Lecture note: Banach's Fixed Point Theorem

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Let X be a nonempty subset of the finite-dimensional Euclidean space $\mathbb{R}^K$, and denote by $\mathbb{B}_0$ the set of all bounded functions $f : X \to \mathbb{R}$. The subset of continuous functions in $\mathbb{B}_0$ is denoted by $\mathbb{B}$.

1. Set $\mathbb{B}_0$ as a Metric Space

Define the function $d : \mathbb{B}_0 \times \mathbb{B}_0 \to \mathbb{R}$ by

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|,$$

which is well defined since all functions in $\mathbb{B}_0$ are bounded.

It is straightforward that d satisfies the following propertes: for all $f, g \in \mathbb{B}_0$, $d(f, g) \geqslant 0$, with equality if, and only if, $f = g$; and $d(f, g) = d(g, f)$. Importantly, note that, for all $f, g, h \in \mathbb{B}_0$,

$$\begin{aligned} d(f, g) &= \sup_{x \in X} |f(x) - g(x)| \\ &\leqslant \sup_{x \in X} \{|f(x) - h(x)| + |h(x) - g(x)|\} \\ &\leqslant \sup_{x \in X} \{|f(x) - h(x)|\} + \sup_{x \in X} |h(x) - g(x)|\} \\ &= d(f, h) + d(h, g), \end{aligned}$$

where the first inequality comes from the triangle inequality of the real numbers, and the second one by definition of the supremum. This property is important, and we will later refer to it as the *triangle inequality* itself.

The four properties of function d make it a metric for universe $\mathbb{B}_0$. This metric is usually called the *sup metric*, and the pair $(\mathbb{B}, d)$ is, by definition a metric space.

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2. Sequences and Convergence in $\mathbb{B}_0$

A sequence in $\mathbb{B}_0$ is a function $\phi : \mathbb{N} \to \mathbb{B}_0$, as in Real Analysis, and we will maintain the simplified notation $(f_n)_{n=1}^{\infty}$, where $f_n = \phi(n)$, for sequences.

The following definitions are verbatim of their analoga in Real Analysis. Sequence $(f_n)_{n=1}^{\infty}$ is said to be *Cauchy* if for all $\varepsilon > 0$, there exists $n^* \in \mathbb{N}$ such that whenever $n, n' \geqslant n^*$, $d(f_n, f_{n'}) < \varepsilon$. The sequence is said to *converge to* $f \in \mathbb{B}_0$ if for all $\varepsilon > 0$, there exists $n^* \in \mathbb{N}$ such that whenever $n \geqslant n^*$, $d(f_n, f) < \varepsilon$. Shorthand notation in the latter case is that $\lim_{n\to\infty} f_n = f$, or simply that $f_n \to f$.

3. Completeness of $(\mathbb{B}, d)$

It is not hard to show that any convergent sequence must also be Cauchy. The following result, which need not be obvious, is that the opposite causality is true as well.

Theorem 1 (Completeness). *Any Cauchy sequence in* $(\mathbb{B}, d)$ *is also convergent.*

For the proof of this result, we will proceed in a series of steps. Before the first one of them, it is important to notice that, if $(f_n)_{n=1}^{\infty}$ is Cauchy, then

$$\forall \varepsilon > 0, \exists n_\varepsilon^* \in \mathbb{N} : \forall n, n' \geqslant n_\varepsilon^*, d(f_n, f_{n'}) < \varepsilon.$$

This implies that

$$\forall \varepsilon > 0, \exists n_\varepsilon^* \in \mathbb{N} : \forall n, n' \geqslant n_\varepsilon^*, \forall x \in X |f_n(x) - f_{n'}(x)| < \varepsilon. \tag{1}$$

Notice that in this statement n_ε^* depends on ε *but not on* x.

Now, fix a Cauchy sequence $(f_n)_{n=1}^{\infty}$. Our goal is to show that this sequence converges to some limit $f \in \mathbb{B}$.

Step 0: Construction of a candidate to be the limit: For each $x \in X$, note that Eq. (1) implies that the sequence $(f_n(x))_{n=1}^{\infty}$, which lies in $\mathbb{R}$, is Cauchy. Since $\mathbb{R}$ is complete, it follows that $f_n(x) \to y_x$, for some $y_x \in \mathbb{R}$. Letting $f(x) = y_x$ for each $x \in X$, we construct a function $f : X \to \mathbb{R}$.

Step 1: $f \in \mathbb{B}_0$*:* Note that, since each f_n is bounded, we have that for some $b_n \in \mathbb{R}$ it is true that $|f_n(x)| \leqslant b_n$ for all $x \in X$. Also, by Eq. (1),

$$\forall n \geqslant n_1^*, \forall x \in X, |f_n(x)| \leqslant b_{n_1^*} + 1.$$

If we now let

$$b = \max\{b_1, \dots, b_{n_1^*-1}, b_{n_1^*} + 1\},$$

it is by construction that $|f_n(x)| \leqslant b$ for all $n \in \mathbb{N}$ and all $x \in X$. By convergence, since limits preserve weak inequalities, $|f(x)| \leqslant b$ for all $x \in X$, so function f is bounded.

Step 2: $f_n \to f$*:* Fix $\varepsilon > 0$ and $n \geqslant n^*_{\varepsilon/3}$. Now, for any $x \in X$, we know that

$$\exists n_x \in \mathbb{N} : \forall n \geqslant n_x, |f_n(x) - f(x)| \leqslant \frac{\varepsilon}{3}.$$

Then, letting $n' = \max\{n^*_{\varepsilon/3}, n_x\}$, it follows from triangle inequality on the real line and construction that

$$\begin{aligned} |f_n(x) - f(x)| &\leqslant |f_n(x) - f_{n'}(x)| + |f_{n'}(x) - f(x)| \\ &< \sup_{x' \in X} |f_n(x') - f_{n'}(x')| + \frac{\varepsilon}{3} \\ &< \frac{2\varepsilon}{3} \end{aligned}$$

Since the latter is true for all $x \in X$, it follows that for all $n \geqslant n^* = n^*_{\varepsilon/3}$,

$$d(f_n, f) = \sup_{x \in X} |f_n(x) - f(x)| \leqslant \frac{2\varepsilon}{3} < \varepsilon.$$

Step 3: $f \in \mathbb{B}$*:* Before proceeding, note that the previous step implies that

$$\forall \varepsilon > 0, \exists \tilde{n}_\varepsilon \in \mathbb{N} : \forall n \geqslant \tilde{n}_\varepsilon, \forall x \in X, |f_n(x) - f(x)| < \varepsilon. \tag{2}$$

Importantly, in this statement $\tilde{n}_\varepsilon$ is, as in Eq. (1), independent of x.

We now want to prove that f is continuous at (any) $x \in X$. To see this, let $\varepsilon > 0$ and note that for $n = \tilde{n}_{\varepsilon/3}$, since f_n is continuous,

$$\exists \delta > 0 : \forall x' \in B_\delta(x) \cap X, |f_n(x') - f_n(x)| < \frac{\varepsilon}{3}.$$

Then, for all $x' \in B_\delta(x) \cap X$,

$$\begin{aligned} |f(x') - f(x)| &\leqslant |f(x') - f_n(x')| + |f_n(x') - f_n(x)| + |f_n(x) - f(x)| \\ &< 2 \sup_{x'' \in X} |f_n(x'') - f(x'')| + \frac{\varepsilon}{3} \\ &< \varepsilon. \end{aligned}$$

Q.E.D.

4. Banach's Fixed Point Theorem

The following theorem is important is the theory of dynamic programming. We say that a function $T : \mathbb{B} \to \mathbb{B}$ is said to be *a contraction* if there exists a number $\alpha < 1$ such that, for all $f, f' \in \mathbb{B}$,

$$d[T(f), T(f')] \leqslant \alpha d(f, f').$$

The infimum of the numbers α for which the latter condition holds is called the *Lipschitz constant*, or *modulus*, of T

Theorem 2 (Banach). *If $T : \mathbb{B} \to \mathbb{B}$ is a contraction, then there exists a function $\bar{f} \in \mathbb{B}$ such that $T(\bar{f}) = \bar{f}$. Such $\bar{f}$ is unique, and for any $f \in \mathbb{B}$, the sequence constructed by letting*

$$f_1 = f \ \textit{and} \ f_n = T(f_{n-1}) \ \ \textit{for all} \ \ n \geqslant 2$$

converges to $\bar{f}$.

Proof: Let α be the modulus of T. Fix some $f \in \mathbb{B}$, and construct a sequence by letting

$$f_1 = f \ \text{ and } \ f_n = T(f_{n-1}) \ \text{ for all } \ n \geqslant 2.$$

By construction, for all $n \in \mathbb{N}$, $d(f_{n+1}, f_n) \leqslant \alpha^{n-1} d(f_2, f_1)$. Since $\alpha < 1$, this implies that $(f_n)_{n=1}^{\infty}$ is Cauchy. By Theorem 1, $f_n \to \bar{f}$ for some $\bar{f} \in \mathbb{B}$.

To see that $T(\bar{f}) = \bar{f}$, note that for all $\varepsilon > 0$, since we can find $n^* \in \mathbb{N}$ such that $d(f_n, \bar{f}) < \varepsilon/2$ whenever $n \geqslant n^*$, then

$$\begin{aligned}
d[T(\bar{f}), \bar{f}] &\leqslant d[T(\bar{f}), f_{n^*+1}] + d(f_{n^*+1}, \bar{f}) \\
&= d[T(\bar{f}), T(f_{n^*})] + d(f_{n^*+1}, \bar{f}) \\
&\leqslant \alpha d(\bar{f}, f_{n^*}) + d(f_{n^*+1}, \bar{f}) \\
&< \frac{1+\alpha}{2} \varepsilon \\
&< \varepsilon.
\end{aligned}$$

Since this is true for all $\varepsilon > 0$, it must be that $d[T(\bar{f}), \bar{f}] = 0$.

All that remains to show is that $\bar{f}$ is unique. To see this, let $\tilde{f} \in \mathbb{B}$ such that $T(\tilde{f}) = \tilde{f}$. Then,

$$d(\bar{f}, \tilde{f}) = d[T(\bar{f}), T(\tilde{f})] \leqslant \alpha d(\bar{f}, \tilde{f}).$$

Since $\alpha < 1$, the latter is possible only if $d(\bar{f}, \tilde{f}) = 0$, so $\tilde{f} = \bar{f}$. *Q.E.D.*