Curso de Análisis Real para Economía

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Exercise 7. *Does the sequence* $(3n/\sqrt{n})_{n=1}^{\infty}$ *converge?*

Exercise 8. *Consider a sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$ *and a number* $a \in \mathbb{R}$. *Prove that if* $a_n \leqslant \alpha$, *for all* $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} a_n = a$, *then* $a \leqslant \alpha$. *Similarly, if* $a_n \geqslant \alpha$, *for all* $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} a_n = a$, *then* $a \geqslant \alpha$.

Exercise 9. *Recall the function* $\delta = \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$ *defined by*

$$\delta(x, y) = \sum_{k=1}^{K} |x_k - y_k|,$$

which was introduced in Exercise 4. Say that a sequence $(x_n)_{n=1}^{\infty}$, *defined in* $\mathbb{R}^K$, goes towards $x \in \mathbb{R}^K$ in a taxi *if for every* $\varepsilon > 0$, *there exists* $n^* \in \mathbb{N}$ *such that, for all* $n \geqslant n^*$, $\delta(x_n, x) < \varepsilon$. *Denote this fact by* $x_n \rightsquigarrow x$.

Argue that if sequence $(x_n)_{n=1}^{\infty}$ *goes towards* x *in a taxi, then it also converges to* $x \in \mathbb{R}^K$.

Exercise 10. *Given a sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$, *define, for each* $n \in \mathbb{N}$, *the number* $\sigma_n = \sum_{m=1}^{n} a_m$, *and call the expression*

$$\sum_{n=1}^{\infty} a_n$$

the infinite series defined by sequence $(a_n)_{n=1}^{\infty}$. *If* $\sigma_n \to \sigma \in \mathbb{R}$, *we say that the series converges to* σ, *and write*

$$\sum_{n=1}^{\infty} a_n = \sigma$$

1. *Prove that if series* $\sum_{n=1}^{\infty} a_n$ *converges, then sequence* a_n *converges to 0.*
2. *The following steps are going to show that the converse statement is not true, as* $a_n \to 0$ *does* not *suffice to imply that the series converges:*
   (a) *argue that sequence*

   $$\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}, \ldots\right)$$

   where, for each $m \in \mathbb{N}$, *the term* $1/m$ *appears* m *times, converges to* 0;

   (b) *argue that, for this sequence,* $(\sigma_n)_{n=1}^{\infty}$ *is unbounded;*

   (c) *argue that the series defined by this sequence does not converge.*