simplex of prices $\chi_e : \Xi \to [0,1]$. That is, given a profile of endowments, $e \in E$, and a measurable set of prices, $C \in \Xi$, $\chi_e(C)$ is the frequency with which prices were observed to lie in C when endowments were e.

Let $\mathcal{U}$ be the class of continuous, strongly concave, strictly monotone functions from $\mathbb{R}^L_+$ into $\mathbb{R}$. For a profile of preferences $e \subseteq (\mathbb{R}^L_{++})^I$ and a profile of preferences $U \in \mathcal{U}^I$, let $W_{U,e}$ denote the set of competitive equilibrium prices.

The dataset $\{E, (\chi_e)_{e\in E}\}$ is rationalizable if the observed distributions of prices can be explained as being induced, via equilibrium, by a probability distribution over the set of profiles of preferences, $\mathcal{U}^I$. In order to deal with multiplicity of equilibria, the assumption made here is that prices are determined *randomly* from within the Walras set of the economy. Since the latter set depends on the profile of endowments, rationalizability would be requiring that there exist, for each $e \in E$, a probability measure[3] $\pi_e : \mathcal{P}(\mathcal{U}^I) \times \Xi \to [0,1]$ such that: (i) for each $C \in \Xi$, $\chi_e(C) = \pi_e(\mathcal{U}^I, C)$; and (ii) for each $\mathcal{V} \in \mathcal{P}(\mathcal{U}^I)$ and each $C \in \Xi$:

$$\pi_e(\mathcal{V}, C) > 0 \Rightarrow (\exists U \in \mathcal{V})(\exists p \in C) : p \in W_{U,e}$$

Intuitively, the first condition says that given an observed profile of endowments, the observed probability that prices lie in a given measurable set of prices is equal to the theoretical probability of that set prices (once preferences are integrated out). That condition imposes none of the principles of individual rationality and market clearing. This is done by the second condition, which requires that, given endowments, the theoretical joint probability of a set of profiles of preferences and a set of prices be positive only if for at least one of the profiles of preferences in the former set, there is a price in the latter set which is Walrasian equilibrium given the endowments.

The two conditions, however, demand too little from the rationalization, as they do not require that preferences be independent from endowments. For that, one must require that the family $(\pi_e)_{e\in E}$ have a common marginal distribution over $\mathcal{U}^I$: that there exist a probability measure $\vartheta : \mathcal{P}(\mathcal{U}^I) \to [0,1]$, such that for every $\mathcal{V} \in \mathcal{P}(\mathcal{U}^I)$ and every $e \in E$, $\vartheta(\mathcal{V}) = \pi_e(\mathcal{V}, \mathcal{S})$.

Under this further requirement, each conditional distribution for prices, $\pi_e(\cdot|U) : \Xi \to [0,1]$, is a random selector over $W_{U,e}$, as defined in Allen (1985): let $\mathcal{F}$ denote the set of probability measures on $\mathcal{S}$, defined over Ξ; given $U \in \mathcal{U}^I$ and $e \in (\mathbb{R}^L_{++})^I$, a random selector is $\varphi \in \mathcal{F}$ such that $\varphi(W_{U,e}) = 1$.[4]

In order to distinguish randomness in preferences from randomness in prices, given preferences, define a finite set, Ω, of states of the world, which only account for changes in the preferences of individuals.

[3] For a given set Z, $\mathcal{P}(Z)$ denotes its power set.

[4] This is under the assumption that $W_{u,e} \in \Xi$. A weaker requirement would be that for every $C \in \Xi$, if $W_{u,e} \subseteq C$, then $\varphi(C) = 1$.

Definition 1. A dataset $\{E, (\chi_e)_{e\in E}\}$ is Ω-rationalizable if there exist a probability measure $\delta : \mathcal{P}(\Omega) \to [0, 1]$, a function $u : \Omega \to \mathcal{U}^I$ and a function[5] $\varphi : u[\Omega] \times E \to \mathcal{F}$ such that for each $e \in E$ and each $C \in \Xi$:[6]

$$\chi_e(C) = \sum_{\omega\in\Omega} \delta(\{\omega\})\varphi(u(\omega), e)(C)$$

and for each $\omega \in \Omega$ and each $e \in E$:

$$\varphi(u(\omega), e)(W_{u(\omega),e}) = 1$$

In the definition, δ is a probability distribution over states of the world, u is a rule that assigns a profile of preferences to each state and φ assigns to each profile of preferences and each profile of endowments a probability distribution over $\mathcal{S}$. The first condition in the definition is, again, that the observed probabilities be explained by δ, u and φ, whereas the second one requires that each distribution over $\mathcal{S}$ have as support the Walras set of its economy.

Condition 1. Throughout the rest of the paper, it is assumed that:

1. Prices can be observed with perfect accuracy: $\Xi = \mathcal{P}(\mathcal{S})$.
2. Only a finite number of profiles of endowments has been observed (as in Brown and Matzkin, 1996): $\#E < \infty$.
3. For each observed profile of endowments, only a finite number of strictly positive prices has been observed to occur:[7]

$$(\forall e \in E) : \#\text{Supp}(\chi_e) < \infty \wedge \chi_e(\text{Supp}(\chi_e)) = 1$$

where

$$\text{Supp}(\chi_e) = \{p \in \mathcal{S} | \chi_e(\{p\}) > 0\} \subseteq \mathbb{R}^L_{++}$$

3. Rationalizable datasets

One observes individual endowments, the supports of the distributions of prices, and the actual probabilities that each one of the prices in these supports attains. The unobservables whose existence one wants to test are the profiles of preferences that occur in each state of the world, the probability that each state of the world attains and the random selectors given utilities and endowments.

[5] Given a function $f : X \to Y$ and a set $Z \subseteq X$, $f[Z]$ denotes the image of Z under f.

[6] Since φ maps $u[\Omega] \times E$ into $\mathcal{F}$, which is a function space, then, $\varphi(u, e) \in \mathcal{F}$ means that $\varphi(u, e)$ is a function from Ξ into $[0, 1]$. For $C \in \Xi$, the value of this function is $\varphi(u, e)(C)$.

[7] The logical connectors 'and' and 'or' will be denoted, respectively, by $\wedge$ and $\vee$. $\neg$ will denote negation of a subsequent sentence. Under the null hypothesis of rationalizability, if one restricts $\mathcal{U}$ to include only differentiable functions with interior contours, this assumption holds generically on endowments. This follows from Debreu (1970).

There exists a third class of variables: those that one would observe if one could obtain individual-level data on an experimental laboratory. There are two groups of data in this set of variables. First are the demands of each individual, for each budget he has faced and for each state of nature.[8] Second, since for each profile of budgets, actual choices depend on the state of the world, then profiles of demands from those budgets are also random variables; if one had access to richer data, one would know the distributions of these variables.

If these two groups of variables were observed, under the null hypothesis of consistency with general equilibrium they would have to satisfy certain conditions. For the first group of variables, the necessary and sufficient conditions are derived from standard revealed-preference theory. For the second group, the relevant condition comes from the theory of revealed-preference under random utility. Finally, if one could also observe random selectors then the restrictions of their definition should be imposed directly.

Theorems 1 and 2 characterize rationalizability in terms of existence of these observable-but-unobserved variables.

Notation 1. Given $\{E, (\chi_e)_{e\in E}\}$, denote, for each $i \in \mathcal{I}$:

$$\mathcal{B}^i = \{B^i \subseteq \mathbb{R}^L_+ | (\exists e \in E)(\exists p \in \mathrm{Supp}(\chi_e)) : B(p, e^i) = B^i\}$$

Denote also:

$$\mathcal{B} = \left\{ B \subseteq (\mathbb{R}^L_+)^I | (\exists e \in E)(\exists p \in \mathrm{Supp}(\chi_e)) : \prod_{i\in\mathcal{I}} B(p, e^i) = B \right\}$$

where for $p \in \mathcal{S}$ and $e^i \in \mathbb{R}^L_+$:

$$B(p, e^i) = \{x \in \mathbb{R}^L_+ | p \cdot x \leqslant p \cdot e^i\}$$

Notation 2. Let $B = \prod_{i\in\mathcal{I}} B^i \in \mathcal{B}$. For each $i \in \mathcal{I}$, let $\Gamma^{i,B^i} \subseteq \mathcal{P}(B^i)\backslash\{\varnothing\}$. Then, denote

$$\Gamma^B = \left\{ C \subseteq B | \left(\exists (C^i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \Gamma^{i,B^i} \right) : \prod_{i\in\mathcal{I}} C^i = C \right\}$$

and denote by Σ^B the σ-algebra generated by Γ^B on B. Moreover, denote:

$$\mathcal{B} \otimes \Sigma = \bigcup_{B\in\mathcal{B}} (\{B\} \times \Sigma^B)$$

Theorem 1. *A dataset* $\{E, (\chi_e)_{e\in E}\}$ is *Ω-rationalizable only if*:

- *For each $i \in \mathcal{I}$, each $B^i \in \mathcal{B}^i$ and each $\omega \in \Omega$, there exists $x^{i,B^i,\omega} \in \mathbb{R}^L_+$.*
- *Defining, for each $i \in \mathcal{I}$ and each $B^i \in \mathcal{B}^i$:*

$$\Gamma^{i,B^i} = \bigcup_{\omega\in\Omega} \{\{x^{i,B^i,\omega}\}\}$$

[8] Under the null hypothesis, some of these exercises may be counterfactual.

then, for each $B \in \mathcal{B}$ *and each* $C \in \Sigma^B$ *there exists* $g_{B,C} \in \mathbb{R}_+$.

- *For each* $\omega \in \Omega$, *there exists* $d_\omega \in \mathbb{R}_+$.
- *For each* $\omega \in \Omega$, *each* $e \in E$ *and each* $p \in \text{Supp}(\chi_e)$, *there exists* $f_{\omega,e,p} \in \mathbb{R}$ *which satisfy the following conditions*:

1. *For each* $i \in \mathcal{I}$, *each* $\omega \in \Omega$ *and each finite sequence* $(B^i_k)^K_{k=1}$ *in* $\mathcal{B}^i$:

$$\left((\forall k \in \{1, \dots, K-1\}) : x^{i,B^i_{k+1},\omega} \in B^i_k\right) \Rightarrow \left(x^{i,B^i_1,\omega} = x^{i,B^i_K,\omega} \vee x^{i,B^i_1,\omega} \notin B^i_K\right)$$

2. *For each* $i \in \mathcal{I}$, *each* $\omega \in \Omega$, *each* $e \in E$ *and each* $p \in \text{Supp}(\chi_e)$:

$$p \cdot x^{i,B(p,e^i),\omega} = p \cdot e^i$$

3. *For each* $B \in \mathcal{B}$:

$$g_{B,B} = 1$$

and for each $B \in \mathcal{B}$ *and each finite disjoint sequence* $(C_k)^K_{k=1}$ *in* Σ^B:

$$\sum_{k=1}^{K} g_{B,C_k} = g_{B,\cup_{k=1}^K C_k}$$

4. *For each finite sequence* $(B_k, C_k)^K_{k=1}$ *in* $\mathcal{B} \otimes \Sigma$, *there exists* $\omega \in \Omega$ *such that*:[9]

$$\sum_{k=1}^{K} g_{B_k,C_k} \leqslant \sum_{k=1}^{K} \mathbf{1}_{C_k}((x^{i,B^i_k,\omega})_{i\in\mathcal{I}})$$

5. *For each* $B \in \mathcal{B}$ *and each* $C \in \Sigma^B$:

$$g_{B,C} = \sum_{\omega\in\Omega} d_\omega \mathbf{1}_C((x^{i,B^i,\omega})_{i\in\mathcal{I}})$$

6. *For each* $e \in E$, *each* $p \in \text{Supp}(\chi_e)$ *and each* $\omega \in \Omega$:

$$\sum_{i\in\mathcal{I}} x^{i,B(p,e^i),\omega} \neq \sum_{i\in\mathcal{I}} e^i \Rightarrow f_{\omega,e,p} = 0$$

7. *For each* $\omega \in \Omega$ *and each* $e \in E$:

$$d_w > 0 \Rightarrow \sum_{p\in\text{Supp}(\chi_e)} f_{\omega,e,p} = 1$$

8. *For each* $e \in E$ *and each* $p \in \text{Supp}(\chi_e)$:

$$\chi_e(\{p\}) = \sum_{\omega\in\Omega} d_\omega f_{\omega,e,p}$$

[9] For a given set Z, $\mathbf{1}_Z$ denotes the indicator function.

Proof. Let δ, u and φ Ω-rationalize $\{E, (\chi_e)_{e\in E}\}$. Define $\forall i \in \mathcal{I}$, $\forall B^i \in \mathcal{B}^i$ and $\forall \omega \in \Omega$:[10]

$$x^{i,B^i,\omega} = \arg\max_{x\in B^i} u^i(\omega)(x)$$

Theorem 2 in Matzkin and Richter (1991) implies Condition 1. Condition 2 follows from the monotonicity of $u^i(\omega)$.

Define $\forall B \in \mathcal{B}$ and $\forall C \in \Sigma^B$:

$$g_{B,C} = \delta(\{\omega \in \Omega | (\arg\max_{x\in B^i} u^i(\omega)(x))_{i\in\mathcal{I}} \in C\})$$

Condition 3 is immediate. Condition 4 follows from Theorem 1 in Carvajal (2003).[11]

Define $\forall \omega \in \Omega$, $d_\omega = \delta(\{\omega\})$. Then,

$$\begin{aligned} g_{B,C} &= \delta(\{\omega \in \Omega | (x^{i,B^i,\omega})_{i\in\mathcal{I}} \in C\}) \\ &= \sum_{\omega\in\Omega:(x^{i,B^i,\omega})_{i\in\mathcal{I}}\in C} \delta(\{\omega\}) = \sum_{\omega\in\Omega} d_\omega \mathbf{1}_C((x^{i,B^i,\omega})_{i\in\mathcal{I}}) \end{aligned}$$

which is Condition 5.

Define $\forall \omega \in \Omega$, $\forall e \in E$ and $\forall p \in \mathrm{Supp}(\chi_e)$, $f_{\omega,e,p} = \varphi(u(\omega), e)(\{p\})$. Suppose that for $\omega \in \Omega$, $e \in E$ and $p \in \mathrm{Supp}(\chi_e)$:

$$\sum_{i\in\mathcal{I}} x^{i,B(p,e^i),\omega} \neq \sum_{i\in\mathcal{I}} e^i$$

By construction, this means that $p \notin W_{u(\omega),e}$ and, then, since $\varphi(u(\omega), e)(W_{u(\omega),e}) = 1$, it follows that $f_{\omega,e,p} = \varphi(u(\omega), e)(\{p\}) = 0$, which proves Condition 6.

Now, let $\tilde{\omega} \in \Omega$ and $e \in E$ and suppose that $d_{\tilde{\omega}} > 0$ and $\sum_{p\in\mathrm{Supp}(\chi_e)} f_{\tilde{\omega},e,p} \neq 1$. Since $\varphi(u(\tilde{\omega}), e) \in \mathcal{F}$, it must be that:

$$\sum_{p\in\mathrm{Supp}(\chi_e)} \varphi(u(\tilde{\omega}), e)(\{p\}) < 1$$

which implies that $\exists C \subseteq \mathcal{S}\backslash\mathrm{Supp}(\chi_e) : \varphi(u(\tilde{\omega}), e)(C) > 0$. Then, since $\delta(\{\tilde{\omega}\}) = d_{\tilde{\omega}} > 0$, it follows that:

$$\chi_e(C) = \sum_{\omega\in\Omega} \delta(\{\omega\})\varphi(u(\omega), e)(C) \geqslant \delta(\{\tilde{\omega}\})\varphi(u(\tilde{\omega}), e)(C) > 0$$

contradicting the fact that $C \subseteq \mathcal{S}\backslash\mathrm{Supp}(\chi_e)$. This implies Condition 7.

Finally, by construction, $\forall e \in E$ and $\forall p \in \mathrm{Supp}(\{p\})$:

$$\chi_e(\{p\}) = \sum_{\omega\in\Omega} \delta(\{\omega\})\varphi(u(\omega), e)(\{p\}) = \sum_{\omega\in\Omega} d_\omega f_{\omega,e,p}$$

which is Condition 8. □

[10] The notational proviso of Note 6 applies here: $u^i(\omega)$ is a function mapping $\mathbb{R}^L_+$ into $\mathbb{R}$, which takes the value $u^i(\omega)(x)$ at $x \in \mathbb{R}^L_+$.

[11] Since $\#\Omega < \infty$, it follows that $\forall B \in \mathcal{B}$, $\#\Gamma^B < \infty$ and, hence, that $\#\Sigma^B < \infty$. Since $\#E < \infty$, and $\forall e \in E$, $\#\mathrm{Supp}(\chi_e) < \infty$ then $\#\mathcal{B} < \infty$ and therefore Condition 1 in Carvajal (2003) is satisfied. Conditions 2 and 3 are also satisfied since $\forall B \in \mathcal{B}$, $B = \prod_{i\in\mathcal{I}} \overrightarrow{B}^i = \prod_{i\in\mathcal{I}} B^i$.

Intuition about the conditions of the theorem is straightforward, considering the following definition of variables. Let $x^{i,B^i,\omega}$ be i's utility-maximizing demand over B^i when ω realizes and let Γ^{i,B^i} be the collection of singleton sets of bundles that maximize i's utility over B^i, considering all possible states in Ω. For each collective budget B, let Σ^B be the σ-algebra induced by the individual collections Γ^{i,B^i}, and for each set of profiles of bundles $C \in \Sigma^B$, let $g_{B,C} \in \mathbb{R}_+$ be the probability that, if each individual chooses from B^i rationally, then the profile of choices lies in C. Finally, for each $\omega \in \Omega$, d_ω is its probability, while $f_{\omega,e,p} \in \mathbb{R}_+$ is the probability assigned to price p by the random selector corresponding to e and the profile of preferences assigned at ω.

Then, given an individual and a state of the world, individual rationality imposes SARP across all possible budgets, which is Condition 1, while strict monotonicity imposes Walras' law, which is Condition 2. For a collective budget $B \in \mathcal{B}$, Condition 3 imposes the second and third axioms of Kolmogorov for probability distributions. Condition 4 is the extension to collective problems of the Axiom of Stochastic Revealed Preference of McFadden and Richter (1990), as proposed by Carvajal (2003). Condition 5 requires that, indeed, probabilities over collective choices be explained by probabilities over states of the world, via individual rationality. For the random selectors, Condition 6 is market clearing while Condition 7 requires that they be probability distributions. Finally, given Ω-rationalizability, probabilities over states of the world and random selectors must explain the observed probabilities accurately: Condition 8.

If collective budgets and states of the world fully discriminate individual behavior, in the sense that for each state of the world there exists a budget for which individual behavior differs from that on all the rest of states, then one does not need probabilities both over states of the world and over collective choices.

Theorem 2. *A dataset* $\{E, (\chi_e)_{e\in E}\}$ *is* Ω*-rationalizable if*:

- *For each* $i \in \mathcal{I}$, *each* $B^i \in \mathcal{B}^i$ *and each* $\omega \in \Omega$, *there exist* $x^{i,B^i,\omega} \in \mathbb{R}^L_+$.
- *Defining, for each* $i \in \mathcal{I}$ *and each* $B^i \in \mathcal{B}^i$:

$$\Gamma^{i,B^i} = \bigcup_{\omega\in\Omega} \{\{x^{i,B^i,\omega}\}\}$$

  then, for each $B \in \mathcal{B}$ *and each* $C \in \Sigma^B$ *there exists* $g_{B,C} \in \mathbb{R}_+$.
- *For each* $\omega \in \Omega$, *each* $e \in E$ *and each* $p \in \operatorname{Supp}(\chi_e)$, *there exists* $f_{\omega,e,p} \in \mathbb{R}_+$, *which satisfy the following conditions*:
  1. *For each* $i \in \mathcal{I}$, *each* $\omega \in \Omega$ *and each finite sequence* $(B^i_k)^K_{k=1}$ *in* $\mathcal{B}^i$:

$$((\forall k \in \{1, \ldots, K-1\}) : x^{i,B^i_{k+1},\omega} \in B^i_k) \Rightarrow (x^{i,B^i_1,\omega} = x^{i,B^i_K,\omega}) \vee (x^{i,B^i_1,\omega} \notin B^i_K)$$

  2. *For each* $i \in \mathcal{I}$, *each* $\omega \in \Omega$, *each* $e \in E$ *and each* $p \in \operatorname{Supp}(\chi_e)$:

$$p \cdot x^{i,B(p,e^i),\omega} = p \cdot e^i$$

  3. *For each* $B \in \mathcal{B}$:

$$g_{B,B} = 1$$

and for each $B \in \mathcal{B}$ and each finite disjoint sequence $(C_k)_{k=1}^{K}$ in Σ^B:

$$\sum_{k=1}^{K} g_{B,C_k} = g_{B,\cup_{k=1}^{K} C_k}$$

4. *For each finite sequence (B_k, C_k) in $\mathcal{B} \otimes \Sigma$, there exists $\omega \in \Omega$ such that*:

$$\sum_{k=1}^{K} g_{B_k,C_k} \leqslant \sum_{k=1}^{K} \mathbf{1}_{C_k}((x^{i,B_k^i,\omega})_{i\in\mathcal{I}})$$

5. *For each $\omega \in \Omega$, there exists $B \in \mathcal{B}$ such that*:

$$\tilde{\omega} \in \Omega\backslash\{\omega\} \Rightarrow (\exists i \in \mathcal{I}) : x^{i,B^i,\tilde{\omega}} \neq x^{i,B^i,\omega}$$

6. *For each $e \in E$, each $p \in$* Supp(χ_e) *and each $\omega \in \Omega$*:

$$\sum_{i\in\mathcal{I}} x^{i,B(p,e^i),\omega} \neq \sum_{i\in\mathcal{I}} e^i \Rightarrow f_{\omega,e,p} = 0$$

7. *For each $\omega \in \Omega$, let $B(\omega) \in \mathcal{B}$ be implicitly defined by*:

$$B = B(\omega) \Leftrightarrow (\forall\tilde{\omega} \in \Omega\backslash\{\omega\})(\exists i \in \mathcal{I}) : x^{i,B^i,\tilde{\omega}} \neq x^{i,B^i,\omega}$$

If:

$$g_{B(\omega),\{(x^{i,B^i(\omega),\omega})_{i\in\mathcal{I}}\}} > 0$$

then, for each $e \in E$:

$$\sum_{p\in\text{Supp}(\chi_e)} f_{\omega,e,p} = 1$$

8. *For each $e \in E$ and each $p \in$* Supp(χ_e):

$$\chi_e(\{p\}) = \sum_{\omega\in\Omega} g_{B(\omega),\{(x^{i,B^i(\omega),\omega})_{i\in\mathcal{I}}\}} f_{\omega,e,p}$$

where $B(\omega) \in \mathcal{B}$ is defined as in Condition 7.

Proof. Given Conditions 1 and 2, it follows from Theorem 2 in Matzkin and Richter (1991) that $\forall i \in \mathcal{I}$ and $\forall \omega \in \Omega$, $\exists U^{i,\omega} \in \mathcal{U}$ such that:

$$(\forall B^i \in \mathcal{B}^i) : \arg\max_{x\in B^i} U^{i,\omega}(x) = \{x^{i,B^i,\omega}\}$$

Define $u : \Omega \to \mathcal{U}^I$ by $u(\omega) = (U^{i,\omega})_{i\in\mathcal{I}}$.

Define, $\forall B \in \mathcal{B}$, the function $\gamma_B : \Sigma^B \to [0, 1]$ by $\gamma_B(C) = g_{B,C}$. It follows from Condition 3 that γ_B is a probability measure and from Condition 4 and Theorem 1 in

Carvajal (2003), that $\exists\delta : \mathcal{P}(\Omega) \to [0,1]$, a probability measure, such that $\forall B \in \mathcal{B}$ and $\forall C \in \Sigma^B$:

$$\delta(\{\omega \in \Omega | (\arg\max_{x\in B^i} u^i(\omega)(x))_{i\in\mathcal{I}} \in C\}) = g_{B,C}$$

Define, $\forall\omega \in \Omega$, $B(\omega)$ as:

$$B(\omega) = B \in \mathcal{B} \Leftrightarrow (\forall\tilde{\omega} \in \Omega\backslash\{\omega\})(\exists i \in \mathcal{I}) : x^{i,B^i,\tilde{\omega}} \neq x^{i,B^i,\omega}$$

which one can do by Condition 5. Then, $\forall\omega \in \Omega$:

$$\begin{aligned} g_{B(\omega),\{(x^{i,B^i(\omega),\omega})_{i\in\mathcal{I}}\}} &= \gamma_{B(\omega)}(\{(x^{i,B^i(\omega),\omega})_{i\in\mathcal{I}}\}) \\ &= \delta(\{\tilde{\omega} \in \Omega | (\arg\max_{x\in B^i(\omega)} u^i(\tilde{\omega})(x))_{i\in\mathcal{I}} \in \{(x^{i,B^i(\omega),\omega})_{i\in\mathcal{I}}\}\}) \\ &= \delta(\{\tilde{\omega} \in \Omega | (\forall i \in \mathcal{I}) : x^{i,B^i(\tilde{\omega}),\omega} = x^{i,B^i(\omega),\omega}\}) = \delta(\{\omega\}) \end{aligned}$$

Now, construct $\varphi : u[\Omega] \times E \to \mathcal{F}$ as follows. Let $u \in u[\Omega]$ and let $e \in E$. By definition and Condition 5, $\#\{\omega \in \Omega | u(\omega) = u\} = 1$. Then, let $\{\omega_u\} = \{\omega \in \Omega | u(\omega) = u\}$.

If $g_{B(\omega_u),\{(x^{i,B^i(\omega_u),\omega_u})_{i\in\mathcal{I}}\}} > 0$, then define $\varphi(u,e) : \Xi \to [0,1]$ as follows:

$$\begin{aligned} &(\forall p \in \text{Supp}(\chi_e)) : \varphi(u,e)(\{p\}) = f_{\omega_u,e,p} \\ &(\forall p \in \mathcal{S}\backslash\text{Supp}(\chi_e)) : \varphi(u,e)(\{p\}) = 0 \\ &(\forall D \in \Xi : \#D \neq 1) : \varphi(u,e)(D) = \textstyle\sum_{p\in D} \varphi(u,e)(\{p\}) \end{aligned}$$

Condition 7 implies that, so defined, $\varphi(u,e) \in \mathcal{F}$.

If, alternatively, $g_{B(\omega_u),\{(x^{i,B^i(\omega_u),\omega_u})_{i\in\mathcal{I}}\}} = 0$, then let $p \in W_{u,e}$, which exists by Arrow and Debreu (1954), and define $\varphi(u,e) : \Xi \to [0,1]$ by:

$$(\forall D \in \Xi) : \varphi(u,e)(D) = \begin{cases} 1, & \text{if } p \in D \\ 0, & \text{otherwise} \end{cases}$$

It only remains to show that, so constructed, u, δ and Ω-rationalize $\{E, (\chi_e)_{e\in E}\}$.

First, let $e \in E$ and $p \in \text{Supp}(\chi_e)$. Then, by Condition 8:

$$\sum_{\omega\in\Omega} \delta(\{\omega\})\varphi(u(\omega),e)(\{p\}) = \sum_{\omega\in\Omega} g_{B(\omega),\{(x^{i,B^i(\omega),\omega})_{i\in\mathcal{I}}\}} f_{\omega,e,p} = \chi_e(\{p\})$$

It follows that $\forall e \in E$ and $\forall C \in \Xi$:

$$\begin{aligned} \chi_e(C) = \chi_e(C \cap \text{Supp}(\chi_e)) &= \sum_{p\in C\cap\text{Supp}(\chi_e)} \chi_e(\{p\}) \\ &= \sum_{p\in C\cap\text{Supp}(\chi_e)} \sum_{\omega\in\Omega} \delta(\{\omega\})\varphi(u(\omega),e)(\{p\}) \\ &= \sum_{p\in C\cap\text{Supp}(\chi_e)} \sum_{\omega\in\Omega:\delta(\{\omega\})>0} \delta(\{\omega\})\varphi(u(\omega),e)(\{p\}) \end{aligned}$$

$$= \sum_{\omega\in\Omega:\delta(\{\omega\})>0}\left(\delta(\{\omega\})\sum_{p\in C\cap \mathrm{Supp}(\chi_e)}\varphi(u(\omega),e)(\{p\})\right)$$
$$= \sum_{\omega\in\Omega:\delta(\{\omega\})>0}\left(\delta(\{\omega\})\sum_{p\in C}\varphi(u(\omega),e)(\{p\})\right)$$
$$= \sum_{\omega\in\Omega:\delta(\{\omega\})>0}\delta(\{\omega\})\varphi(u(\omega),e)(C) = \sum_{\omega\in\Omega}\delta(\{\omega\})\varphi(u(\omega),e)(C)$$

where the sixth step follows from the fact that, by Condition 7, $\forall\omega \in \Omega$ such that $\delta(\{\omega\}) > 0$, if $p \notin \mathrm{Supp}(\chi_e)$ then $\varphi(u(\omega), e)(\{p\}) = 0$.

Second, fix $\omega \in \Omega$ such that $g_{B(\omega),\{(x^{i,B^i(\omega),\omega})_{i\in\mathcal{I}}\}} > 0$ and let $e \in E$. Suppose that $p \notin W_{u(\omega),e}$. If $p \notin \mathrm{Supp}(\chi_e)$, it follows by construction that $\varphi(u(\omega), e)(\{p\}) = 0$. Now, if $p \in \mathrm{Supp}(\chi_e)\backslash W_{u(\omega),e}$, it follows that $\sum_{i\in\mathcal{I}} x^{i,B(p,e^i),\omega} \neq \sum_{i\in\mathcal{I}} e^i$ and, hence, from Condition 6, one has that $\varphi(u(\omega), e)(\{p\}) = f_{\omega,e,p} = 0$. It then follows that $\varphi(u(\omega), e)(\mathcal{S}\backslash W_{u(\omega),e}) = 0$ and that $\varphi(u(\omega), e)(W_{u(\omega),e}) = 1$.

By construction, the same conclusion applies $\forall\omega \in \Omega$ such that $g_{B(\omega),\{(x^{i,B^i(\omega),\omega})_{i\in\mathcal{I}}\}} = 0$. □

4. Non-rationalizable datasets

The results of the previous section characterize rationalizability via the existence of unobserved variables satisfying certain conditions. In principle, it could happen that such variables always exist, and hence that the general equilibrium hypothesis is irrefutable. The results of Section 5 show that there are testable restrictions purely on observed data. The examples obtained in this section show that those quantifier-free restrictions are not tautological and that the hypothesis is refutable.

There are two types of examples, corresponding to the non-existence of particular unobserved variables. The first type of example has to do with demands that satisfy SARP and arises solely from the supports of the observed distributions of prices. If this were the only kind of example, one could suspect that there is a stronger version of the results of Section 3 which does not involve the actual values of the probabilities and that, therefore, the testable restrictions obtained here are just a relaxation of the ones found by Brown and Matzkin (allowing for $\#\Omega$-many instances of SARP per individual, instead of just one). The second type of example shows that this is not the case: even under consistent supports, there are values of the actual probabilities which are impossible to rationalize.[12]

4.1. Inconsistent supports

Consider Fig. 1, where endowments $e, e' \in (\mathbb{R}^L_{++})^2$ and prices $\bar{p}, \bar{p}' \in \mathcal{S}$ are illustrated. Suppose that the supports of χ_e and $\chi_{e'}$ are as in Fig. 2.

[12] The existence of this example implies that the existence clauses for $g_{B,C}$, d_ω and $f_{\omega,e,p}$ in Theorems 1 and 2 were not trivial.

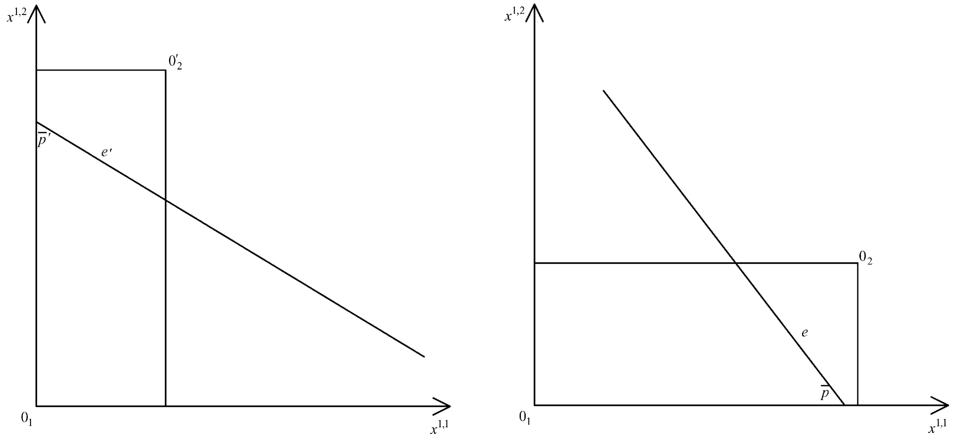

Fig. 1. Bounds on the supports of observed distributions of prices.

To see that for no Ω can these data be Ω-rationalized, argue by contradiction: suppose that for some Ω these data are rationalizable. Let $p \in \text{Supp}(\chi_e)$. By definition, there must exist $\omega \in \Omega$ such that $\delta(\{\omega\}) > 0$ and $p \in W_{u(\omega),e}$. Fix one such ω. Since $\delta(\{\omega\}) > 0$, there must exist $p' \in W_{u(\omega),e'}$ such that $p' \in \text{Supp}(\chi_{e'})$. But, then, consider Fig. 3. Whatever $p \in \text{Supp}(\chi_e)$ and $p' \in \text{Supp}(\chi_{e'})$ are, it is impossible that consumer 1 satisfy the weak axiom of revealed preferences, and therefore it cannot be that $p \in W_{u(\omega),e}$ and $p' \in W_{u(\omega),e'}$.

This example is extreme in that all the prices in $\text{Supp}(\chi_e)$ are inconsistent with all the prices in $\text{Supp}(\chi_{e'})$. It suffices that there exists one price in either one of the supports which is inconsistent with all the prices in the other support.

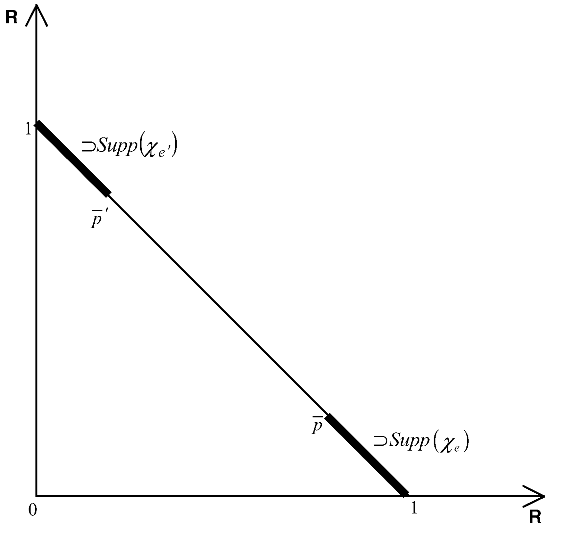

Fig. 2. The supports of observed distributions of prices.

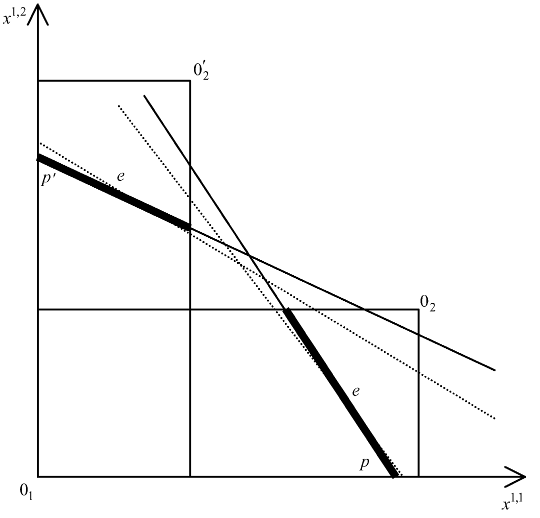

Fig. 3. Prices with positive probability imply violations of WARP.

4.2. Inconsistent probabilities

Consider Fig. 4. Suppose that endowments e and e' and associated distributions of prices χ_e and $\chi_{e'}$ have been observed such that $\text{Supp}(\chi_e) = \text{Supp}(\chi_{e'}) = \{\hat{p}, \tilde{p}\}$. Notice that $\text{Supp}(\chi_e)$ and $\text{Supp}(\chi_{e'})$ are consistent. Nonetheless, the following claim establishes that not all values of $\chi_{e'}(\{\hat{p}\})$ and $\chi_e(\{\tilde{p}\})$ can be rationalized for given set of events Ω. The claim is based on Fig. 5 and is to apply only for this example.

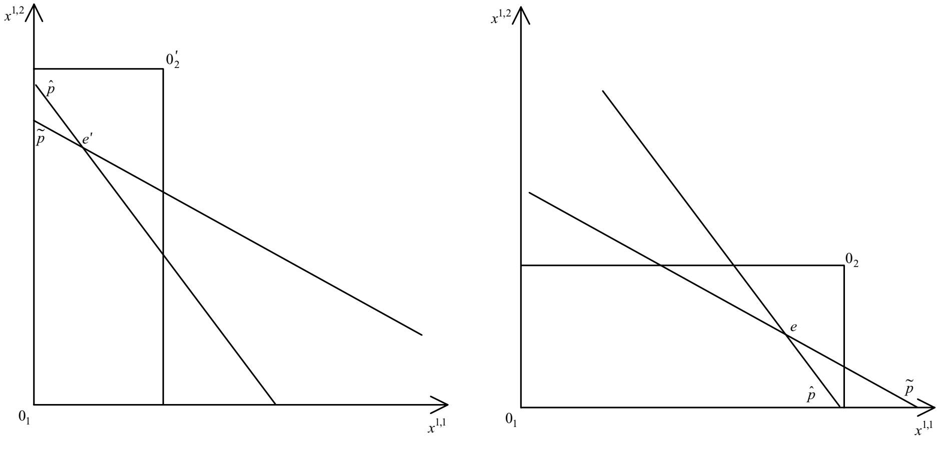

Fig. 4. Observed endowments and prices.

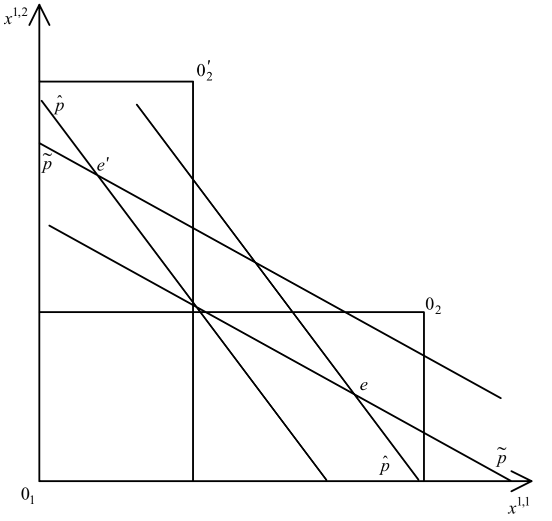

Fig. 5. Overlapping of observed endowments and prices.

Claim 1. *For every set of events Ω, dataset $\{\{e, e'\}, (\chi_e, \chi_{e'})\}$ is Ω-rationalizable only if* $\chi_{e'}(\{\hat{p}\}) + \chi_e(\{\tilde{p}\}) \geqslant 1$.

Proof. Without loss of generality, suppose that $\{\{e, e'\}, (\chi_e, \chi_{e'})\}$ is Ω-rationalized by u, δ and φ, such that the support of δ is Ω. Denote:

$$\Omega_1 = \{\omega \in \Omega | \tilde{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e}\}$$
$$\Omega_2 = \{\omega \in \Omega | \hat{p} \in W_{u(\omega),e} \wedge \hat{p} \in W_{u(\omega),e'}\}$$
$$\Omega_3 = \{\omega \in \Omega | \hat{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e}\}$$

By SARP, $\Omega_1 \cap \Omega_2 = \varnothing$. Now, suppose that $\omega \in \Omega \backslash (\Omega_1 \cup \Omega_2)$. Then,

$$\neg((\tilde{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e}) \vee (\hat{p} \in W_{u(\omega),e} \wedge \hat{p} \in W_{u(\omega),e'}))$$

which is:

$$\neg(\tilde{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e}) \wedge \neg(\hat{p} \in W_{u(\omega),e} \wedge \hat{p} \in W_{u(\omega),e'})$$

or, equivalently:

$$(\tilde{p} \notin W_{u(\omega),e'} \vee \tilde{p} \notin W_{u(\omega),e}) \wedge (\hat{p} \notin W_{u(\omega),e} \vee \hat{p} \notin W_{u(\omega),e'})$$

This implies that:

$$\begin{aligned}&(\tilde{p} \notin W_{u(\omega),e'} \wedge \hat{p} \notin W_{u(\omega),e})\\ &\quad \vee (\tilde{p} \notin W_{u(\omega),e'} \wedge \hat{p} \notin W_{u(\omega),e'}) \vee (\tilde{p} \notin W_{u(\omega),e} \wedge \hat{p} \notin W_{u(\omega),e})\\ &\quad \vee (\tilde{p} \notin W_{u(\omega),e} \wedge \hat{p} \notin W_{u(\omega),e'})\end{aligned}$$

and, therefore, given the dataset, that:

$$(\hat{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e})$$
$$\vee(\hat{p} \in W_{u(\omega),e'} \wedge \hat{p} \notin W_{u(\omega),e'}) \vee (\hat{p} \in W_{u(\omega),e} \wedge \hat{p} \notin W_{u(\omega),e})$$
$$\vee(\hat{p} \in W_{u(\omega),e} \wedge \tilde{p} \in W_{u(\omega),e'})$$

The second and third sentences are self-contradictory, whereas the fourth one is impossible by SARP. Hence, $(\hat{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e})$ and $\omega \in \Omega_3$. This proves that $\Omega_1 \cup \Omega_2 \cup \Omega_3 = \Omega$.

Suppose now that $\omega \in \Omega_1 \backslash \Omega_3$. Then,

$$(\tilde{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e}) \wedge \neg(\hat{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e})$$

which implies that:

$$(\tilde{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e}) \wedge (\hat{p} \notin W_{u(\omega),e'} \vee \tilde{p} \notin W_{u(\omega),e})$$

and hence that:

$$\tilde{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e} \wedge \hat{p} \notin W_{u(\omega),e'}$$

Moreover, by WARP, since $\tilde{p} \in W_{u(\omega),e'}$, it follows that:

$$\tilde{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e} \wedge \hat{p} \notin W_{u(\omega),e'} \wedge \hat{p} \notin W_{u(\omega),e}$$

which implies that:

$$\varphi(u(\omega), e')(\{\tilde{p}\}) = 1$$
$$\varphi(u(\omega), e')(\{\hat{p}\}) = 0$$
$$\varphi(u(\omega), e)(\{\tilde{p}\}) = 1$$
$$\varphi(u(\omega), e)(\{\hat{p}\}) = 0$$

By a symmetric argument, for $\omega \in \Omega_2 \backslash \Omega_3$ it is true that:

$$\varphi(u(\omega), e)(\{\hat{p}\}) = 1$$
$$\varphi(u(\omega), e)(\{\tilde{p}\}) = 0$$
$$\varphi(u(\omega), e')(\{\hat{p}\}) = 1$$
$$\varphi(u(\omega), e')(\{\tilde{p}\}) = 0$$

Consider now the case when $\omega \in \Omega_3$. By definition, $(\hat{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e})$ whereas, by SARP $\neg(\tilde{p} \in W_{u(\omega),e'} \wedge \hat{p} \in W_{u(\omega),e})$, which means that:

$$(\hat{p} \in W_{u(\omega),e'} \wedge \tilde{p} \in W_{u(\omega),e}) \wedge (\tilde{p} \notin W_{u(\omega),e'} \vee \hat{p} \notin W_{u(\omega),e})$$

Hence,

$$(\varphi(u(\omega), e')(\{\hat{p}\}) = 1 \wedge \varphi(u(\omega), e')(\{\tilde{p}\}) = 0) \vee (\varphi(u(\omega), e)(\{\tilde{p}\}) = 1 \wedge \varphi(u(\omega), e)(\{\hat{p}\}) = 0)$$

Now, by Ω-rationalizability, the previous results imply that:

$$\begin{aligned}
\chi_{e'}(\{\hat{p}\}) &= \sum_{\omega\in\Omega} \delta(\{\omega\})\varphi(u(\omega), e')(\{\hat{p}\}) \\
&= \sum_{\omega\in\Omega_1\backslash\Omega_3} \delta(\{\omega\})\varphi(u(\omega), e')(\{\hat{p}\}) + \sum_{\omega\in\Omega_2\backslash\Omega_3} \delta(\{\omega\})\varphi(u(\omega), e')(\{\hat{p}\}) \\
&\quad + \sum_{\omega\in\Omega_3} \delta(\{\omega\})\varphi(u(\omega), e')(\{\hat{p}\}) \\
&= \sum_{\omega\in\Omega_2\backslash\Omega_3} \delta(\{\omega\}) + \sum_{\omega\in\Omega_3} \delta(\{\omega\})\varphi(u(\omega), e')(\{\hat{p}\})
\end{aligned}$$

whereas, similarly:

$$\chi_e(\{\tilde{p}\}) = \sum_{\omega\in\Omega_1\backslash\Omega_3} \delta(\{\omega\}) + \sum_{\omega\in\Omega_3} \delta(\{\omega\})\varphi(u(\omega), e)(\{\tilde{p}\})$$

Then,

$$\begin{aligned}
\chi_{e'}(\{\hat{p}\}) + \chi_e(\{\tilde{p}\}) &= \sum_{\omega\in\Omega_2\backslash\Omega_3} \delta(\{\omega\}) + \sum_{\omega\in\Omega_1\backslash\Omega_3} \delta(\{\omega\}) \\
&\quad + \sum_{\omega\in\Omega_3} \delta(\{\omega\})(\varphi(u(\omega), e')(\{\hat{p}\}) + \varphi(u(\omega), e)(\{\tilde{p}\})) \\
&\geqslant \sum_{\omega\in\Omega_2\backslash\Omega_3} \delta(\{\omega\}) + \sum_{\omega\in\Omega_1\backslash\Omega_3} \delta(\{\omega\}) + \sum_{\omega\in\Omega_3} \delta(\{\omega\}) = 1
\end{aligned}$$

where the inequality comes from the fact that, as implied by previous results, $\forall\omega \in \Omega_3$, $\varphi(u(\omega), e')(\{\hat{p}\}) + \varphi(u(\omega), e)(\{\tilde{p}\}) \geqslant 1$. □

5. Quantifier-free testable restrictions

This section shows that given (i) a set of states of the world, (ii) an observed set of profiles of endowments, and (iii) for each observed profile of endowments, the set of observed prices, there exist restrictions (free of existential quantifiers) on the values of the probabilities that these prices can take. The general functional form of these restrictions is also determined.

For the result, a new characterization of rationalizability is introduced. This characterization is less interesting by itself, as it fails to uncover a fundamental feature of the theory of random preferences, namely the randomness of choices.

Theorem 3. *A dataset $\{E, (\chi_e)_{e\in E}\}$ is Ω-rationalizable if, and only if*:

- *For each $i \in \mathcal{I}$, each $B^i \in \mathcal{B}^i$ and each $\omega \in \Omega$, there exist $x^{i,B^i,\omega} \in \mathbb{R}^L_+$, $\lambda^{i,B^i,\omega} \in \mathbb{R}_{++}$ and $V^{i,B^i,\omega} \in \mathbb{R}$.*
- *For each $\omega \in \Omega$, there exists $d_\omega \in \mathbb{R}_+$.*

- *For each* $\omega \in \Omega$, *each* $e \in E$ *and each* $p \in \text{Supp}(\chi_e)$, *there exists* $f_{\omega,e,p} \in \mathbb{R}_+$, *such that*:
  1. *For each* $i \in \mathcal{I}$, *each* $\omega \in \Omega$, *each* $e, \tilde{e} \in E$, *each* $p \in \text{Supp}(\chi_e)$ *and each* $\tilde{p} \in \text{Supp}(\chi_{\tilde{e}})$:
  $$V^{i,B(\tilde{p},\tilde{e}^i),\omega} \leqslant V^{i,B(p,e^i),\omega} + \lambda^{i,B(p,e^i),\omega} p \cdot (x^{i,B(\tilde{p},\tilde{e}^i),\omega} - x^{i,B(p,e^i),\omega})$$
  with strict inequality if:
  $$x^{i,B(\tilde{p},\tilde{e}^i),\omega} \neq x^{i,B(p,e^i),\omega}$$
  2. *For each* $i \in \mathcal{I}$, *each* $\omega \in \Omega$, *each* $e \in E$ *and each* $p \in \text{Supp}(\chi_e)$:
  $$p \cdot x^{i,B(p,e^i),\omega} = p \cdot e^i$$
  3. $$\sum_{\omega\in\Omega} d_\omega = 1$$
  4. *For each* $\omega \in \Omega$, *each* $e \in E$ *and each* $p \in \text{Supp}(\chi_e)$:
  $$\sum_{i\in\mathcal{I}} x^{i,B(p,e^i),\omega} \neq \sum_{i\in\mathcal{I}} e^i \Rightarrow f_{\omega,e,p} = 0$$
  5. *For each* $\omega \in \Omega$ *and each* $e \in E$:
  $$d_\omega > 0 \Rightarrow \sum_{p\in\text{Supp}(\chi_e)} f_{\omega,e,p} = 1$$
  6. *For each* $e \in E$ *and each* $p \in \text{Supp}(\chi_e)$:
  $$\chi_e(\{p\}) = \sum_{\omega\in\Omega} d_\omega f_{\omega,e,p}$$

Proof. See Appendix A. □

The following theorem shows that there do exist quantifier-free testable restrictions, which have the form of polynomial inequalities.

Theorem 4. *Suppose that* Ω *and* $E \in (\mathbb{R}^L_{++})^I$ *are given. Suppose also that, for each* $e \in E$, *the set* $\text{Supp}(\chi_e) \subseteq \mathcal{S}$ *is fixed and let* Ψ *be the set of vectors*:

$$((\chi_{e,p})_{p\in\text{Supp}(\chi_e)})_{e\in E} \in \prod_{e\in E} [0,1]^{\#\text{Supp}(\chi_e)}$$

such that the dataset:

$$\left\{E, \left(C \mapsto \sum_{p\in C} \chi_{e,p}\right)_{e\in E}\right\}$$

is Ω*-rationalizable.*[13] Ψ *is a semialgebraic set.*

---

[13] The notation $C \mapsto \sum_{p\in C} \chi_{e,p}$ means that the function $\chi_e : \Xi \to [0,1]$ is constructed as:

$$(\forall C \in \Xi) : \chi_e(C) = \sum_{p\in C} \chi_{e,p}$$

Proof. See Appendix A. □

Since equilibria exist, it follows that the set Ψ need not be empty. The first example of Section 4 shows that such set may be empty and, more interestingly, the second example shows that, when non-empty, the set Ψ may be a proper subset of:

$$\prod_{e\in E}[0,1]^{\#\mathrm{Supp}(\chi_e)}$$

Then, there do exist testable restrictions on the set Ψ only, and these restrictions take, in abstract, the form of polynomial inequalities.

6. Concluding remarks

The goal of this paper has been to argue that general equilibrium theory is refutable, even without observation of individual choices and allowing individual preferences to vary randomly. This result goes in line with the ones of Brown and Matzkin (1996). My results, however, overcome the criticism, common in mathematical psychology, of the assumption of invariant preferences, which is present in the work of Brown and Matzkin via their application of revealed-preference theory.

Given a finite economy, if one observes a finite set of profiles of individual endowments and a probability distribution of prices for each one of the profiles, then there exists an exhaustive set of necessary conditions that have to be satisfied for the data to be consistent with general equilibrium theory, given a set of possible states of the world. These restrictions were studied here in two instances. First, a characterization of the definition of consistency of data and theory was given via existence of individual contingent demands and of probabilistic distributions of choices and equilibrium prices. Secondly, it was argued that the existential quantifiers can be eliminated, and that the conditions of the first characterization have an equivalent in terms of conditions purely on the data. The latter conditions were not explicitly obtained and only their abstract mathematical form could be determined. However, it was also shown that they are not vacuous: they constitute a test of the consistency of data and general equilibrium theory with power to refute this hypothesis.

I have assumed that, as the state of the world changes, individuals realize that their preferences change and choose accordingly. An alternative interpretation is that individuals, although endowed with one preference relation, are unclear about their preferences and act accordingly to their perceptions of these preferences, which depend on the state of the world. This interpretation can be easily accommodated by my results. However, in both interpretations, if there are additional hypotheses about how different states of the world affect individual preferences, they need to be incorporated in the theory, since the results here allow for a very broad class of preferences. The conditions argued here should continue to be necessary, but the arguments for sufficiency may not suffice for these additional hypotheses, in which case the list of restrictions given here will no longer be exhaustive.

Acknowledgements

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Appendix A

Proof of Theorem 3. Necessity can be argued in a way similar to Theorem 1. Sufficiency is obtained as follows.

By Conditions 1 and 2 and Theorem 2 in Matzkin and Richter (1991) it follows that $\forall i \in \mathcal{I}, \forall \omega \in \Omega, \exists U^{i,\omega} \in \mathcal{U}$ such that:

$$(\forall e \in E)(\forall p \in \mathrm{Supp}(\chi_e)) : x^{i,B(p,e^i),\omega} = \underset{x \in B(p,e^i)}{\arg\max}\, U^{i,\omega}(x)$$

Define the function $u : \Omega \to \mathcal{U}^I$ by $(\forall \omega \in \Omega) : u(\omega) = (U^{i,\omega})_{i \in \mathcal{I}}$.

Let $S = \#\Omega \in \mathbb{N}$ and denumerate $\Omega = \{\omega_1, \omega_2, \ldots, \omega_S\}$. Consider the following algorithm.

Algorithm 1. Input: Ω

1. $s = 1$, $\Theta = \varnothing$.
2. If $(\exists \tilde{\omega} \in \Theta) : u(\tilde{\omega}) = u(\omega)$, then $\theta = \varnothing$ and go to 4.
3. $\theta = \{\omega_s\}$.
4. $\Theta = \Theta \cup \theta$.
5. If $s = S$, then $\tilde{\Omega} = \Theta$ and stop.
6. $s = s + 1$ and go to 2.

Output: $\tilde{\Omega}$

The output of the algorithm, $\tilde{\Omega} \subseteq \Omega$, has the properties that:

$$(\forall \omega, \tilde{\omega} \in \tilde{\Omega}) : \omega \neq \tilde{\omega} \Rightarrow u(\omega) \neq u(\tilde{\omega})$$

$$(\forall \omega \in \Omega \backslash \tilde{\Omega})(\exists \tilde{\omega} \in \tilde{\Omega}) : u(\tilde{\omega}) = u(\omega)$$

Define now the function $\delta : \mathcal{P}(\Omega) \to \mathbb{R}$ as follows:

$$(\forall \omega \in \tilde{\Omega}) : \delta(\{\omega\}) = d_\omega + \sum_{\tilde{\omega} \in \Omega \backslash \tilde{\Omega} : u(\tilde{\omega}) = u(\omega)} d_{\tilde{\omega}}$$

$$(\forall \omega \in \Omega \setminus \tilde{\Omega}) : \delta(\{\omega\}) = 0$$

$$(\forall \Phi \in \mathcal{P}(\Omega) : \#\Phi \neq 1) : \delta(\Phi) = \sum_{\omega \in \Phi} \delta(\{\omega\})$$

By Condition 3, it follows that δ is a probability measure over Ω.

Define the function $\varphi : u[\Omega] \times E \to \mathcal{F}$ as follows. Fix $u \in u[\Omega]$ and $e \in E$. By the second property of $\tilde{\Omega}$, $\{\omega \in \Omega | u(\omega) = u\} \cap \tilde{\Omega} \neq \varnothing$. Let $\omega_u \in \{\omega \in \Omega | u(\omega) = u\} \cap \tilde{\Omega}$. By the first property of $\tilde{\Omega}$, $\forall \tilde{\omega} \in \tilde{\Omega} \setminus \{\omega_u\}$, $u(\omega) \neq u$, from where $\#(\{\omega \in \Omega | u(\omega) = u\} \cap \tilde{\Omega}) = 1$ and, hence, ω_u is defined with no ambiguity. If $\delta(\{\omega_u\}) > 0$, then define $\varphi(u, e) : \Xi \to [0, 1]$ as:

$$(\forall p \in \text{Supp}(\chi_e)) : \varphi(u, e)(\{p\}) = \frac{d_{\omega_u} f_{\omega_u,e,p} + \sum_{\omega \in \Omega \setminus \tilde{\Omega}: u(\omega) = u} d_\omega f_{\omega,e,p}}{\delta(\{\omega_u\})}$$
$$(\forall p \in \mathcal{S} \setminus \text{Supp}(\chi_e)) : \varphi(u, e)(\{p\}) = 0$$
$$(\forall C \in \Xi : \#C \neq 1) : \varphi(u, e)(C) = \sum_{p \in C} \varphi(u, e)(\{p\})$$

By construction:

$$\begin{aligned}
\varphi(u, e)(\mathcal{S}) &= \sum_{p \in \mathcal{S}} \varphi(u, e)(\{p\}) = \sum_{p \in \text{Supp}(\chi_e)} \varphi(u, e)(\{p\}) \\
&= \sum_{p \in \text{Supp}(\chi_e)} \frac{d_{\omega_u} f_{\omega_u,e,p} + \sum_{\omega \in \Omega \setminus \tilde{\Omega}: u(\omega) = u} d_\omega f_{\omega,e,p}}{\delta(\{\omega_u\})} \\
&= \frac{d_{\omega_u} \sum_{p \in \text{Supp}(\chi_e)} f_{\omega_u,e,p} + \sum_{\omega \in \Omega \setminus \tilde{\Omega}: u(\omega) = u} (d_\omega \sum_{p \in \text{Supp}(\chi_e)} f_{\omega,e,p})}{\delta(\{\omega_u\})} \\
&= \frac{d_{\omega_u} + \sum_{\omega \in \Omega \setminus \tilde{\Omega}: u(\omega) = u} d_\omega}{\delta(\{\omega_u\})} = 1
\end{aligned}$$

where the fifth equality follows from Condition 5. This and the construction imply that $\varphi(u, e) \in \mathcal{F}$. If, alternatively, $\delta(\{\omega_u\}) = 0$, then define $\varphi(u, e) \in \mathcal{F}$ and in the Proof of Theorem 2.

The functions u, δ and φ Ω-rationalize the dataset $\{E, (\chi_e)_{e \in E}\}$.

First, let $e \in E$ and $C \in \Xi$. Then,

$$\begin{aligned}
\sum_{\omega \in \Omega} \delta(\{\omega\}) \varphi(u(\omega), e)(C) &= \sum_{\omega \in \Omega: \delta(\{\omega\}) > 0} \delta(\{\omega\}) \varphi(u(\omega), e)(C) \\
&= \sum_{\omega \in \tilde{\Omega}: \delta(\{\omega\}) > 0} \sum_{p \in C} \delta(\{\omega\}) \varphi(u(\omega), e)(\{p\}) \\
&= \sum_{\omega \in \tilde{\Omega}: \delta(\{\omega\}) > 0} \sum_{p \in C} \left(d_\omega f_{\omega,e,p} + \sum_{\tilde{\omega} \in \Omega \setminus \tilde{\Omega}: u(\tilde{\omega}) = u(\omega)} d_{\tilde{\omega}} f_{\tilde{\omega},e,p} \right) \\
&= \sum_{p \in C} \sum_{\omega \in \Omega} d_\omega f_{\omega,e,p} = \sum_{p \in C} \chi_e(\{p\}) = \chi_e(C)
\end{aligned}$$

where the fourth step follows from the properties of $\tilde{\Omega}$ and the fact that $\forall \omega \in \tilde{\Omega}, \delta(\{\omega\}) = 0$ implies that $d_\omega = 0$, and the previous to last step follows from Property 6.

Now, fix $\omega \in \Omega$ and $e \in E$. Suppose that for some $p \in \mathcal{S}$, $\varphi(u(\omega), e)(\{p\}) > 0$. By the second property of $\tilde{\Omega}$, $\exists \tilde{\omega} \in \tilde{\Omega}$ such that $u(\tilde{\omega}) = u(\omega)$. If $\delta(\tilde{\omega}) = 0$, it follows by construction that $p \in W_{u(\tilde{\omega}),e} = W_{u(\omega),e}$. If, on the other hand, $\delta(\tilde{\omega}) > 0$, then, by construction:

$$(\exists \hat{\omega} \in \Omega) : u(\hat{\omega}) = u(\tilde{\omega}) \wedge d_{\hat{\omega}} > 0 \wedge f_{\hat{\omega},e,p} > 0$$

By Condition 4, $\sum_{i \in \mathcal{I}} x^{i,B(p,e^i),\hat{\omega}} = \sum_{i \in \mathcal{I}} e^i$ and, hence, by construction, $p \in W_{u(\hat{\omega}),e} = W_{u(\tilde{\omega}),e} = W_{u(\omega),e}$. This implies that $\varphi(u(\omega), e)(\mathcal{S} \backslash W_{u(\omega),e}) = 0$, or that $\varphi(u(\omega), e)(W_{u(\omega),e}) = 1$.

Lemma 1. *Let $A \subseteq \mathbb{R}^{K_1} \times \mathbb{R}^{K_2}$, where $K_1, K_2 \in \mathbb{N}$, be a semialgebraic set and let $\overrightarrow{A}^1$ be its projection into $\mathbb{R}^{K_1}$, defined as*:

$$\overrightarrow{A}^1 = \{x \in \mathbb{R}^{K_1} | (\exists y \in \mathbb{R}^{K_2}) : (x, y) \in A\}$$

Then, $\overrightarrow{A}^1$ is semialgebraic.

Proof. Define the function $\eta_1 : \mathbb{R}^{K_1} \times \mathbb{R}^{K_2} \to \mathbb{R}^{K_1}$ by $\eta_1(x, y) = x$. Its graph, $G(\eta_1) = (\mathbb{R}^{K_1} \times \mathbb{R}^{K_2}) \times \mathbb{R}^{K_1}$ is semialgebraic. Since A is semialgebraic, it follows from the Tarski–Seidenberg theorem (see Theorem 8.6.6 in Mishra, 1993) that:

$$\begin{aligned}&\{x \in \mathbb{R}^{K_1} | (\exists (x', y) \in A) : \eta_1(x', y) = x\} \\ &\quad = \{x \in \mathbb{R}^{K_1} | (\exists (x', y) \in A) : x' = x\} = \{x \in \mathbb{R}^{K_1} | (\exists y \in \mathbb{R}^{K_2}) : (x, y) \in A\} = \overrightarrow{A}^1\end{aligned}$$

is semialgebraic. □

Proof of Theorem 5. Define the functions $\text{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ by:

$$\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

and $\overrightarrow{\text{sgn}} : \mathbb{R}^L \to \{-1, 0, 1\}^L$ by $\overrightarrow{\text{sgn}}(x) = (\text{sgn}(x_l))_{l=1}^L$.

It follows from Theorem 3 that:

$$((\chi_{e,p})_{p \in \text{Supp}(\chi_e)})_{e \in E} \in \Psi \subseteq \prod_{e \in E} [0, 1]^{\#\text{Supp}(\chi_e)}$$

if, and only if, there exists a vector:

$$\zeta = \begin{pmatrix} ((((x^{i,B(p,e^i),\omega})_{p \in \text{Supp}(\chi_e)})_{e \in E})_{\omega \in \Omega})_{i \in \mathcal{I}} \\ ((((\lambda^{i,B(p,e^i),\omega})_{p \in \text{Supp}(\chi_e)})_{e \in E})_{\omega \in \Omega})_{i \in \mathcal{I}} \\ ((((V^{i,B(p,e^i),\omega})_{p \in \text{Supp}(\chi_e)})_{e \in E})_{\omega \in \Omega})_{i \in \mathcal{I}} \\ (((f_{\omega,e,p})_{p \in \text{Supp}(\chi_e)})_{e \in E})_{\omega \in \Omega} \\ (d_\omega)_{\omega \in \Omega} \end{pmatrix}$$

in the Cartesian product of the sets:

$$\left(\left(\prod_{e\in E}(\mathbb{R}^L_+)^{\#\mathrm{Supp}(\chi_e)}\right)^{\#\Omega}\right)^I$$

$$\left(\left(\prod_{e\in E}(\mathbb{R}_{++})^{\#\mathrm{Supp}(\chi_e)}\right)^{\#\Omega}\right)^I$$

$$\left(\left(\prod_{e\in E}(\mathbb{R})^{\#\mathrm{Supp}(\chi_e)}\right)^{\#\Omega}\right)^I$$

$$\left(\prod_{e\in E}[0,1]^{\#\mathrm{Supp}(\chi_e)}\right)^{\#\Omega}$$

$$[0,1]^{\#\Omega}$$

such that:

$$(((\chi_{e,p})_{p\in\mathrm{Supp}(\chi_e)})_{e\in E},\zeta)$$

satisfies the following conditions:

(i) $\forall i\in\mathcal{I}, \forall\omega\in\Omega, \forall e,\tilde{e}\in E, \forall p\in\mathrm{Supp}(\chi_e)$ and $\forall\tilde{p}\in\mathrm{Supp}(\chi_{\tilde{e}})$:

$$\begin{aligned}&\mathrm{sgn}(V^{i,B(\tilde{p},\tilde{e}^i),\omega}-V^{i,B(p,e^i),\omega}-\lambda^{i,B(p,e^i),\omega}p\cdot(x^{i,B(\tilde{p},\tilde{e}^i),\omega}-x^{i,B(p,e^i),\omega}))\\&\quad=-1\vee(\mathrm{sgn}(V^{i,B(\tilde{p},\tilde{e}^i),\omega}-V^{i,B(p,e^i),\omega})\\&\quad=0\wedge\overrightarrow{\mathrm{sgn}}(x^{i,B(\tilde{p},\tilde{e}^i),\omega}-x^{i,B(p,e^i),\omega})=(0)_{l=1}^L)\end{aligned}$$

(ii) $\forall i\in\mathcal{I}, \forall\omega\in\Omega, \forall e\in E$ and $\forall p\in\mathrm{Supp}(\chi_e)$:

$$\mathrm{sgn}(p\cdot e^i-p\cdot x^{i,B(p,e^i),\omega})=0$$

(iii) $\mathrm{sgn}(\sum_{\omega\in\Omega}d_\omega-1)=0.$

(iv) $\forall\omega\in\Omega, \forall e\in E$ and $\forall p\in\mathrm{Supp}(\chi_e)$:

$$\overrightarrow{\mathrm{sgn}}(f_{\omega,e,p}(\sum_{i\in\mathcal{I}}e_i-\sum_{i\in\mathcal{I}}x^{i,B(p,e^i),\omega}))=(0)_{l=1}^L$$

(v) $\forall\omega\in\Omega$ and $\forall e\in E$:

$$\mathrm{sgn}(d_\omega(\sum_{p\in\mathrm{Supp}(\chi_e)}f_{\omega,e,p}-1))=0$$

(vi) $\forall e \in E$ and $\forall p \in \mathrm{Supp}(\chi_e)$:

$$\mathrm{sgn}(\chi_{e,p} - \sum_{\omega \in \Omega} d_\omega f_{\omega,e,p}) = 0$$

Consider the set of vectors:

$$(((\chi_{e,p})_{p \in \mathrm{Supp}(\chi_e)})_{e \in E}, \zeta)$$

that satisfy conditions (i)–(vi). By definition, such set is semialgebraic. By Lemma 1, the projection of this set into $\prod_{e \in E} \mathbb{R}^{\#\mathrm{Supp}(\chi_e)}$, which is precisely Ψ, is also semialgebraic. □

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