Curso de Análisis Real para Economía

Escuela de Verano de Bogotá 2022

Exercise 1. *Let* U *be a universe. For any collection* $\mathcal{X}$ *of subsets of* U*, define*

$$\bigcup_{X\in\mathcal{X}} X = \{x \in U \mid \exists X \in \mathcal{X} : x \in X\} \quad \textit{and} \quad \bigcap_{X\in\mathcal{X}} X = \{x \in U \mid \forall X \in \mathcal{X}, x \in X\}.$$

Argue that if $\mathcal{X} = \varnothing$*, then* $\cup_{X\in\mathcal{X}} X = \varnothing$ *and* $\cap_{X\in\mathcal{X}} X = U$*.*

Exercise 2. *Formulate and prove generalized De Morgan's laws that apply to general collections of sets.*

Exercise 3. *Given sets* $A, B \subseteq X$*, prove that*

1. $A \cap B \subseteq A$ *and* $A \subseteq A \cup B$*;*
2. $(A \cap B = \varnothing \Leftrightarrow A \subseteq B^c)$*,* $(A \cap B = A \Leftrightarrow A \subseteq B)$ *and* $(A \cup B = A \Leftrightarrow B \subseteq A)$*.*

Exercise 4. *Let* $K \in \mathbb{N}$ *be fixed, and define the function* $\delta : \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$ *by*

$$\delta(x, y) = \sum_{k=1}^{K} |x_k - y_k|.$$

Argue that δ *satisfies the following properties (which mean that it is a metric for* $\mathbb{R}^K$*):*

1. *for all* $x, y \in \mathbb{R}^K$*,* $\delta(x, y) \geqslant 0$*;*
2. *for all* $x, y \in \mathbb{R}^K$*,* $\delta(x, y) = \delta(y, x)$*;*
3. $\delta(x, y) = 0$ *when, and only when,* $x = y$*; and*
4. *for all* $x, y, z \in \mathbb{R}^K$*,* $\delta(x, y) \leqslant \delta(x, z) + \delta(z, y)$*.*

Exercise 5. *Prove that if* $x \in \mathbb{R}_{++}$ *and* $y \in \mathbb{R}$ *is such that* $|y - x| < x$*, then* $y \in \mathbb{R}_{++}$*. Also prove that if* $x \in \mathbb{R}_{--}$ *and* $y \in \mathbb{R}$ *is such that* $|y - x| < -x$*, then* $y \in \mathbb{R}_{--}$*.*

Exercise 6. *Does the sequence* $(1/\sqrt{n})_{n=1}^{\infty}$ *have a limit? Is it Cauchy? How about the sequence* $(3n/(n + \sqrt{n}))_{n=1}^{\infty}$*?*

Exercise 7. *Does the sequence* $(3n/\sqrt{n})_{n=1}^{\infty}$ *converge?*

Exercise 8. *Consider a sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$ *and a number* $a \in \mathbb{R}$. *Prove that if* $a_n \leqslant \alpha$, *for all* $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} a_n = a$, *then* $a \leqslant \alpha$. *Similarly, if* $a_n \geqslant \alpha$, *for all* $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} a_n = a$, *then* $a \geqslant \alpha$.

Exercise 9. *Recall the function* $\delta = \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$ *defined by*

$$\delta(x, y) = \sum_{k=1}^{K} |x_k - y_k|,$$

which was introduced in Exercise 4. Say that a sequence $(x_n)_{n=1}^{\infty}$, *defined in* $\mathbb{R}^K$, goes towards $x \in \mathbb{R}^K$ in a taxi *if for every* $\varepsilon > 0$, *there exists* $n^* \in \mathbb{N}$ *such that, for all* $n \geqslant n^*$, $\delta(x_n, x) < \varepsilon$. *Denote this fact by* $x_n \rightsquigarrow x$.

Argue that if sequence $(x_n)_{n=1}^{\infty}$ *goes towards* x *in a taxi, then it also converges to* $x \in \mathbb{R}^K$.

Exercise 10. *Given a sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$, *define, for each* $n \in \mathbb{N}$, *the number* $\sigma_n = \sum_{m=1}^{n} a_m$, *and call the expression*

$$\sum_{n=1}^{\infty} a_n$$

the infinite series defined by sequence $(a_n)_{n=1}^{\infty}$. *If* $\sigma_n \to \sigma \in \mathbb{R}$, *we say that the series converges to* σ, *and write*

$$\sum_{n=1}^{\infty} a_n = \sigma$$

1. *Prove that if series* $\sum_{n=1}^{\infty} a_n$ *converges, then sequence* a_n *converges to 0.*
2. *The following steps are going to show that the converse statement is not true, as* $a_n \to 0$ *does* not *suffice to imply that the series converges:*

   (a) argue that sequence

   $$\left(1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \ldots, \tfrac{1}{m}, \tfrac{1}{m}, \ldots, \tfrac{1}{m}, \ldots\right)$$

   where, for each $m \in \mathbb{N}$, *the term* $1/m$ *appears* m *times, converges to* 0;

(b) *argue that, for this sequence,* $(\sigma_n)_{n=1}^{\infty}$ *is unbounded;*

(c) *argue that the series defined by this sequence does not converge.*

Exercise 11. *Prove that if there exists a sequence* $(x_n)_{n=1}^{\infty}$ *defined in* $X \setminus \{\bar{x}\}$ *that converges to a point* $\bar{x}$*, then* $\bar{x}$ *is a limit point of* X.

Exercise 12. *Consider a function* $f : X \to \mathbb{R}$*, where* $X \subseteq \mathbb{R}^K$*. Suppose that* $\bar{x} \in \mathbb{R}^K$ *is a limit point of* X *and that* $\bar{y} \in \mathbb{R}$*. Argue that* $\lim_{x\to\bar{x}} f(x) = \bar{y}$ *if for every sequence* $(x_n)_{n=1}^{\infty}$ *such that* $x_n \in X \setminus \{\bar{x}\}$*, for all* $n \in \mathbb{N}$*, and that* $\lim_{n\to\infty} x_n = \bar{x}$*, one has that* $\lim_{n\to\infty} f(x_n) = \bar{y}$.

Exercise 13. *Suppose that* $X = \mathbb{R}$ *and* $f : X \to \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} 1/x, & \text{if } x \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

What is $\lim_{x\to 5} f(x)$*? What is* $\lim_{x\to 0} f(x)$*?*

Exercise 14. *Let* $f, g : X \to \mathbb{R}$*. Let* $\bar{x}$ *be a limit point of* X*. Suppose that for numbers* $\bar{y}_1, \bar{y}_2 \in \mathbb{R}$ *one has that* $\lim_{x\to\bar{x}} f(x) = \bar{y}_1$ *and* $\lim_{x\to\bar{x}} g(x) = \bar{y}_2$*. Argue that*

$$\lim_{x\to\bar{x}} (f+g)(x) = \bar{y}_1 + \bar{y}_2.$$

Exercise 15. *Prove the following statement: if* $(x_n)_{n=1}^{\infty}$ *is a sequence defined on a compact set* X*, then it has a subsequence that converges to a point in* X.

Exercise 16. *We say that point* x *is* an interior point of the set X*, if there is some* $\varepsilon > 0$ *for which* $B_\varepsilon(x) \subseteq X$*. The set of all interior points of* X *is called* the interior of X*, and is usually denoted* $\text{int}(X)$*. Note that* $\text{int}(X) \subseteq X$.

1. *Show that for every* X*,* $\text{int}(X)$ *is open.*

2. *Show that* X *is open if, and only if,* $\text{int}(X) = X$.

3. *Prove that if* $x \in \text{int}(X)$*, then* x *is a limit point of* X.

Exercise 17. *Given a set* $X \subseteq \mathbb{R}^K$, *we define its* closure, *denoted by* $\text{cl}(X)$, *as the set*

$$\text{cl}(X) = \{x \in \mathbb{R}^K \mid \forall \varepsilon > 0, B_\varepsilon(x) \cap X \neq \varnothing\}.$$

1. *Prove that, given a set* $X \subseteq \mathbb{R}^K$, $x \in \text{cl}(X)$ *if, and only if, there exists a sequence* $(x_n)_{n=1}^\infty$ *in* X *such that* $x_n \to x$.

2. *Prove that for every set* $X \subseteq \mathbb{R}^K$, $X \subseteq \text{cl}(X)$.

3. *Prove that* X *is closed if, and only if,* $X = \text{cl}(X)$.

Exercise 18. *Let* $X \subseteq \mathbb{R}^K$ *be fixed. A subset* $A \subseteq X$ *is called* dense *(in* X*) if* $X \subseteq \text{cl}(A)$. *Prove that if* A *is dense, then any point in* X *is* either *an element of* A or *a limit point of* A.

Exercise 19. *A point* $x \in \mathbb{R}^K$ *is said to be* in the boundary *of set* $X \subseteq \mathbb{R}^K$, *if for all* $\varepsilon > 0$, $B_\varepsilon(x) \cap X \neq \varnothing$ *and* $B_\varepsilon(x) \cap X^c \neq \varnothing$. *Let* $\text{bd}(X)$ *be the set of all points in the boundary of* X.[1] *Argue that* $\text{bd}(X) = \text{cl}(X) \setminus \text{int}(X)$.

Exercise 20. *Argue that function* $f : \mathbb{R}^K \to \mathbb{R}$ *is continuous if for all open set* $U \subseteq \mathbb{R}$, *one has that* $f^{-1}[U]$ *is open.*

Exercise 21. *Let* $X \subseteq \mathbb{R}^K$ *be non-empty and* $f : X \to \mathbb{R}$. *Argue that if for every open set* $U \subseteq \mathbb{R}$, *there exists an open set* $O \subseteq \mathbb{R}^K$ *such that* $f^{-1}[U] = O \cap X$, *then* f *is continuous.*

Exercise 22. *Suppose that* $X \subseteq \mathbb{R}$ *is open, and fix a point* $x \in X$ *and* $\varepsilon > 0$ *such that* $B_\varepsilon(x) \subseteq X$. *Given a function* $f : X \to \mathbb{R}$, *define* $\delta : B'_\varepsilon(0) \to \mathbb{R}$ *by*

$$\delta(h) = \frac{f(x+h) - f(x)}{h}.$$

Function $f : X \to \mathbb{R}$ *is* differentiable at x *if for some* $\ell \in \mathbb{R}$ *it is true that* $\lim_{h \to 0} \delta(h) = \ell$.

Argue that if $f : X \to \mathbb{R}$, *is differentiable at* x, *then it is continuous at* x.[2]

[1] Alternative notation is X^∂.

[2] Hint: when $x \neq \bar{x}$,

$$f(x) = f(\bar{x}) + \frac{f(x) - f(\bar{x})}{x - \bar{x}} \cdot (x - \bar{x}).$$

Exercise 23. *Consider the problem of a consumer who must choose a bundle of* $L \in \mathbb{N}$ *perfectly divisible commodities. Assume that this individual can only consume positive amounts of these goods, so that her consumption space is* $X = \mathbb{R}^L_{++}$. *The individual's preferences are a complete, reflexive and transitive binary relation* $\succsim$ *on* X, *with* $\succ$ *and* $\sim$ *defined as usual:* $x \succ x'$ *if it is not true that* $x' \succsim x$; *and* $x \sim x'$ *if it is true that* $x \succsim x'$ *and that* $x' \succsim x$.

In this setting, $\succsim$ *is said to be* strictly monotone *if* $x > x'$ *implies* $x \succ x'$, *and* strongly convex *if for any* x, *any* $x' \neq x$ *such that* $x \succsim x'$, *and any* $0 < \alpha < 1$, *it is true that* $\alpha x + (1-\alpha)x' \succ x'$. *It is* continuous *if the weak preference relation is preserved at the limit: for every pair of convergent sequences* $(x_n)_{n=1}^{\infty}$ *and* $(x'_n)_{n=1}^{\infty}$ *defined in* X *and satisfying that* $x_n \succsim x'_n$ *at all* $n \in \mathbb{N}$, *one has that*

$$\lim_{n\to\infty} x_n \succsim \lim_{n\to\infty} x'_n.$$

Finally, relation $\succsim$ *is* represented by function $u : X \to \mathbb{R}$ *if* $u(x) \geqslant u(x')$ *occurs when, and only when,* $x \succsim x'$. *The following steps prove that if* $\succsim$ *is strictly monotone, strongly convex and continuous, then it can be represented by a continuous utility function.*

1. *Fix* $x \in X$, *and define the sets*

   $$B = \{t \in \mathbb{R}_+ \mid te \succsim x\} \text{ and } W = \{t \in \mathbb{R}_+ \mid x \succsim te\},$$

   where $e = (1, \ldots, 1)$. *Argue there exist numbers* $\bar{t}$ *and* $\underline{t}$ *such that* $B = [\underline{t}, \infty)$ *and* $W = [0, \bar{t}]$.

2. *Argue that* $B \cap W \neq \varnothing$.

3. *Argue that* $B \cap W$ *is a singleton set.*

4. *Define* $u(x)$ *as the number for which* $u(x)e \sim x$. *Argue that this assignment constitutes a function.*

5. *Argue that that* u *represents* $\succsim$.

6. *Argue that for every pair of numbers* $a, b \in \mathbb{R}_{++}$,

   $$u^{-1}[(a,b)] = \{x \in \mathbb{R}^L_+ \mid x \succsim be\}^c \cap \{x \in \mathbb{R}^L_+ \mid ae \succsim x\}^c.$$

7. *Argue that for all numbers* $a, b \in \mathbb{R}_+$*, set* $u^{-1}[(a, b)]$ *is open.*

8. *Conclude that* u *is continuous.*

Exercise 24. *Let* $X \subseteq \mathbb{R}^K$*, where* $K \in \mathbb{N}$*, be non-empty, and suppose that functions* $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$ *are continuous and satisfy that* $f(x) \leqslant g(x)$ *for all* $x \in X$*. Argue that the correspondence* $\Gamma : X \twoheadrightarrow \mathbb{R}$*, defined by*

$$\Gamma(x) = [f(x), g(x)]$$

is non-empty- and compact-valued, and lower hemicontinuous.

Exercise 25. *Let* $X \subseteq \mathbb{R}^K$*, where* $K \in \mathbb{N}$*, be non-empty, and suppose that functions* $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$ *are bounded and continuous, and satisfy that* $f(x) \leqslant g(x)$ *for all* $x \in X$*. Argue that the correspondence* $\Gamma : X \twoheadrightarrow \mathbb{R}$*, defined by*

$$\Gamma(x) = [f(x), g(x)]$$

is (non-empty- and compact-valued, and) upper hemicontinuous.

Exercise 26. *Consider a society populated by a finite number of individuals who trade a finite number of commodities,* $\ell = 1, \ldots, L$*. Denote the* $(L-1)$-dimensional simplex *by*

$$\Delta, = \left\{p \in \mathbb{R}^L_+ \mid \textstyle\sum_\ell p_\ell = 1\right\},$$

and let the aggregate excess demand function be $Z : \Delta \to \mathbb{R}^L$*.*[3]

A vector of competitive equilibrium prices is a root *of the aggregate excess demand function, namely* $p \in \Delta$ *for which* $Z(p) = 0$*. The economy is said to be* determinate *if every vector of (competitive) equilibrium prices is* locally unique,

[3] Formally, let consumer i's utility function and endowment be, respectively, $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ and $w^i \in \mathbb{R}^L_+$. The aggregate excess demand function is $Z : \Delta \to \mathbb{R}^L$, is defined by

$$Z(p) = \textstyle\sum_i [x^i(p) - w^i],$$

where

$$x^i(p) = \operatorname{argmax}_x \left\{u^i(x) : p \cdot x \leqslant p \cdot w^i\right\}.$$

Assume that this function is well defined.

in the sense that for all $p \in \Delta$ *for which* $Z(p) = 0$*, there exists a number* $\varepsilon > 0$ *such that for all* $p' \in B'_\varepsilon(p) \cap \Delta$, $Z(p') \neq 0$*. Assuming that the demand function is continuous, the following steps show that determinate economies have finitely many equilibria.*

1. *Argue that any sequence of equilibrium prices has a convergent subsequence, and that the limit of that subsequence is a vector of equilibrium prices too.*

2. *Argue that if one can construct a sequence of* distinct *vectors of equilibrium prices, then there exists some* $p \in \Delta$ *such that:* (i) $Z(p) = 0$*, and* (ii) *for every* $\varepsilon > 0$ *there exists* $p' \in B'_\varepsilon(p) \cap \Delta$ *for which* $Z(p') = 0$*.*

3. *Argue that, as a consequence, if the economy is determinate, there exist only finitely many vectors* $p \in \Delta$ *such that* $Z(p) = 0$*.*

Exercise 27. *Consider a two-person simultaneous-move game, where each player* $i = 1, 2$ *chooses an action* s^i *from a predetermined set* Σ^i*. A pair of strategies* (s^1, s^2) *is a* Nash equilibrium in pure strategies *if, for each* i, s^i *solves the problem*

$$\max_{\hat{s} \in \Sigma^i} u^i(\hat{s}, s^{\neg i}),$$

where $\neg i$ *is used to denote the agent other than* i*. The steps below prove the following theorem:*

> **Theorem** (Glicksberg). *Suppose that for both* i*, set* $\Sigma^i \subset \mathbb{R}$ *is compact and convex, and function* u^i *is concave in* s^i *and continuous. Then, the game has a Nash equilibrium in pure strategies.*

1. *For each* i*, define the correspondence* $\sigma^i : \Sigma^{\neg i} \twoheadrightarrow \Sigma^i$*, by*

$$\sigma^i(s^{\neg i}) = \text{argmax}_{\hat{s} \in \Sigma^i} u^i(\hat{s}, s^{\neg i}). \qquad (*)$$

   Argue that each σ^i *is nonempty-, compact- and convex valued.*

2. *Argue that, moreover, each* σ^i *is upper hemicontinuous.*

3. *Define* $\sigma : \Sigma^1 \times \Sigma^2 \twoheadrightarrow \Sigma^1 \times \Sigma^2$ *by* $\sigma(s^1, s^2) = \sigma^1(s^2) \times \sigma^2(s^1)$. *Argue that this correspondence has a fixed point.*

4. *Conclude that this proves Glicksberg's theorem.*

Exercise 28. *Let the matrix* $R = (r^1, r^2, \ldots, r^A)$*, of dimensions* $S \times A$*, be a financial market. Denote the set of prices that allow no arbitrage opportunities by*

$$Q_1 = \{q \in \mathbb{R}^A \mid R\vartheta > 0 \implies q\vartheta > 0\};$$

and denote the set of rationalizable asset prices by

$$Q_2 = \{q \in \mathbb{R}^A \mid \exists \pi \in \mathbb{R}^S_{++} : \pi R = q\}.$$

1. *Show that* Q_1 *is non-empty and convex, and is a (positive) cone.*[4]

2. *Argue that* Q_2 *is non-empty and convex, and is a (positive) cone.*

3. *Argue that* $Q_2 \subseteq Q_1$

Exercise 29. *The space of sequences of real numbers can be written as* $\mathbb{R}^\infty$*, where each element, for the sake of clarity, will be written as*

$$\vec{x} = (x_1, x_2, \ldots) = (x_m)_{m=1}^\infty.$$

Let $\mathcal{B} \subseteq \mathbb{R}^\infty$ *denote the subset that contains all bounded sequences in* $\mathbb{R}$*. A sequence* $(\vec{x}_n)_{n=1}^\infty$ *defined in* $\mathcal{B}$ *is a sequence of sequences, where*

$$\vec{x}_n = (x_{n,1}, x_{n,2}, \ldots) = (x_{n,m})_{m=1}^\infty$$

is a bounded sequence defined in $\mathbb{R}$*.*

Sequence $(\vec{x}_n)_{n=1}^\infty$ *is said to* converge to $\vec{x}$ pointwise *if for each* $m \in \mathbb{N}$*,*

$$\lim_{n\to\infty} x_{n,m} = x_m.$$

[4] That is, for all $q \in Q_1$, for all $\alpha \in \mathbb{R}_{++}$, $\alpha q \in Q_1$

It is said to converge to $\vec{x}$ uniformly *if for every* $\varepsilon > 0$ *there exists an* $n^* \in \mathbb{N}$ *such that, for all* $n \geqslant n^*$,

$$\sup_{m \in \mathbb{N}} |x_{n,m} - x_m| < \varepsilon.$$

The following exercises will show that uniform convergence implies, but is not implied by, pointwise convergence.

1. *Argue that if* $(\vec{x}_n)_{n=1}^{\infty}$ *converges to* $\vec{x}$ *uniformly, then it converges to* $\vec{x}$ *pointwise.*

2. *Consider the sequence* $(\vec{x}_n)_{n=1}^{\infty}$ *constructed as follows:*

$$x_{n,m} = \begin{cases} 1, & \textit{if } m = n; \\ 0 & \textit{otherwise.} \end{cases}$$

   Argue that this sequence converges to $\vec{x} = (0)_{m=1}^{\infty}$ *pointwise, but not uniformly.*

Exercise 30. *Let* $X \subseteq \mathbb{R}^K$*, and denote by* $\mathbb{B}$ *the set of all continuous, bounded functions* $f : X \to \mathbb{R}$*. Equip this set with the sup metric* d*. Argue that if* $T : \mathbb{B} \to \mathbb{B}$ *is a contraction, then for any* $f \in \mathbb{B}$*, the sequence constructed by letting*

$$f_1 = f \ \textit{ and } \ f_n = T(f_{n-1}) \ \textit{ for all } \ n \geqslant 2$$

is Cauchy.

Exercise 31. *Let* $B \subseteq \mathbb{R}^K$ *be closed, and suppose that* $f : B \to B$ *is such that, for some number* $\alpha < 1$*, we have that for all* $x, x' \in B$,

$$\|f(x) - f(x')\| \leqslant \alpha \|x - x'\|.$$

1. *Argue that there exists* $\bar{x} \in B$ *such that* $f(\bar{x}) = \bar{x}$.

2. *Argue that the* $\bar{x}$ *found before is unique.*

3. *Argue that for any* $x \in B$*, the sequence constructed by letting*

$$x_1 = x \ \textit{ and } \ x_n = f(x_{n-1}) \ \textit{ for all } \ n \geqslant 2$$

   converges to $\bar{x}$.