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A simple(r) Lindahl solution to the provision of public goods with warm-glow: Efficiency and implementation[☆]

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A R T I C L E I N F O

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A B S T R A C T

We provide a simple(r) solution to the problem of efficiently providing public goods in a warm-glow economy. Compared with Allouch (2013), our solution is closer to Lindahl (1958), which requires only one personalized price for each consumer. This innovation is important in that it makes the techniques developed to deal with the pure public goods provision applicable to the warm-glow case. As an application, we show that under our solution concept, the implementation mechanism of Tian (1989) can be modified to implement Lindahl allocations.

1. Introduction

In the classical model of public goods provision, consumers care about the aggregate amount of public goods only, and the identity of the agent who funded it is irrelevant. Samuelson (1954) characterized the conditions for Pareto efficiency in the classical public goods model, and showed that competitive markets fail these conditions due to free riding.[1] Lindahl (1958) and Foley (1970) argued that the Pareto efficient level of public goods can be achieved through Lindahl equilibrium with each consumer having one personalized price for the public goods.

However, as Warr (1983) and Bergstrom et al. (1986) among others demonstrated, the classical public goods model implies that government provision will crowd out private contribution dollar for dollar, and the public goods provision is neutral to the income redistribution up to the set of contributors not changing. Andreoni (1988) showed that the classical public goods model implies that virtually no one makes contribution in a large economy. These model predictions are in contrast with evidence, as documented by Andreoni (1989).

Based on the earlier work by Becker (1974) and Cornes and Sandler (1984), Andreoni (1989) and Andreoni (1990) proposed the "warm-glow" model to explain the consumers' public goods contribution behavior better. In the warm-glow model, the consumers' altruism is impure: the reason why they make public goods contributions can be social pressure, guilt, sympathy or simply a desire for the warm-glow feeling. In this sense, the model is one of "impure" public goods. From the perspective of a consumer with warm-glow preferences, her contribution is not a perfect substitute for other consumers' contribution, which implies that the crowding-out is not complete, income redistribution is not neutral and free-riding is less severe. The warm-glow model is considered to be more consistent with observed consumer behavior and is widely used in public economics.

Allouch (2013) extended Lindahl's solution to achieve Pareto efficient provision of public goods in a warm-glow economy. In his *warm-glow equilibrium*, each consumer has two personalized Lindahl prices for each public good: one for her own contribution and the other for the other consumers' aggregate contribution.[2]

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[1] Actually the competitive allocation is generically constrained suboptimal as showed by Geanakoplos and Polemarchakis (2008).

[2] Carvajal and Song (2018) showed that the warm-glow model under the equilibrium concept of Allouch (2013) has strong nonparametric testable implications.

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In this paper, we provide an alternative solution to the problem of efficient public goods provision in a decentralized competitive market *à la* Lindahl when there is warm-glow. Our argument is based on the following observation. A consumer's provision of an "impure" public good enters her utility twice—both as a private good and as a pure public good. For private goods, every consumer consumes different bundles and pays the same prices. For pure public goods, every consumer consumes the same bundle and pays different Lindahl prices. Therefore, in our *warm-glow Lindahl equilibrium* each impure public good has two market prices: one as a private good and the other as a pure public good. We interpret the first price as the price for the appropriation right of each unit of the impure public good, and the second one as the price of contributing to its communal provision. Each consumer pays the same price for appropriation rights. Each consumer also has a personalized price, the sum of which is the production price for the supply of public good. We show that warm-glow Lindahl equilibrium achieves Pareto efficiency, and that any Pareto efficient allocation can be supported as a warm-glow Lindahl equilibrium. Under standard assumptions, a warm-glow Lindahl equilibrium exists and lies in the core of the economy.

The innovation of our solution is that, compared with Allouch (2013), it requires less prices to support efficient allocations: for each consumer, only one personalized price is needed. In this sense our solution is closer to Lindahl (1958). This innovation is important in that it makes many techniques developed to deal with case of pure public goods provision applicable the warm-glow case.

As Arrow (1970) pointed out, however, the Lindahl solution does not satisfy the incentive compatibility constraints of Hurwicz (1972), and consumers have no incentive to truthfully reveal their valuation for the public goods. Groves and Ledyard (1977) invented a mixed competitive mechanism, in which the private goods are allocated by competitive markets and the government designs a game to allow consumers to determine the aggregate supply of pure public goods and the taxes imposed on each consumer. The equilibria of this mechanism are Pareto optimal, though not necessarily Lindahl equilibria. Later research has refined the original work of Groves and Ledyard. In particular, Tian (1989) designed a game form to allocate both private and pure public goods to achieve Lindahl equilibria, and the mechanism is single-valued, feasible and continuous.[3]

Since our solution to efficient public goods provision with warm-glow is close to Lindahl (1958), we show that Tian's mechanism for pure public goods provision can be modified to implement Pareto efficient provision of public goods with warm-glow.[4]

2. Environment

Consider an economy with L private goods and K public goods. There are $I \geq 2$ consumers in the economy, denoted by $i = 1, \ldots, I$. Denote by $x^i \in \mathbb{R}^L_+$ consumer i's private goods consumption, and by $y^i \in \mathbb{R}^K_+$ her public goods provision.

Each consumer is endowed with a warm-glow utility function, $u^i(x^i, y^i, \sum_j y^j)$, and an initial bundle of private goods, $w^i \in \mathbb{R}^L_{++}$. There are no initial endowments of public goods, which instead are produced from private goods. The public goods production set is denoted by $\mathbb{Y}$. Its typical element is (X, Y), where $X \in \mathbb{R}^L_-$ is the private goods input and $Y \in \mathbb{R}^K_+$ is the public goods output.[5]

For each individual i, the utility function u^i is assumed to be continuous, strictly increasing, and quasi-concave on $\mathbb{R}^{L+2K}_+$. The production set $\mathbb{Y}$ is assumed to be a closed, convex cone, satisfying $(\mathbb{R}^L_-, 0) \subseteq \mathbb{Y}$ and such that for any $Y \in \mathbb{R}^K_+$, there is an $X \in \mathbb{R}^L_-$ such that $(X, Y) \in \mathbb{Y}$.

Throughout the paper, we will use boldface to denote profiles of variables.[6] Thus, an *allocation* is a triple $(\mathbf{x}, \mathbf{y}, Y) \in \mathbb{R}^{I(L+K)+K}_+$; it is *feasible* if $Y = \sum_i y^i$ and $(\sum_i(x^i - w^i), Y) \in \mathbb{Y}$; it is *Pareto efficient* if it is feasible and there does not exist another feasible allocation $(\tilde{x}, \tilde{y}, \tilde{Y})$ such that $u^i(\tilde{x}^i, \tilde{y}^i, \tilde{Y}) \geq u^i(x^i, y^i, Y)$ for each consumer i, with strict inequality at least for one.

Definition 1. A *warm-glow Lindahl equilibrium* is a tuple $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{Y}, p, \mathbf{q}, Q, r)$, where $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{Y})$ is a feasible allocation, $p \in \mathbb{R}^L_+$ denotes the prices of private goods, each $q^i \in \mathbb{R}^K_+$ is the vector of personalized consumption price for public goods, $Q \in \mathbb{R}^K_+$ is the production price for public goods, and $r \in \mathbb{R}^K_+$ is the price for appropriation rights over public goods, such that:

1. the firm maximizes profits: for all $(X, Y) \in \mathbb{Y}$,
$$p \cdot X + (Q + r) \cdot Y \leq p \cdot \sum_i (\bar{x}^i - w^i) + (Q + r) \cdot \bar{Y},$$
where $Q = \sum_i q^i$; and
2. each consumer maximizes her utility:
$$p \cdot \bar{x}^i + r \cdot \bar{y}^i + q^i \cdot \bar{Y} \leq p \cdot w^i,$$
and
$$u^i(x, y, Y) > u^i(\bar{x}^i, \bar{y}^i, \bar{Y}) \Rightarrow p \cdot x + r \cdot y + q^i \cdot Y > p \cdot w^i,$$

In economies with warm-glow effects, the public goods contain both a proprietary component, y^i, and a communal component, Y. In the latter decentralized competitive market, these two components are traded separately: all individuals pay a common price r_k for the appropriation of each unit of public good k, whereas each i contributes q^i_k per unit of the whole communal provision of that good.[7] The firm receives the aggregate of the personalized contributions, Q_k, as well as the appropriation price, r_k, per unit of good k.

3. Pareto efficiency

We now show that the concept of warm-glow Lindahl equilibrium retains the two fundamental theorems of welfare economics.

Theorem 1. *Any warm-glow Lindahl equilibrium allocation is Pareto efficient.*

Proof. Suppose, by way of contradiction, that there exists a warm-glow Lindahl equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{Y}, p, \mathbf{q}, Q, r)$ whose allocation is not efficient. Let $(\tilde{x}, \tilde{y}, \tilde{Y})$ be feasible and Pareto superior.

From utility maximization, it must be true that $p \cdot \tilde{x}^i + r \cdot \tilde{y}^i + q^i \cdot \tilde{Y} \geq p \cdot w^i$ for each consumer i, with strict inequality at least for one of them. Summing the above inequalities across consumers,

[3] Chen (2008) provided experimental evidence on the performance of these mechanisms.

[4] Carvajal and Song (2020) constructed a mechanism to implement efficient public goods provision with warm-glow under Allouch's equilibrium concept. That construction is quite different from the current mechanism since it has to take two personalized prices into account. Besides, there is resource waste out of equilibrium, which is not the case in the current mechanism.

[5] We do not need to assume a distribution for the profits of the firm, since we will assume that this technology displays constant returns to scale and maximized profits will be zero.

[6] For example, $\mathbf{x} = (x^1, \ldots, x^I)$.

[7] In the classic public goods model, consumers derive utility from using the communal provision of public goods only, and therefore $r = 0$.

we have

$$p \cdot \sum_i \tilde{x}^i + r \cdot \sum_i \tilde{y}^i + \sum_i q^i \cdot \tilde{Y} > p \cdot \sum_i w^i.$$

Given that $\sum_i \tilde{y}^i = \tilde{Y}$ and $\sum_i q^i = Q$, the above inequality implies that

$$p \cdot \sum_i \tilde{x}^i + (r+Q) \cdot \tilde{Y} > p \cdot \sum_i w^i,$$

contradicting the firms' profit maximization.

Theorem 2. *If the feasible allocation $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{Y})$ is Pareto efficient, then there exists a price system $(p, \mathbf{q}, Q, r)$ such that:*

1. *for all $(X, Y) \in \mathbb{Y}$,*
$$p \cdot X + (Q+r) \cdot Y \le p \cdot \sum_i (\bar{x}^i - w^i) + (Q+r) \cdot \bar{Y},$$
where $Q = \sum_i q^i$; and
2. *for each consumer i,*
$$u^i(x, y, Y) > u^i(\bar{x}^i, \bar{y}^i, \bar{Y}) \Rightarrow p \cdot x^i + r \cdot y^i + q^i \cdot Y > p \cdot \bar{x}^i + r \cdot \bar{y}^i + q^i \cdot \bar{Y}.$$

Proof. As with the previous theorem, the proof is standard. Let $\mathcal{F}$ denote the set of arrays $(X, Y, \dots, Y)$, where Y appears $I+1$ times, for some $(X, Y) \in \mathbb{Y}$. This is a convex cone, and is non-empty.

Define also as $\mathcal{S}$ the set of arrays $(X, Y, Y^1, \dots, Y^I) \in \mathbb{R}^{L+K(I+1)}$ such that there exist $\mathbf{x}$ and $\mathbf{y}$ for which: $\sum_i (x^i - w^i) = X$; $\sum_i y^i = Y$; and for all i, $u^i(x^i, y^i, Y^i) > u^i(\bar{x}^i, \bar{y}^i, \bar{Y})$. This set is convex and non-empty.

Since $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{Y})$ is efficient, set $\mathcal{F}$ and $\mathcal{S}$ are disjoint, so, by the separating hyperplane theorem, there exist a vector $(p, \mathbf{q}, r) \neq 0$ and a scalar α such that

$$\forall (X, Y, Y, \dots, Y) \in \mathcal{F},\ p \cdot X + (r + \sum_i q^i) \cdot Y \le \alpha, \quad (1)$$

while

$$\forall (X, Y, Y^1, \dots, Y^I) \in \bar{\mathcal{S}},\ p \cdot X + r \cdot Y + \sum_i q^i \cdot Y^i \ge \alpha, \quad (2)$$

where $\bar{\mathcal{S}}$ denotes the closure of $\mathcal{S}$.

By continuity of the utility functions, $(\bar{X}, \bar{Y}, \bar{Y}, \dots, \bar{Y}) \in \mathcal{F} \cap \bar{\mathcal{S}}$, so

$$p \cdot \sum_i (\bar{x}^i - w^i) + (\sum_{i=1}^n q^i + r) \cdot \bar{Y} = \alpha,$$

which, together with Eq. (1), yields the first claim of the theorem.

Since $(0, 0, \dots, 0) \in \mathcal{F}$, which is a cone, we can take $\alpha = 0$ in Eqs (1) and (2). Moreover, by monotonicity of all utility functions one can prove that $p > 0$.

Now, suppose $u^j(x^j, y^j, Y^j) > u^j(\bar{x}^j, \bar{y}^j, \bar{Y}^j)$ for some individual j. Let $(x^i, y^i) = (\bar{x}^i, \bar{y}^i)$ for all $i \neq j$, and note that

$$\left(\sum_{i=1}^I (x^i - w^i), \sum_{i=1}^I y^i, Y^1, \dots, Y^n \right) \in \bar{\mathcal{S}}.$$

It must be true that

$$p \cdot x^j + r \cdot y^j + q^j \cdot Y^j \ge p \cdot \bar{x}^j + r \cdot \bar{y}^j + \bar{q}^j \cdot \bar{Y}. \quad (3)$$

Suppose that equality holds in this equation. We can find another point, near $(\tilde{x}^j, \tilde{y}^j, \tilde{Y}^j)$, that is be preferred by j to $(\bar{x}^j, \bar{y}^j, \bar{Y})$ but has lower value. This corresponds to a point in $\bar{\mathcal{S}}$ that has a smaller inner product with $(p, r, \mathbf{q})$ than $(\sum_{i=1}^I \bar{x}^i, \sum_{i=1}^n \bar{y}^i, \bar{Y}, \dots, \bar{Y})$, contradicting Eq. (2). Therefore, Eq. (3) must hold with inequality, which gives the second claim of the theorem.

The existence of the warm-glow Lindahl equilibrium can by proved by modifying the argument of Foley (1970). Foley's argument can be applied to show that the warm-glow Lindahl equilibrium allocation is in the core of the economy.

4. Implementation

Since consumers have no incentive to truthfully reveal their valuation for the public goods, a natural question is how to design a mechanism to implement Pareto efficient allocations in a warm-glow economy. In this section, we show that the mechanism in Tian (1989) can be modified to fully implement the set of warm-glow Lindahl equilibria. For this result, we maintain the following assumptions: $I \ge 3$; each function $u^i(x, y, Y)$ is continuous, strictly increasing and strictly quasi-concave in (x, Y), non-decreasing and strictly quasi-concave in y, and satisfies that for any $\bar{x} \in \partial R^L_+$, any $x \in R^L_{++}$, any $y \le Y$ and any $\bar{y} \le \bar{Y}$, $u^i(x, y, Y) > u^i(\bar{x}, \bar{y}, \bar{Y})$.

4.1. Mechanisms

A mechanism (or game form) fixes a message space for each individual and an outcome function defined over the product of the message spaces. Let M^i denote the message space of individual i, with generic element m^i. Let $M = \prod_i M^i$ and denote the outcome function $\varphi : M \to \mathbb{R}^{(K+L)I}$ as

$$\varphi(\mathbf{m}) = \left(x^i(\mathbf{m}), y^i(\mathbf{m}) \right)_{i=1}^I .$$

The mechanism is said to be *feasible* if this allocation satisfies

$$\left(\sum_i [x^i(\mathbf{m}) - \omega^i], \sum_i y^i(\mathbf{m}) \right) \in \mathbb{Y}$$

for all profiles of messages.

4.2. Nash Implementation

Given the mechanism, we further denote, for each individual,

$$\varphi^i(\mathbf{m}) = \left(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m}) \right).$$

The mechanism defines a game, with strategy spaces given by the message spaces and payoff functions $\upsilon^i(\mathbf{m}) = u^i(\varphi^i(\mathbf{m}))$.

The set of allocations that are attained at the Nash equilibria of the game defined by the mechanism, given the economy, is the set *implemented* by the mechanism (in Nash equilibrium).

In what follows, we will further denote $Y^{\neg i}(\mathbf{m}) = \sum_{j \neq i} y^j(\mathbf{m})$ and $\mathbf{m}^{\neg i} = (m^j)_{j \neq i}$.

4.3. The Tian's mechanism

Fix the matrix

$$\beta = \begin{pmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & 1 & 0 & 2-I \\ 2-I & 2-I & \cdots & \cdots & 2-I & 2-I & 0 \end{pmatrix}_{I \times I},$$

noting that its rank is $I-1$ and that $\sum_{i=1}^I \beta_{ij} = 0$, $\beta_{ii} = 0$.

The message space for all individuals is

$$M^i = \mathbb{R}^L_{++} \times \mathbb{R}^K_{++} \times \mathbb{R}^K_{++} \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^K_{++} \times \mathbb{R}^L_{+} \times \mathbb{R}_{++},$$

with typical element $m^i = (p^i, r^i, Q^i, a^i, b^i, c^i, d^i, n^i)$.

Then:

1. The market prices are constructed as follows: first, let

$$s^i(\mathbf{m}) = \sum_{j,h\neq i} \|(p^j, r^j, Q^j) - (p^h, r^h, Q^h)\|$$

and $s(\mathbf{m}) = \sum_i s^i(\mathbf{m})$; then, if $s(\mathbf{m}) = 0$, let $p(\mathbf{m}) = p^1$, $r(\mathbf{m}) = r^1$ and $Q(\mathbf{m}) = Q^1$; otherwise,

$$[p(\mathbf{m}), r(\mathbf{m}), Q(\mathbf{m})] = \frac{1}{s(\mathbf{m})} \sum_i s^i(\mathbf{m})(p^i, r^i, Q^i).$$

2. For each individual, there is one personalized price for each public good k:

$$q^i_k(\mathbf{m}) = \frac{1}{I} Q_k(\mathbf{m}) - \sum_{j=1}^{i} \beta_{ij} a^j_k. \tag{4}$$

Then, by construction, $\sum_i q^i_k(\mathbf{m}) = Q_k(\mathbf{m})$.

3. The set of bundles of public goods that can be produced and is affordable by all individuals, as a result of the prices, is the set of $Y \in R^K_+$ for which
   (a) there exists a profile $\mathbf{y}$ such that $Y = \sum_i y^i$,
   $$p(\mathbf{m}) \cdot \omega^i - r(\mathbf{m}) \cdot y^i - q^i(\mathbf{m}) \cdot Y \geq 0,$$
   for all i; and
   (b) $(-\sum_i \omega^i, Y) \in \mathbb{Y}$.

   Denoting this set by $\mathcal{B}(\mathbf{m})$, the actual supply of public goods is

$$Y(\mathbf{m}) = \operatorname*{argmin}_Y \left\{ \| \sum_i b^i - Y \| : Y \in \mathcal{B}(m) \right\},$$

and its allocation is, for each good k,

$$y^i_k(\mathbf{m}) = \frac{c^i_k}{\sum_j c^j_k} Y_k(\mathbf{m}).$$

4. There is a *tax* levied on each individual i at a rate

$$T^i(\mathbf{m}) = r(\mathbf{m}) \cdot y^i(\mathbf{m}) + q^i(\mathbf{m}) \cdot Y(\mathbf{m}).$$

5. The set of bundles of private goods that can be produced and is affordable by individual i is $\mathcal{B}^i(\mathbf{m})$, defined as

$$\{x^i \in R^L_+ \mid p(\mathbf{m}) \cdot x^i \leq p(\mathbf{m}) \cdot \omega^i - T^i(\mathbf{m}) \text{ and } (x^i - \sum_j \omega^j, Y(\mathbf{m})) \in \mathbb{Y}\},$$

and the bundle of private goods for individual i claim is

$$\hat{x}^i(\mathbf{m}) = \operatorname*{argmin}_x \left\{ \|x - d^i\| : x \in \mathcal{B}^i(\mathbf{m}) \right\}. \tag{5}$$

In order to determine the actual allocation of the private good define a "shrinking factor"

$$N(\mathbf{m}) = \min \left\{ N \in \mathbb{R}_{++} : (\sum_i \frac{n^i}{N} \hat{x}^i(\mathbf{m}) - \sum_i \omega^i, Y(\mathbf{m})) \in \mathbb{Y} \text{ and } N \geq n^i \text{ for all } i \right\}, \tag{6}$$

and then let

$$x^i(\mathbf{m}) = \frac{n^i}{N(\mathbf{m})} \hat{x}^i(\mathbf{m}). \tag{7}$$

Lemma 1. *The mechanism gives non-negative consumption to all individuals, is feasible and satisfies:*

1. *If $u^i(\hat{x}^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})) > u^i(x, y, Y)$, then individual i can choose a message $\hat{m}$ such that $\upsilon^i(\hat{m}, \mathbf{m}^{-i}) > u^i(x, y, Y)$.*
2. *Suppose that $x^i(\mathbf{m}) \in R^L_{++}$ for all individuals. If for some i there exists (x, y, Y) with $Y = \sum_j y^j$ such that $p(\mathbf{m}) \cdot x \leq p(\mathbf{m}) \cdot \omega^i - T^i(\mathbf{m})$, and*
   $$u^i(x, y, Y) > u^i(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})),$$
   then there exists a message that i can send, $\hat{m}$, such that $\upsilon^i(\hat{m}, \mathbf{m}^{-i}) > \upsilon^i(\mathbf{m})$.

Proof. Non-negativity of individual consumption and feasibility of the individual bundles follow from the construction of $Y(\mathbf{m})$ and $x^i(\mathbf{m})$. For property 1, since $N(\hat{n}, \mathbf{m}^{-i}) \geq \hat{n}$, individual i can announce a large enough $\hat{n}$ to make $x^i(\hat{m}, \mathbf{m}^{-i}) \approx \hat{x}^i(\mathbf{m})$, which, suffices for the result, by continuity of υ^i.

For property 2, define, for $\lambda \in (0, 1)$, the convex combination

$$(x_\lambda, y_\lambda, Y_\lambda) = \lambda(x, y, Y) + (1 - \lambda)[x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})],$$

and note that

$$p(\mathbf{m}) \cdot x_\lambda \leq p(\mathbf{m}) \cdot \omega^i - T^i(\mathbf{m}),$$

by construction (given the definition of $x^i(\mathbf{m})$).

Define the message $\hat{m}$ as: $\hat{p} = p^i$, $\hat{r} = r^i$, $\hat{Q} = Q^i$, $\hat{a} = a^i$, $\hat{b} = Y_\lambda - \sum_{j\neq i} b^j$, $\hat{d} = x_\lambda$, and $\hat{c}$ such that

$$\frac{\hat{c}_k}{\hat{c}_k + \sum_{j\neq i} c^j_k} Y_{\lambda,k} = y_{\lambda,k}$$

for each public good. If $x^j(\mathbf{m}) \in R^L_{++}$ for all individuals, and λ is small enough, then $p(\mathbf{m}) \cdot \omega^j - r(\mathbf{m}) \cdot y^j_\lambda - q^j(\mathbf{m}) \cdot Y_\lambda > 0$ for all j and $(x^i_\lambda - \sum_j \omega^j, Y_\lambda) \in \mathbb{Y}$. And if $\hat{n}$ is large enough, by continuity $\bar{x}^i(\hat{m}, \mathbf{m}^{-i}) = x_\lambda$, and $Y(\hat{m}, \mathbf{m}^{-i}) = Y_\lambda$, which further implies that $y^i(\hat{m}, \mathbf{m}^{-i}) = y_\lambda$.

To prove that message $\hat{m}$ is a beneficial deviation for agent i, it suffices to notice that

$$\upsilon^i(\hat{m}, \mathbf{m}^{-i}) = u^i(x_\lambda, y_\lambda, Y_\lambda) > u^i(x^i(\mathbf{m}), y^i(\mathbf{m}), Y(\mathbf{m})) = \upsilon^i(\mathbf{m}),$$

by strict quasi-concavity of u^i.[8]

Lemma 2. *Let $\bar{\mathbf{m}}$ be a Nash equilibrium of the game induced by the mechanism. Then for all individuals:*

1. *$x^i(\bar{\mathbf{m}}) \gg 0$;*
2. *$p(\bar{\mathbf{m}}) \cdot x^i(\bar{\mathbf{m}}) + r(\bar{\mathbf{m}}) \cdot y^i(\bar{\mathbf{m}}) + q^i(\bar{\mathbf{m}}) \cdot Y(\bar{\mathbf{m}}) = p(\bar{\mathbf{m}}) \cdot \omega^i$; and*
3. *$N(\bar{\mathbf{m}}) = n^i$ and thus $x^i(\bar{\mathbf{m}}) = \hat{x}^i(\bar{\mathbf{m}})$.*

Proof. For property 1, suppose, by way of contradiction, that $x^i(\bar{\mathbf{m}}) = 0$ for some i. Consider the following message $\hat{m} = (\hat{p}, \hat{r}, \hat{Q}, \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{n})$ that individual i can play: $\hat{p} = \bar{p}$, $\hat{r} = \bar{r}$, $\hat{Q} = \bar{Q}$, $\hat{a} = \bar{a}$, $\hat{c} = \bar{c}$, $\hat{n} = \bar{n}$, and

$$\hat{b} = -\sum_{j\neq i} y^j(\bar{\mathbf{m}}),$$

$$\hat{d} = x^i,$$

where $x^i \gg 0$ and $p(\bar{\mathbf{m}}) \cdot x^i \leq p(\bar{\mathbf{m}}) \cdot \omega^i$. Then $\hat{x}^i(\hat{m}, \bar{\mathbf{m}}^{-i}) = x^i$ and $Y(\hat{m}, \bar{\mathbf{m}}^{-i}) = 0$. By the interiority assumption, $\upsilon^i(\hat{m}, \bar{\mathbf{m}}^{-i}) > \upsilon^i(\bar{\mathbf{m}})$, so $\bar{\mathbf{m}}$ cannot be a Nash equilibrium.

For property 2, suppose, by way of contradiction, that $p(\bar{\mathbf{m}}) \cdot x^i(\bar{\mathbf{m}}) + r(\bar{\mathbf{m}}) \cdot y^i(\bar{\mathbf{m}}) + q^i(\bar{\mathbf{m}}) \cdot Y(\bar{\mathbf{m}}) < p(\bar{\mathbf{m}}) \cdot \omega^i$ for some i. Then there

[8] The argument is a little more subtle than it seems. If $(x, Y) \neq [x^i(\mathbf{m}), Y(\mathbf{m})]$, strict quasi-concavity of u^i in its first and third arguments yields the inequality. Else, the result follows from the (weaker) quasi-concavity property imposed on the second argument of u^i.

exists (x^i, y^i, Y) which is budget feasible, and strictly preferred. But property 2 in Lemma 1 implies $\bar{\mathbf{m}}$ is not a Nash equilibrium.

For property 3, suppose $N(\bar{\mathbf{m}}) > n^i$ for some individual i. Then

$$x^i(\bar{\mathbf{m}}) = \frac{n^i}{N}\hat{x}^i(\bar{\mathbf{m}}) < \hat{x}^i(\bar{\mathbf{m}}),$$

and therefore

$$p(\bar{\mathbf{m}}) \cdot x^i(\bar{\mathbf{m}}) + r(\bar{\mathbf{m}}) \cdot y^i(\bar{\mathbf{m}}) + q^i(\bar{\mathbf{m}}) \cdot Y(\bar{\mathbf{m}}) < p(\bar{\mathbf{m}}) \cdot \omega^i,$$

which contradicts property 2.

Theorem 3. *The set of allocations implemented by the mechanism is the (complete) set of Lindahl equilibrium allocations.*

Proof. We first argue that if $\bar{\mathbf{m}}$ is a Nash equilibrium, the resulting allocation corresponds to a Lindahl equilibrium. This is now straightforward: by property 2 in Lemma 2, the firm's zero profit condition holds; by construction, the resulting allocations are feasible, and by property 2 in Lemma 1, each $(x^i(\bar{\mathbf{m}}), y^i(\bar{\mathbf{m}}), Y(\bar{\mathbf{m}}))$ must solve the problem

$$\max_{x,y,Y} \left\{u^i(x, y, Y) : p(\bar{\mathbf{m}}) \cdot x + r(\bar{\mathbf{m}}) \cdot y + q^i(\bar{\mathbf{m}}) \cdot Y \leq p(\bar{\mathbf{m}}) \cdot \omega^i\right\}. \quad (8)$$

We now argue that for any Lindahl equilibrium, $[P, (q^i)_{i=1}^I, r, Q, \bar{x}, \bar{y}, \bar{X}, \bar{Y}]$ there is a Nash equilibrium that implements its allocation. To see this, first let $(\mathbf{a}, \mathbf{b})$ solve the following linear equation system:

$$\sum_i b^i = \bar{Y},$$

for all i

$$q_k^i = \frac{1}{I}Q_k - \sum_{j=1}^{i} \beta_{ij} a_k^j,$$

and

$$b_k^i \bar{Y}_k - \sum_j b_k^j \bar{y}_k^i = 0,$$

also for each i and each k. Besides, let $p^i = p$, $r^i = r$, $Q^i = Q$, $c^i = b^i$, $d^i = \bar{x}^i$, and $n^i = 1$. With these numbers, construct the profile of strategies $\mathbf{m}$. By construction,

$$[p(\mathbf{m}), r(\mathbf{m}), Q(\mathbf{m})] = (p, r, Q),$$

while

$$(x^i(\mathbf{m}), y^i(\mathbf{m}), q^i(\mathbf{m})) = (\bar{x}^i, \bar{y}^i, q^i),$$

and $\sum_i q^i(\mathbf{m}) = Q$ for all individuals. For all deviations $\hat{m}$,

$$\begin{aligned} \upsilon^i(\hat{m}, \mathbf{m}^{-i}) &= u^i(x^i(\hat{m}, \mathbf{m}^{-i}), y^i(\hat{m}, \mathbf{m}^{-i}), Y(\hat{m}, \mathbf{m}^{-i})) \\ &\leq u^i(\bar{x}^i, \bar{y}^i, \bar{Y}) \\ &= \upsilon^i(\mathbf{m}), \end{aligned}$$

since, in particular, the choice of $\hat{m}$ cannot affect the individual's personalized prices.

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