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SHORT COURSE ON REAL ANALYSIS FOR ECONOMICS

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Preliminary concepts

1.1 Some Useful Notation

THE FOLLOWING NOTATION, which is standard in mathematics, will be used throughout these lecture notes: the symbol $\exists$ means *there exists*; the symbol $\forall$ means *for all*; the symbol : will be used to mean *such that*.[1]

[1] Soon we will introduce an alternative notation for *such that*, which we will use, for clarity, in particular cases.

EXAMPLE 1.1.1. *The statement* $\forall x \neq 0, \exists y \neq 0 : y \cdot x = 1$ *means that for every number* x*, different from zero, there exists some number* y*, also different from zero, such that the product* $y \cdot x$ *equals 1.*

The symbol $\Rightarrow$ means *implies*. For example, $A \Rightarrow B$ is shorthand for saying that statement A is a sufficient condition for B (and B a necessary condition for A). The symbol $\Leftrightarrow$ means *if, and only if*. For instance, $A \Leftrightarrow B$ means that statements A and B are equivalent. The symbol $\wedge$ will mean *and*, whereas $\vee$ will mean *or*. The symbol $\neg$ will be used to negate a statement. In some cases, we will use parenthesis to clarify the notation.

EXAMPLE 1.1.2 (The Contrapositive Principle). $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$.

EXAMPLE 1.1.3. $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$*, while* $\neg(A \vee B) \Leftrightarrow (\neg A \wedge \neg B)$.

The Contrapositive Principle is very useful for mathematical proofs. A closely related method of proof is the one of *arguments by contradiction*, in which, in order to show that $A \Rightarrow B$, one shows that $(A \wedge \neg B)$ is impossible.[2]

[2] These are alternatives to the method of *direct proof*, in which in order to prove $A \Rightarrow B$ one finds collection of statements (whether definitions, axioms or previously proven theorems) of the form $A_{m-1} \Rightarrow A_m$, for each $m \in \{1, \ldots, n\}$, such that $A_0 = A$ and $A_n = B$. Then, one has the following reasoning:

$$A = A_0 \Rightarrow A_1 \Rightarrow \ldots$$
$$\Rightarrow A_{n-1} \Rightarrow A_n = B.$$

We take the natural numbers as preliminary of our theory, and denote their collection by $\mathbb{N} = \{1, 2, \ldots\}$. Historically, the existence of these numbers has been taken for granted, for it was «natural» for people to use them to count.[3]

[3] Other numbers – namely 0, the negative integers, the rational numbers, the irrational numbers and the complex numbers – were more problematic, even controversial, and had to be constructed on the basis of the natural numbers.

1.2 Set Theoretical Concepts

BY *set*, WE MEAN A *collection* of objects, which are called the set's *elements*. Of course, this is not a bona fide definition, since the concept «collection» has not been defined either. Rather that trying to define a collection, we will take the concept of set as a primitive of our theory. The idea that is important, though, is that a set is completely defined by its elements, no matter what means one uses to describe them.

Following standard notation, we use $x \in X$ to denote that object x is an element of set X.[4] Also, we use $|$ to signify *such that* in the definition of a set.

[4] This statement is also read as «x belongs to X.»

EXAMPLE 1.2.1 (The Archimedean Property for $\mathbb{N}$). $\forall x \in \mathbb{N}, \exists y \in \mathbb{N} : y > x$.

EXAMPLE 1.2.2. *We can define the set* X *as the subset of elements of set* $\mathbb{N}$ *that are greater than some given* y*, by saying that* $X = \{x \in \mathbb{N} \mid x > y\}$*. Or, we can write the set* $Y = \{1, 4, 9, 16, 25, \ldots\}$ *as* $Y = \{y \in \mathbb{N} \mid \exists x \in \mathbb{N} : x \cdot x = y\}$.

DEFINITION. *The empty set,* $\varnothing$*, is the set that contains no elements.*[5]

We say that $X \subseteq Y$ whenever $x \in X \Rightarrow x \in Y$. When this is the case, we say that X *is a subset* of Y. When we have that $X \subseteq Y$ and $Y \subseteq X$, we say that $X = Y$.

THEOREM 1.2.1. *For every set* X*,* $\varnothing \subseteq X$ *and* $X \subseteq X$.[6]

[5] Notice that we say "*the* empty set", rather than "*an* empty set". The reason is that, since a set is completely defined by its elements, there exists only one empty set, no matter how one ends up finding it!

[6] Unless stated otherwise, it is expected that the student will be able to prove all the theorems in the notes. In the case of results that are not too advanced, if a proof is not provided in the notes then it is suggested as an exercise. If a proof is too advanced, a reference where it can be found is given.

1.2.1 Operations

Let us fix some set X. Relative to this *universe*, we can define the elementary set operations: given sets $A, B \subseteq X$, we define their *intersection*,

$$A \cap B = \{x \in X \mid x \in A \wedge x \in B\};$$

their *union*,

$$A \cup B = \{x \in X \mid x \in A \vee x \in B\};$$

the *complement of* B *relative to* A,

$$A \setminus B = \{x \in X \mid x \in A \wedge x \notin B\};$$

and the *complement of* A, $A^c = X \setminus A$. Note that if $A, B \subseteq Y$, then the first three operations do not change when defined relative to a different universe Y, but the fourth one does.

THEOREM 1.2.2. *Given sets* $A, B \subseteq X$,

1. $A \cap B \subseteq A$ *and* $A \subseteq A \cup B$;
2. $A \subseteq B \Leftrightarrow B^c \subseteq A^c$;
3. $(A^c)^c = A$, $\varnothing^c = X$ *and* $X^c = \varnothing$;
4. $A \cup A^c = X$ *and* $A \cap A^c = \varnothing$;
5. $A \setminus B = A \cap B^c$;
6. $(A \cap B = \varnothing \Leftrightarrow A \subseteq B^c)$, $(A \cap B = A \Leftrightarrow A \subseteq B)$ *and* $(A \cup B = A \Leftrightarrow B \subseteq A)$.

Proof. The proof is left as an exercise. For illustration purposes, let us prove the second statement. Since it is an «if and only if» statement, it is best to give the two proofs: one for necessity and one for sufficiency, independently.

Let us first prove the «if» part. Suppose that $B^c \subseteq A^c$. Then,

$$x \in B^c \Rightarrow x \in A^c,$$

which means, by definition, that

$$(x \in X \wedge x \notin B) \Rightarrow (x \in X \wedge x \notin A),$$

and hence, by Example 1.1.2, that

$$\neg(x \in X \wedge x \notin A) \Rightarrow \neg(x \in X \wedge x \notin B).$$

Then,

$$(x \notin X \vee x \in A) \Rightarrow (x \notin X \vee x \in B),$$

and, therefore,

$$x \in A \Rightarrow (x \notin X \vee x \in B),$$

but then, since $A \subseteq X$, we must conclude that

$$x \in A \Rightarrow x \in B.$$

Now we must prove the «only if» part. Suppose that $A \subseteq B$. If $B^c = \varnothing$, then we are done, by Theorem 1.2.1. Otherwise, suppose that $x \in B^c$. Then, $x \in X$ and $x \notin B$. Since $A \subseteq B$, we must have that $x \notin A$. Then, we have $x \in X$ and $x \notin A$, which means that $x \in A^c$. Since this is true for all $x \in B^c$, we can conclude that $B^c \subseteq A^c$, as needed.[7] □

THEOREM 1.2.3. *Given sets* $A, B, C \subseteq X$*,*

1. $A \cup (B \cup C) = (A \cup B) \cup C = (A \cup C) \cup B$*;*

2. $A \cap (B \cap C) = (A \cap B) \cap C = (A \cap C) \cap B$*;*

3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *and* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$*;*

4. $(A \cap B) \cup (A \setminus B) = A$.

Proof. Again, the argument is left as an exercise, but for illustration purposes we prove the last statement. By definition, we have to show that $(A \cap B) \cup (A \setminus B) \subseteq A$ and that $A \subseteq (A \cap B) \cup (A \setminus B)$. For the first inclusion,[8] suppose that $x \in (A \cap B) \cup (A \setminus B)$. By definition, then, either $x \in (A \cap B)$ or $x \in (A \setminus B)$. If the former statement is true, then the fact that $x \in A$ follows from statement 1 in Theorem 1.2.2. If the latter is true, then the fact that $x \in A$ follows by definition. For the second inclusion, consider $x \in A$. Since $A \subseteq X$, then $x \in X$. Obviously, either $x \in B$ or $x \notin B$. Since $B \subseteq X$, then either $x \in B$ or $x \in B^c$. In the first case, $x \in A \cap B$. In the second $x \in A \cap B^c = A \setminus B$, by statement 5 in Theorem 1.2.2. □

Given the first result of this theorem, if we just define

$$A \cup B \cup C = \{x \in X \mid x \in A \vee x \in B \vee x \in C\},$$

we get that $A \cup (B \cup C) = (A \cup B) \cup C = (A \cup C) \cup B = A \cup B \cup C$. The same can be done for the intersection, and for collections or more than three sets.

THEOREM 1.2.4 (DeMorgan's laws). *Given sets* $A, B \subseteq X$*, we have that* $(A \cap B)^c = A^c \cup B^c$ *and* $(A \cup B)^c = A^c \cap B^c$.

Proof. Let us prove only the first law, leaving the second one as an exercise. Suppose first that $x \in (A \cap B)^c$. Then $x \in X$ and $x \notin (A \cap B)$. The latter implies that $\neg(x \in A \wedge x \in B)$, which is the same as saying that $(x \notin A \vee x \notin B)$. Since $x \in X$,

[7] Indeed, this was a long and rather pedantic argument, for a result that is relatively simple. As these notes progress, we will present the arguments in a more efficient way.
Note that the argument used to prove sufficiency (if) uses the contrapositive principle, whereas the necessity (only if) part argues by contradiction (where?). As part of the exercise, you may want to try the sufficiency part using a contradiction argument and the necessity part via the contrapositive principle.

[8] Note that if $(A \cap B) \cup (A \setminus B) = \varnothing$, we are done by Theorem 1.2.1. It is this this kind of immediate step that will be omitted henceforth in the notes.

we have that $(x \in A^c \vee x \in B^c)$, so that $x \in A^c \cup B^c$ and $(A \cap B)^c \subseteq A^c \cup B^c$. Now, suppose that $x \in A^c \cup B^c$. Then, $(x \in A^c \vee x \in B^c)$, which means that

$$((x \in X \wedge x \notin A) \vee (x \in X \wedge x \notin B)).$$

It follows that $x \in X$ and $\neg(x \in A \wedge x \in B)$, so that $x \in X$ and $\neg(x \in A \cap B)$ or that $x \in (A \cap B)^c$. Then, we can conclude that $A^c \cup B^c \subseteq (A \cap B)^c$. □

All these results exist in far more general versions. In most cases, however, the ideas behind their proofs are the same as the ones given here.

Exercise 1.2.1 (Generalized DeMorgan's laws). *A general version of DeMorgan's laws will later prove to be useful. Formulate and prove a law that applies to more general collections of sets (and not just to two-set collections).*

A *partition* of a set $A \subseteq X$ is a collection $\mathcal{B}$ of subsets of X such that: (i) $B, B' \in \mathcal{B}$ and $B \neq B'$ imply that $B \cap B' = \varnothing$; and (ii) $\cup_{B \in \mathcal{B}} B = A$.

1.3 The Principle of Mathematical Induction

We already know some standard techniques to prove statements of the form $A \Rightarrow B$. In general, in order to prove that in some specific mathematical context statement B is true, we can use these techniques, using as "A" the whole mathematical structure that the context has. For a particular type of problems, though, there exists a technique that usually proves to be very effective. Consider first the following axiom:

Axiom 1.3.1 (The Principle of Mathematical Induction). *Let $P(n)$ be a statement that is defined for each natural number $n \in \mathbb{N}$. If $P(1)$ is true and*

$$P(n) \Rightarrow P(n+1),$$

then $P(n)$ is true for all $n \in \mathbb{N}$.

Now, suppose that we want to show that for all $n \in \mathbb{N}$, the statement $B(n)$ is true. Then, it follows from Axiom 1.3.1 that all we need to show is that $B(1)$ is true, and that (for all $n \in \mathbb{N}$) *if* statement $B(n)$ is true, then so is statement $B(n+1)$.

It turns out that the principle of mathematical induction is equivalent to another, equally intuitive, axiom.

Axiom 1.3.2 (The Principle of Well Ordering). *Every non-empty set of natural numbers has a smallest element.*

Theorem 1.3.1. *Axiom 1.3.1 is true if, and only if, so is Axiom 1.3.2.*

Proof. For the «if» part, we can argue by contradiction: let us assume that while Axiom 1.3.2 is true, Axiom 1.3.1 is not. Then, there exists a family of statements, $P(n)$, defined for all natural number, that satisfies three properties: $P(1)$ is true, $P(n) \Rightarrow P(n+1)$, and $P(n^*)$ is not true, for some $n^* \in \mathbb{N}$. Since $P(1)$ is true, it must be that $n^* > 1$, while, by the third property,

$$\{n \in \mathbb{N} \mid P(n) \text{ is not true }\} \neq \varnothing.$$

By Axiom 1.3.2, this latter set has a smallest element, which we can call $\hat{n}$. By construction, since $\hat{n}$ is the smallest natural for which $P(n)$ is false, it must be that $P(\hat{n}-1)$ is true. Then, by the second condition, $P(\hat{n})$ is also true, which gives us a contradiction.

For the «only if» part, define the following statement for each natural number n:

$$P(n) = [(S \subseteq \mathbb{N} \wedge S \cap \{1, \dots, n\} \neq \varnothing) \Rightarrow S \text{ has a smallest element}].$$

Note that $P(1)$ is true, for if $S \cap \{1\} \neq \varnothing$, then $1 \in S$. Now, suppose that $P(n)$ is true, and that $S \cap \{1, \dots, n+1\} \neq \varnothing$. If $S \cap \{1, \dots, n\} = \varnothing$, then $n+1$ is the smallest element of S. Otherwise, $S \cap \{1, \dots, n\} \neq \varnothing$, which implies by $P(n)$ that S has a smallest element. This means that $P(n) \Rightarrow P(n+1)$, and then, by Axiom 1.3.1, it follows that $P(n)$ is true for all $n \in \mathbb{N}$. But this implies Axiom 1.3.2, for if $S \neq \varnothing$, then there is some $n \in \mathbb{N}$ for which $S \cap \{1, \dots, n\} \neq \varnothing$. □

1.4 Binary Relations

A BINARY RELATION IS the result of all pairwise comparisons between the elements of a set, according to some criterion. Formally, *binary relation* R on set X is a subset of $X \times X$. It tells us whether pairs of elements (x, x') stand in the relation or not.[9] The set-theoretical notation should be that if they stand in the relation one says that $(x, x') \in R$, whereas $(x, x') \notin R$ signifies that they do not, but more usual notation is to write that xRx' in the former case, and that $\neg xRx'$ in the latter.

[9] It should be clear that the order of the elements in the pair matters.

EXAMPLE 1.4.1. *$>$, $\geqslant$ and $=$ are all binary relations on $\mathbb{N}$. If X is a set and $\mathcal{P}$ is the collection of all subsets of X, then $\subseteq$, $\subset$ and $=$ are all binary relations on $\mathcal{P}$. One can also define a binary relation on $\mathcal{P}$, by saying that ARB if, and only if, $A \cap B \neq \varnothing$.*

Binary relation R on X is *complete* if for any $x, x' \in X$, either xRx', $x'Rx$ or $x = x'$;[10] it is *reflexive* if for any x, xRx, and *irreflexive* if for any x, $\neg xRx$; it is *symmetric* if xRx' implies that $x'Rx$, *asymmetric* if xRx' implies $\neg x'Rx$, and *antisymmetric* if xRx' and $x'Rx$ together imply that $x = x'$; it is *transitive* if xRx' and $x'Rx''$ together imply xRx''.

[10] Or both. In these notes, we do not treat the connector «or» as exclusive.
When the binary relation is not complete, it is usually said that it is *partial*, and less often that it is *incomplete*.

If R is reflexive and transitive, it is said to be *a pre-order on* X. If a pre-order is antisymmetric, it is further called *an order*.

EXAMPLE 1.4.2. *Recall the notation of the previous example. Note that $\geqslant$ is a complete order on $\mathbb{N}$, and that $\subseteq$ is only an order on $\mathcal{P}$. Note that $>$ on $\mathbb{N}$ is irreflexive and asymmetric. Note that the binary relation defined on $\mathcal{P}$ by saying that ARB if, and only if, $A \cap B \neq \varnothing$ is not transitive.*

EXERCISE 1.4.1. *For each one of the relations introduced in Example 1.4.1, argue which properties it satisfies, and which not.*

If R is a binary relation on X and $A \subseteq X$, we define

$$R[A] = \{x \in X \mid \exists x' \in A : x'Rx\}$$

and

$$R^{-1}[A] = \{x \in X \mid \exists x' \in A : xRx'\}.$$

The *domain* of R is the set $R^{-1}[X]$, and its *range* is $R[X]$.[11]

If relation R is reflexive, symmetric and transitive, we say that it is *an equivalence relation on* X.[12] In such case, a subset A of X is said to be *an* R*-equivalence class* if there exists some $x \in X$ such that $R[x] = A$.[13] The importance of equivalence classes is that they form a partition of the space on which they are defined, as the following theorem shows.

EXERCISE 1.4.2. *Prove that if* R *is an equivalence on* X*, then* $x' \in R[x]$ *implies that* $R[x'] = R[x]$.

THEOREM 1.4.1. *Let* R *be an equivalence on* X *and let* $\mathcal{B}$ *be the collection of all* R*-equivalence classes.* $\mathcal{B}$ *is a partition of* X.

Proof. By reflexivity of R, we have that $x \in R[x] \in \mathcal{B}$, which implies that $x \in \cup_{B \in \mathcal{B}} B$.

On the other hand, suppose that $B, B' \in \mathcal{B}$, $B \neq B'$. We need to show that $B \cap B' = \varnothing$. In order to argue by contradiction, let us assume that $\bar{x} \in B \cap B'$. Let x and x' be such that $R[x] = B$ and $R[x'] = B'$. Since $\bar{x} \in R[x]$ and $\bar{x} \in R[x']$, by Exercise 1.4.2 we have that $R[\bar{x}] = R[x]$ and $R[\bar{x}] = R[x']$, wich is impossible since $B \neq B'$. □

1.5 Application: Preferences

CONSIDER A SITUATION IN WHICH a person faces a nonempty set X of alternatives. We refer to X as the *choice space*. A problem we study in economics[14] is how the person makes her choice, when she is allowed to pick one alternative from X, or perhaps from a subset of it. The key element in our analysis of the person's choice is to model «what she wants». For us, the individual's preferences are subjective judgments about the relative desirability of the available choices: given two alternatives, preferences are defined by her answer to the question *Is the first alternative at least as good as the second one?* Formally, then, the decision-maker's *preferences* are a binary relation $\succsim$ defined on the choice set: given a pair of alternatives x and x', we write $x \succsim x'$ if, according to the person's tastes, x is at least as good as x'.[15] In economics binary relation $\succsim$ is said to be *rational* if it is complete, reflexive and transitive.[16]

Fix a rational binary relation $\succsim$, and define the following (induced) binary relations on the choice set: the *strict preference* relation $\succ$, by saying $x \succ x'$ if it is not true that $x' \succsim x$; and the *indifference* relation $\sim$, by saying $x \sim x'$ if it is true that $x \succsim x'$ and that $x' \succsim x$. Formally, $\succ$ corresponds to the *asymmetric part* of $\succsim$, while $\sim$ is its *symmetric part*.

EXERCISE 1.5.1. *Argue that* $\succ$ *is transitive, but not reflexive, and that* $\sim$ *is reflexive, symmetric and transitive. Could these relations be complete? Could they be rational?*

Importantly, it follows from the previous exercise that relation $\sim$ is an equivalence, and then, from Theorem 1.4.1, that it partitions X by the equivalence classes it defines.

1.6 Functions

FIX TWO NONEMPTY SETS, X and Y. A *function* f, defined from set X into set Y, and denoted $f : X \to Y$, is a rule that assigns to each $x \in X$ a unique $f(x) \in Y$. Here,

[11] Equivalently, the domain is

$$\{x \in X \mid \exists x' \in X : xRx'\};$$

and the range is

$$\{x \in X \mid \exists x' \in X : x'Rx\}.$$

[12] Do you see the rationale behind this name?

[13] More correct notation would be $R[\{x\}]$, since x is an element of X and not a subset of it. We simplify the notation and hope that no confusion arises.

[14] More specifically, in the field of economics called Decision Theory.

[15] We take the person's preferences as exogenous, in the sense that we do not explain where they come from. Instead, we concentrate on the problem of studying the individual's behavior given her preferences, under the assumption that these preferences will *not* be affected by the person's choices.

[16] Decision-makers with incomplete preferences may find instances in which they are unable to choose: they are simply unable to make a value judgments about the relative (subjective) quality of two alternatives. Reflexivity is consistent with our interpretation of at-least-as-good preference. People with non-transitive preferences are open to full rent extraction, as a person could find a cycle of choices for which the person is willing to pay a positive premium at each step. In economics, one usually assumes that the decision-maker under consideration has rational preferences, although in some cases (e.g. very complicated problems) it may be reasonable to consider that individual's preferences are incomplete; also, some cases of non-transitive preferences are sometimes observed in real life.

set X is said to be the *domain* of f, and Y its *co-domain* or *target set.* If $f : X \to Y$, and $A \subseteq X$, the *image of* A *under* f, denoted $f[A]$, is the set

$$f[A] = \{y \in Y \mid \exists x \in A : f(x) = y\}.$$

In particular, $f[X]$ is called the *range* of f.

Function $f : X \to Y$ is said to be *onto*, or *surjective*, if $f[X] = Y$; it is said to be *one-to-one*, or *injective*, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. A function is said to be *a one-to-one correspondence*, or *bijective*, if it is both onto and one-to-one.

If $f : X \to Y$, and $B \subseteq Y$, the *inverse image of* B *under* f is the set

$$f^{-1}[B] = \{x \in X \mid f(x) \in B\}.$$

THEOREM 1.6.1. *The function* $f : X \to Y$ *is onto if, and only if, for all* $B \subseteq Y$, $B \neq \varnothing$, $f^{-1}[B] \neq \varnothing$.

If $f : X \to Y$ is a one-to-one correspondence, the *inverse function* $f^{-1} : Y \to X$ is implicitly defined by $\{f^{-1}(y)\} = f^{-1}[\{y\}]$.[17]

[17] Notice that this would not have been be a bona fide definition, had we forgotten to say that f is a one-to-one correspondence. What could have gone wrong?

1.7 Application: Representation of Preferences

IT IS MOST USUAL IN ECONOMICS to represent a decision-maker's preferences by a function that gives a higher value the more the person likes an alternative. That is, we say that the binary relation $\succsim$ on X is *represented by function* $U : X \to \mathbb{R}$ if $U(x) \geqslant U(x')$ occurs when, and only when, $x \succsim x'$; we say that $\succsim$ is *representable* if there is some function U that represents it. Notice that the representation of $\succsim$ by a utility function amounts to attributing real numbers to the equivalence classes defined by $\sim$, in such a way that higher numbers are assigned to classes that are «more prefered» by the individual.

The function U that represents $\succsim$ is called *utility function.* Notice that if a preference relation is representable, then there are infinitely many different utility functions that represent it. All these representations will have the same contour sets,[18] but may give nontrivially different utility levels.[19] It is for this reason that interpersonal comparisons of utility are considered problematic.

[18] That is, the same ordinal information

[19] Which means different cardinal information.

EXERCISE 1.7.1. *Argue that representability implies rationality. Do you think that rationality implies representability?*

In some cases the existence of a utility function that represents an individual's preferences is very easy to establish. For example, if X is finite, then any complete preference relation on it will be representable. But there are also well-known cases of preference relations that cannot be represented by utility functions, because they partition the choice space in «too many,» ordered equivalence classes.

The general problem of what preference relations can be represented by functions is beyond these lectures, but a canonical example from economics is given in Chapter 4.

1.8 Countable Sets

AS NATURAL AS THE DEFINITION of the natural numbers is the definition of a finite set: set S is *finite* if there exist a natural number n and an onto function $f : \{1, \ldots, n\} \to S$.

Now, there are sets that are infinite, but we can still «count» them, even if it would take us forever to do so. Formally, we say that set S is *countable* if there exists an onto function $f : \mathbb{N} \to S$.

EXAMPLE 1.8.1. *Any finite set is countable, obviously. The set of natural numbers is countable, also obviously. Maybe more surprisingly, given that one of them contains more numbers than the other, is that the set of integers,* $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$, *is also countable. To see why this is the case, it suffices to consider the following mapping:*

$$\begin{aligned} 1 &\mapsto 0 \\ 2 &\mapsto 1 \\ 3 &\mapsto -1 \\ 4 &\mapsto 2 \\ 5 &\mapsto -2 \\ &\vdots \end{aligned}$$

This mapping constitutes a function from $\mathbb{N}$ *onto* $\mathbb{Z}$, *so the latter is indeed countable.*[20]

[20] If you want to be really formal, just define the function $f : \mathbb{N} \to \mathbb{Z}$ as:

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even;} \\ -(n-1)/2, & \text{otherwise.} \end{cases}$$

It should be clear that this map is onto.

THEOREM 1.8.1. *If an infinite set* S *is countable, then there exists a bijection* $f : \mathbb{N} \to S$.

Proof. Since S is countable, there exists a surjective function $\phi : \mathbb{N} \to S$. Now, construct the following mapping, recursively: First, let $f(1) = \phi(1)$. Then, for any $n \in \mathbb{N}$, consider the set

$$M_n = \left\{ m \in \mathbb{N} \mid \forall n' \in \{1, \ldots, n-1\}, \phi(m) \neq f(n') \right\}.$$

Since S is infinite and ϕ is onto, this set is nonempty and, then, by the principle of well ordering, it contains a smallest element, which we can denote as m_n. Then, we define $f(m) = \phi(m_n)$.

In order to complete the proof, we must show that, so constructed, f is both one-to-one and onto. To see that it is onto, consider any $s \in S$. Since ϕ is onto, we can find some $n \in \mathbb{N}$ for which $\phi(n) = s$. Let n_s be the smallest such n, which exists by Axiom 1.3.2. Then, for all $n < n_s$ we have that $f(n') \neq \phi(n_s)$, which implies that n_s is the smallest element of M_{n_s} and, then, $f(n_s) = \phi(n_s) = s$.

Now, to see that f is one-to-one, consider two distinct natural numbers n and n', assuming, with no loss of generality, that $n > n'$. By construction, $f(n) = \phi(m_n)$, for some $m_n \in M_n$. It is immediate that $f(n) = \phi(m_n) \neq f(n')$. $\square$

EXERCISE 1.8.1. *Argue that a set is countable if, and only if, every nonempty subset of it is also countable.*

It is also interesting to know which sets are not countable. In the next chapter, we introduce the set of real numbers, which is one such set. Another example, which is classical in mathematics, is that the set of infinite strings of two distinct objects, say the numbers 0 and 1, is not countable.[21]

[21] The argument is due to Cantor, and is extremely elegant: if such set were countable, we could organize it in the order given by the bijection of Proposition 1.8.1 – think of it as having the strings stacked one over the other – now, construct a string by going up diagonally and taking the element different than the one on the corresponding string; it follows by construction that this new string is different from all the other ones, so it is not the image of any natural number, which is impossible because the mapping was supposed to be onto.

2
Introduction to Real Analysis

2.1 Natural and Real Numbers

In a Real Analysis course, we would be concerned with the construction of the set of real numbers, $\mathbb{R}$. One way to do this is to take as a primitive the set $\mathbb{N}$ of natural numbers, then construct the set of rational numbers, and finally fill in the holes that are left by the latter (i.e., the irrational numbers). Once $\mathbb{R}$ is constructed, it is shown that it can be completely characterized by three groups of axioms. Although we will take as given the existence and properties of $\mathbb{R}$, we now recall the first two groups of axioms exhibited by $\mathbb{R}$.[1]

[1] You will see how trivial these two groups of axioms look; that's why we need not spend much time on them, and can take them as given. In Chapter 3, we will introduce the third axiom (Axiom 3.3.1), which is satisfied by $\mathbb{R}$, but not by the set of rational numbers.

Axiom 2.1.1 (Field Axioms). *Given numbers $x, y, z \in \mathbb{R}$,*

1. $x + y = y + x$ *and* $(x + y) + z = x + (y + z)$ *;*
2. *there exists a number,* $0 \in \mathbb{R}$*, such that* $w + 0 = w$ *for all* $w \in \mathbb{R}$*;*
3. *there exists a number* $w \in \mathbb{R}$ *for which* $x + w = 0$*;*
4. $xy = yx$ *and* $(xy)z = x(yz)$*;*
5. *there exists a number,* $1 \in \mathbb{R}$*, such that* $1w =$ *for all* $w \in \mathbb{R}$*;*
6. *if* $x \neq 0$*, there exists* $w \in \mathbb{R}$ *for which* $xw = 1$*; and*
7. $x(y + z) = xy + xz$.

We will denote by $\mathbb{R}_+$ the set of nonnegative real numbers and by $\mathbb{R}_{++}$ the set of positive real numbers. Accordingly, we denote $\mathbb{R}_- = \mathbb{R} \setminus \mathbb{R}_{++}$ and $\mathbb{R}_{--} = \mathbb{R} \setminus \mathbb{R}_+$.

Axiom 2.1.2 (Order Axioms). *Given numbers* $x, y \in \mathbb{R}_{++}$ *and* $z \in \mathbb{R}$*,* $x + y \in \mathbb{R}_{++}$*,* $xy \in \mathbb{R}_{++}$*,* $-x \notin \mathbb{R}_{++}$ *and, either* $z \in \mathbb{R}_{++}$*, or* $-z \in \mathbb{R}_{++}$*, or* $z = 0$*.*

2.2 Metric Spaces

Fix a nonempty set X. A *metric* for X is a function $d : X \times X \to \mathbb{R}$ that satisfies:

1. for all $x, x' \in X$, $d(x, x') \geqslant 0$;
2. $d(x, x') = 0 \iff x = x'$;

3. for all $x, x' \in X$, $d(x, x') = d(x', x)$; and

4. for all $x, x', x'' \in X$, $d(x, x') + d(x', x'') \geqslant d(x, x'')$.

The fourth of these properties is called *the triangle inequality*. The third property is referred to as symmetry. Somewhat surprisingly, the first property is redundant in the definition, as it is implied by the other three.

EXERCISE 2.2.1. *Argue that* $d : X \times X \to \mathbb{R}$ *is a metric if, and only if, it satisfies properties* 2 *to* 4 *above.*

EXAMPLE 2.2.1. *The usual way to measure how «far from 0» a real number is, is by its* absolute value, $|x|$, *which is defined as* x *if* $x \geqslant 0$, *and as* $-x$ *otherwise.*[2] *Importantly, given numbers* $x, y \in \mathbb{R}$, $|x + y| \leqslant |x| + |y|$. *To see this, note first that if* $x + y \geqslant 0$, *we have that* $|x + y| = x + y \leqslant |x| + |y|$, *by definition and the second property above. Alternatively, when* $x + y < 0$, *we have that*

$$|x + y| = -(x + y) = (-x) + (-y) \leqslant |-x| + |-y| = |x| + |y|,$$

by definition and the second and fourth properties. It then follows that the function $(x, x') \mapsto |x - x'|$ *is a metric for* $\mathbb{R}$.

EXERCISE 2.2.2. *Prove that if* $x \in \mathbb{R}_{++}$ *and* $y \in \mathbb{R}$ *is such that* $|y - x| < x$, *then* $y \in \mathbb{R}_{++}$. *Also prove that if* $x \in \mathbb{R}_{--}$ *and* $y \in \mathbb{R}$ *is such that* $|y - x| < -x$, *then* $y \in \mathbb{R}_{--}$.

2.3 Finite-Dimensional Euclidean Spaces

FOR A NUMBER $K \in \mathbb{N}$, the K-dimensional real (Euclidean) space is the K-fold Cartesian product of $\mathbb{R}$. We denote this space by $\mathbb{R}^K$, so that $x \in \mathbb{R}^K$ is $(x_1, x_2, \ldots, x_K)$.[3] Now, in order to measure how far from 0 (that is, from $(0, 0, \ldots, 0)$) an element x of $\mathbb{R}^K$ is, we use the *Euclidean norm*, which is defined as[4]

$$\|x\| = \left(\sum_{k=1}^{K} x_k^2\right)^{1/2} = (x \cdot x)^2.$$

It is obvious that when $K = 1$ the Euclidean norm corresponds to the absolute value. More importantly, it is also clear that for every $x \in \mathbb{R}^K$, one has that $\|x\| \geqslant 0$; $\|x\| = 0 \Leftrightarrow x = 0$; $-y \leqslant x \leqslant y \Rightarrow \|x\| \leqslant \|y\|$; and $\|x\| = \|-x\|$. The crucial property, finally, is that Triangle Inequality also holds in $\mathbb{R}^K$.

LEMMA 2.3.1 (Triangle Inequality in $\mathbb{R}^K$). *Given* $x, y \in \mathbb{R}^K$, $\|x + y\| \leqslant \|x\| + \|y\|$.

Proof. A well-established result in mathematics, called the *Cauchy-Schwartz Inequality*, states that[5]

$$\Big(\sum_k x_k y_k\Big)^2 \leqslant \sum_k x_k^2 \times \sum_k y_k^2.$$

[2] Technically speaking, the absolute value is a *norm*, and, when used, it defines $\mathbb{R}$ as a normed vector space. Four properties of the absolute value (and of *any* norm) are straightforward:
(1) for all $x \in \mathbb{R}$, $|x| \geqslant 0$;
(2) for all $x \in \mathbb{R}$, $|x| \geqslant x$;
(3) if $x \in \mathbb{R}$, and $y \in \mathbb{R}_+$ are such that $-y \leqslant x \leqslant y$, then $|x| \leqslant |y|$; and
(4) for all $x \in \mathbb{R}$, $|x| = |-x|$.

[3] Similar notation as above is used for orthants of $\mathbb{R}^K$.

[4] If you want to avoid confusion, you can be explicit about the dimension for which the norm is being used, by adopting the notation $\|\cdot\|_K$ instead. Also, we will simplify the notation by not always writing the limits in the index of a summation, when it is obvious what these limits are; for instance, we may write

$$\left(\sum_k x_k^2\right)^{1/2}$$

for the definition that follows.

[5] The proof of this is not very hard. In vector terms, we need to show that

$$x \cdot y \leqslant \|x\| \times \|y\|.$$

If $y = 0$, the result is obvious. Else, define

$$\delta = x - \frac{x \cdot y}{y \cdot y} y,$$

and note that $\delta \cdot y = 0$. Now,

$$\begin{aligned}\|x\|^2 &= x \cdot x \\ &= \left(\delta + \frac{x \cdot y}{y \cdot y} y\right) \cdot \left(\delta + \frac{x \cdot y}{y \cdot y} y\right) \\ &= \delta \cdot \delta + \left(\frac{x \cdot y}{y \cdot y}\right)^2 y \cdot y \\ &\geqslant \frac{(x \cdot y)^2}{y \cdot y} \\ &= \frac{(x \cdot y)^2}{\|y\|^2}.\end{aligned}$$

Then,

$$\begin{aligned}\|x+y\|^2 &= \sum_k x_k^2 + 2\sum_k x_k y_k + \sum_k y_k^2 \\ &\leqslant \sum_k x_k^2 + 2\big(\sum_k x_k^2\big)^{1/2}\big(\sum_k y_k^2\big)^{1/2} + \sum_k y_k^2 \\ &= (\|x\| + \|y\|)^2,\end{aligned}$$

which implies the result we want. □

As the other properties are straightforward, the important implication of this lemma is that function $(x, x') \mapsto \|x - x'\|$ is a metric for $\mathbb{R}^K$. This function is, indeed, known as the *Euclidean metric*.

2.4 Sequences

A FINITE SEQUENCE IN $\mathbb{R}^K$ IS a function $f : X \to \mathbb{R}^K$, where, for some $n^* \in \mathbb{N}$, we have $X = \{n \in \mathbb{N} \mid n \leqslant n^*\}$. An *infinite sequence* in $\mathbb{R}^K$ is a function $f : \mathbb{N} \to \mathbb{R}^K$.[6]

If no confusion is likely, the space in which a sequence lies is omitted. In the cases of finite sequences, it is usual to express them extensively as $(a_1, a_2, \ldots, a_{n^*})$, where $a_n = f(n)$, for $n \in X$. [7] Similarly, we can express infinite sequences as $(a_1, a_2, \ldots)$ or $(a_n)_{n=1}^{\infty}$, where $a_n = f(n)$, for $n \in \mathbb{N}$.

EXAMPLE 2.4.1. *Suppose that* $n^* = 5$, *so that* $X = \{1, 2, 3, 4, 5\}$, *and* $f(n) = n^2 - 1$, *for* $n \in X$*; then, we can express the finite sequence as* $(0, 3, 8, 15, 24)$ *or* $(n^2 - 1)_{n=1}^{5}$. *Or suppose that* $f(n) = (\sqrt{n}, 1/n, 3)$, *for all* $n \in \mathbb{N}$*; then, we can express the infinite sequence as* $((1, 1, 3), (\sqrt{2}, 1/2, 3), (\sqrt{3}, 1/3, 3), \ldots)$ *or* $(\sqrt{n}, 1/n, 3)_{n=1}^{\infty}$.

Following common usage in the literature, when referring to an infinite sequence, henceforth we will simply say «a sequence.»

Using the structure that a sequence has, $(a_n)_{n=1}^{\infty}$ is said to be *nondecreasing* if for all $n \in \mathbb{N}$, $a_{n+1} \geqslant a_n$, and *nonincreasing* if for all $n \in \mathbb{N}$, $a_{n+1} \leqslant a_n$. If all the inequalities in the first definition are strict, the sequence is *increasing*, while if all the inequalities in the second definition are strict, the sequence is *decreasing*.[8] Sequence $(a_n)_{n=1}^{\infty}$ is *bounded above* if there exists $\bar{a} \in \mathbb{R}$ such that $a_n \leqslant \bar{a}$ for all $n \in \mathbb{N}$. It is *bounded below* if there exists $\bar{a} \in \mathbb{R}$ such that $a_n \geqslant \bar{a}$ for all $n \in \mathbb{N}$, and is *bounded* if it is bounded above and below.

[6] It is important to note that a sequence has more structure than a set (i.e. it is more complicated). Remember that a set is completely defined by its elements, no matter how they are described. For example, the set $\{0, 3, 8, 15, 24\}$ is the same as the set $\{24, 15, 8, 3, 0\}$. However, the sequences $(0, 3, 8, 15, 24)$ and $(24, 15, 8, 3, 0)$ are clearly different: in a sequence, the order matters!

[7] We have already introduced finite sequences: an element of $\mathbb{R}^K$ is nothing but a sequence in $\mathbb{R}$, with $n^* = K$. In other words, shorthand for $(a_1, a_2, \ldots, a_{n^*})$ is simply $(a_n)_{n=1}^{n^*}$.

[8] Note that when we say $x \leqslant y$ with $x, y \in \mathbb{R}^K$, we are expressing K inequalities: that $x_k \leqslant y_k$ for every $k = 1, \ldots, K$. This implies that for every $x, y \in \mathbb{R}$, either $x \leqslant y$ or $x \geqslant y$, but the same is not true in higher-dimensional spaces. Hence, the previous concepts are more useful in $\mathbb{R}$ than in $\mathbb{R}^K$ for $K \geqslant 2$.

2.5 Cauchy Sequences and Subsequences

A TYPE OF SEQUENCE THAT IS very useful is the given by the following definition: a sequence $(a_n)_{n=1}^{\infty}$ is *Cauchy* if for all $\varepsilon > 0$ there exists some $n^* \in \mathbb{N}$ for which $\|a_{n_1} - a_{n_2}\| < \varepsilon$ for all $n_1, n_2 \geqslant n^*$.

Given a sequence $(a_n)_{n=1}^{\infty}$, sequence $(b_m)_{m=1}^{\infty}$ is *a subsequence of* $(a_n)_{n=1}^{\infty}$ if there exists an increasing sequence $(n_m)_{m=1}^{\infty}$ such that $n_m \in \mathbb{N}$ and $b_m = a_{n_m}$ for all $m \in \mathbb{N}$. That is, a subsequence is a selection of some (possibly all) members of the original sequence, preserving their original order.

EXAMPLE 2.5.1. *Consider the sequence* $(1/\sqrt{n})_{n=1}^{\infty}$*, and note that* $(1/\sqrt{2n+5})_{n=1}^{\infty}$ *is a subsequence of the former. To see why, consider the sequence* $(n_m)_{m=1}^{\infty} = (2m+5)_{m=1}^{\infty}$*. Note however that none of* $(1/n)_{n=1}^{\infty}$ *and* $(1/(2n-5))_{n=1}^{\infty}$ *is a subsequence of the other one. Note also that neither* $(n-2)_{n=1}^{\infty}$ *nor* $((n-2)^2)_{n=1}^{\infty}$ *are subsequences of each other.*

EXERCISE 2.5.1. *Is* $(1/\sqrt{n})_{n=1}^{\infty}$ *a subsequence of* $(1/n)_{n=1}^{\infty}$*? How about the other way around?*

2.6 Limits

VERY OFTEN ONE NEEDS to understand the behaviour of a sequence as it advances along its members, *ad infinitum*. Alternatively, one may need to understand the values that a function takes as one gets arbitrarily close, within the function's domain, to a given point.

2.6.1 Limits of sequences

Point $a \in \mathbb{R}^K$ is *the limit of sequence* $(a_n)_{n=1}^{\infty}$ if for all $\varepsilon > 0$ there exists some $n^* \in \mathbb{N}$ for which one has that $\|a_n - a\| < \varepsilon$ for all $n \geqslant n^*$.[9,10]

EXERCISE 2.6.1. *Does the sequence* $(1/\sqrt{n})_{n=1}^{\infty}$ *have a limit? Is it Cauchy? How about* $(3n/(n+\sqrt{n}))_{n=1}^{\infty}$*?*

Sequence $(a_n)_{n=1}^{\infty}$ is said to be *convergent* when it has a limit $a \in \mathbb{R}^K$, in which case one also says that the sequence *converges* to the limit point. When $(a_n)_{n=1}^{\infty}$ converges to a, the following notation is also sometimes used: $a_n \to a$, or $\lim_{n\to\infty} a_n = a$.

EXERCISE 2.6.2. *Does* $((-1)^n)_{n=1}^{\infty}$ *converge? Does* $(-1/n)_{n=1}^{\infty}$*?*

It is convenient to allow ∞ and $-\infty$ to be limits of sequences. Thus, we extend the definition as follows: for a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$, we say that $\lim_{n\to\infty} a_n = \infty$ when for all $\Delta > 0$ there exists some $n^* \in \mathbb{N}$ for which $a_n > \Delta$ for all $n \geqslant n^*$; we say that $\lim_{n\to\infty} a_n = -\infty$ when $\lim_{n\to\infty}(-a_n) = \infty$.

EXERCISE 2.6.3. *Does the sequence* $(3n/\sqrt{n})_{n=1}^{\infty}$ *have a limit? Does it converge?*

The importance of concepts introduced in the previous section is given by the following theorems.

THEOREM 2.6.1. *Sequence* $(a_n)_{n=1}^{\infty}$ *converges to* $a \in \mathbb{R}^K$ *if, and only if, every subsequence of* $(a_n)_{n=1}^{\infty}$ *converges to* a.

Proof. Sufficiency is immediate, for any sequence is a subsequence of itself.

For necessity, note first that if $(n_m)_{m=1}^{\infty}$ is an increasing sequence of natural numbers, then,[11]

$$n_m \geqslant m \text{ for all } m \in \mathbb{N}. \tag{$*$}$$

Now, let $(b_m)_{m=1}^{\infty}$ be a subsequence of $(a_n)_{n=1}^{\infty}$, fix the increasing sequence $(n_m)_{m=1}^{\infty}$ of natural numbers for which $b_m = a_{n_m}$, and assume that $a_n \to a \in \mathbb{R}^K$. Fix $\varepsilon > 0$. Since $a_n \to a$, there exists some $n^* \in \mathbb{N}$ for which $\|a_n - a\| < \varepsilon$ whenever $n \geqslant n^*$. By $(*)$, it is immediate that if $m \geqslant n^*$, then $\|b_m - a\| = \|a_{n_m} - a\| < \varepsilon$. □

[9] Note that we said «*the* limit», and not «*a* limit». This is correct, because if a sequence has a limit, this limit is unique. It is a very good exercise to prove this!

[10] Obviously, the sequence $(a_n)_{n=1}^{\infty}$ is defined in $\mathbb{R}^K$ as well. Otherwise, expressions like $\|a_n - a\|$ would not make any sense.

[11] The argument for this is by Axiom 1.3.1: (i) Obviously, $n_1 \geqslant 1$, since $n_1 \in \mathbb{N}$. (ii) Suppose that $n_m \geqslant m$ for a given m; Since $(n_m)_{m=1}^{\infty}$ is increasing, $n_{m+1} > n_m$, so $n_{m+1} \geqslant n_m + 1 \geqslant m + 1$.

THEOREM 2.6.2. *Sequence* $(a_n)_{n=1}^{\infty}$ *converges if, and only if, it is a Cauchy sequence.*

Theorem 2.6.2 is important, for it constitutes the key step in the construction of the real numbers. The «only if» part is an interesting exercise, and the student should be able to do it. The «if» argument, on the other hand, is very complicated and is based on the fact that the Reals are constructed as the «completion» of the holes left in the line by the Rationals.[12]

The next results study the connection between boundedness and convergence of sequences.

THEOREM 2.6.3. *If sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is convergent, then it is bounded.*

We do not prove this theorem, but the argument is one that the student should by now be confident in giving. The theorem shows that boundedness is a necessary condition for convergence. The following result states that in some cases it is also sufficient.

THEOREM 2.6.4. *If sequence* $(a_n)_{n=1}^{\infty}$ *is non-decreasing and bounded above, or non-increasing and bounded below, it is convergent.*

Proof. We only argue the case for non-decreasing sequences, for the other case is identical, and assume that $K = 1$ in order to make the argument easier to see. For this, it suffices to show that if a sequence $(a_n)_{n=1}^{\infty}$ is non-decreasing and does not converge, then it cannot be bounded above.

Since $(a_n)_{n=1}^{\infty}$ does not converge, it follows from Theorem 2.6.2 that it cannot be Cauchy. Then, there is some $\varepsilon > 0$ such that for all $n^* \in \mathbb{N}$ one can find $n', n'' \geqslant n^*$, $n'' > n'$, such that $|a_{n'} - a_{n''}| \geqslant \varepsilon$. Using the fact that $(a_n)_{n=1}^{\infty}$ is non-decreasing, we can write the latter inequality as $a_{n''} \geqslant a_{n'} + \varepsilon$, given that $n'' > n'$.

Now, we construct a subsequence of $(a_n)_{n=1}^{\infty}$ as follows. Using $n^* = 1$, it is immediate that there exist $n_1, n_2 \in \mathbb{N}$, $n_2 > n_1$, for which $a_{n_2} \geqslant a_{n_1} + \varepsilon$. Then, using $n^* = n_2 + 1$, we can find $n_4 > n_3 > n_2$ for which $a_{n_4} \geqslant a_{n_3} + \varepsilon$, and so on: for each even m, we find $n_{m+2} > n_{m+1} > n_m$ such that $a_{n_{m+2}} \geqslant a_{n_{m+1}} + \varepsilon$.

Since $(n_m)_{m=1}^{\infty}$ is increasing, it follows that $(b_m)_{m=1}^{\infty} = (a_{n_m})_{m=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. Since the latter is non-decreasing, we have, by construction, that

$$b_1 < b_1 + \varepsilon \leqslant b_2 \leqslant b_3 < b_3 + \varepsilon \leqslant b_4 \leqslant \ldots,$$

which implies that for all $m \geqslant 2$,

$$b_m \geqslant \begin{cases} b_1 + \frac{m}{2}\varepsilon, & \text{if } m \text{ is even;} \\ b_1 + \frac{m-1}{2}\varepsilon, & \text{otherwise.} \end{cases}$$

Since $\varepsilon > 0$, the latter implies that $b_m \to \infty$, which means that $(a_n)_{n=1}^{\infty}$ is unbounded above. □

This result shows that, for monotone sequences, boundedness suffices for convergence: if a sequence $(a_n)_{n=1}^{\infty}$ is monotone and bounded, then it is convergent. It is easy to see that, in the absence of monotonicity, boundedness does not suffice for convergence, but the following theorem, will prove to be important, proves that boundedness guarantees the existence of convergent subsequences, in all cases.

[12] The idea is as follows. Say that two Cauchy sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ in $\mathbb{R}$ *are Cauchy-equivalent* if for all $\varepsilon > 0$ there exists some $n^* \in \mathbb{N}$ for which $|a_n - b_n| < \varepsilon$ for all $n \geqslant n^*$. It is easy to see that this relation is reflexive, symmetric and transitive. Now, let $[(a_n)_{n=1}^{\infty}]$ denote the *equivalence class of* $(a_n)_{n=1}^{\infty}$, which is the set of all sequences that are Cauchy-equivalent to $(a_n)_{n=1}^{\infty}$.

Then, one can *construct* the Reals as the set of equivalence classes of Cauchy sequences. That is, one defines the Reals as the set of numbers x with the property that there exists a Cauchy sequence $(a_n)_{n=1}^{\infty}$ of rational numbers that is Cauchy-equivalent to the sequence $(x, x, x, \ldots)$.

The proof of Theorem 2.6.2 then proceeds in three steps: First, one proves that any Cauchy sequence of rational numbers converges to its equivalence class. Second, one proves that if a Cauchy sequence has a convergent subsequence, then the sequence itself converges to the same limit as the subsequence. Then, one argues Theorem 2.6.2, using the previous results and arguing that every Cauchy sequence has a convergent subsequence. For a detailed argument, see, for example, Corbae et al (2009).

THEOREM 2.6.5 (Bolzano-Weierstrass). *If sequence* $(a_n)_{n=1}^{\infty}$ *is bounded, then it has a convergent subsequence.*

Proof. A formal proof is deferred to Chapter 3.[13] □

EXERCISE 2.6.4. *Given a sequence* $(x_n)_{n=1}^{\infty}$ *defined in* $\mathbb{R}$*, number* $x^* \in \mathbb{R}$ *is said to be its* limit superior *if*

$$\forall \varepsilon > 0, \exists n^* \in \mathbb{N} : \forall n \geqslant n^*, x_n < x^* + \varepsilon$$

and

$$\forall \varepsilon > 0, \forall n \in \mathbb{N}, \exists n' \geqslant n : x_{n'} > x^* - \varepsilon.$$

Number $x_* \in \mathbb{R}$ *is the sequence's* limit inferior *if*

$$\forall \varepsilon > 0, \exists n_* \in \mathbb{N} : \forall n \geqslant n_*, x_n > x_* - \varepsilon$$

and

$$\forall \varepsilon > 0, \forall n \in \mathbb{N}, \exists n' \geqslant n : x_{n'} < x_* + \varepsilon.$$

When they exist, these numbers are denoted, respectively, as $\limsup_{n\to\infty} x_n = x^*$ *and* $\liminf_{n\to\infty} x_n = x_*$*.*

1. *Does the existence of the limit superior of a sequence guarantee that its limit inferior also exists?*
2. *Argue that if* $\limsup_{n\to\infty} x_n = x^*$*, then there exists a subsequence* $(x_{n_m})_{m=1}^{\infty}$ *of* $(x_n)_{n=1}^{\infty}$ *that converges to* x^**.*
3. *Argue that, when they both exist,* $\limsup_{n\to\infty} x_n \geqslant \liminf_{n\to\infty} x_n$*.*
4. *Give an example of a sequence for which the previous inequality is strong.*
5. *Argue that if* $\lim_{n\to\infty} x_n = \bar{x}$*, then* $\limsup_{n\to\infty} x_n = \bar{x}$*.*
6. *Argue that if*

$$\limsup_{n\to\infty} x_n = \bar{x} = \liminf_{n\to\infty} x_n,$$

then $\lim_{n\to\infty} x_n = \bar{x}$*.*

2.6.2 Limits of functions

Let $x \in \mathbb{R}^K$ and $\delta > 0$. The *open ball of radius* δ *around* x, denoted $B_\delta(x)$, is the set

$$B_\delta(x) = \{y \in \mathbb{R}^K \mid \|y - x\| < \delta\}.$$

The *punctured open ball of radius* δ *around* x is the set $B'_\delta(x) = B_\delta(x) \setminus \{x\}$. A point $\bar{x} \in \mathbb{R}^K$ is *a limit point of* $X \subseteq \mathbb{R}^K$ if for all $\varepsilon > 0$, $B'_\varepsilon(\bar{x}) \cap X \neq \varnothing$.

EXERCISE 2.6.5. *Prove that a point* $\bar{x}$ *is a limit point of* X *if, and only if, there exists a sequence* $(x_n)_{n=1}^{\infty}$ *defined in* $X \setminus \{\bar{x}\}$ *that converges to* $\bar{x}$*.*

Another type of limit has to do with functions, although not directly with sequences.

DEFINITION 2.6.1. *Consider a function* $f : X \to \mathbb{R}$*, where* $X \subseteq \mathbb{R}^K$*. Suppose that* $\bar{x} \in \mathbb{R}^K$ *is a limit point of* X *and that* $\bar{y} \in \mathbb{R}$*. We say that* $\lim_{x\to\bar{x}} f(x) = \bar{y}$

[13] An informal argument, for sequences in $\mathbb{R}$, is as follows: if $(a_n)_{n=1}^{\infty}$ is bounded, then it lies in some bounded interval I_1. Slice that interval in halves. At least one of the halves will contain infinitely many terms of the sequence. Call that interval I_2, slice it in halves, and let I_3 be a half that contains infinitely many elements ... By doing this indefinitely, we construct intervals $I_1, I_2, \ldots$ such that for all $n \in \mathbb{N}$, I_n contains infinitely many terms of the sequence and $I_{n+1} \subseteq I_n$. By construction, we can find a subsequence $(x_{n_m})_{m=1}^{\infty}$ such that for all $m \in \mathbb{N}$, $a_{n_m} \in I_m$. This subsequence is Cauchy, because, by construction, our sequence of intervals is shrinking to zero diameter as m goes to ∞; by Theorem 2.6.2, it must be convergent.

when for all $\varepsilon > 0$ *there exists* $\delta > 0$ *for which one has that* $|f(x) - \bar{y}| < \varepsilon$ *for all* $x \in B'_{\delta}(\bar{x}) \cap X$.

It is important to notice that we do not require $\bar{x} \in X$ in our previous definition, so that $f(\bar{x})$ need not be defined. Also, one should notice that even if $\bar{x} \in X$, $\bar{x}$ is not always a limit point of X, in which case the definition does not apply. Finally, notice that even if $\bar{x} \in X$ and $\bar{x}$ is a limit point of X, it need not be the case that $\lim_{x \to \bar{x}} f(x) = f(\bar{x})$.

DEFINITION 2.6.2. *Consider a function* $f : X \to \mathbb{R}$, *where* $X \subseteq \mathbb{R}^K$. *Suppose that* $\bar{x} \in \mathbb{R}^K$ *is a limit point of* X. *We say that* $\lim_{x \to \bar{x}} f(x) = \infty$ *when for all* $\Delta > 0$, *there exists* $\delta > 0$ *for which one has that* $f(x) \geqslant \Delta$ *for all* $x \in B'_{\delta}(\bar{x}) \cap X$. *We say that* $\lim_{x \to \bar{x}} f(x) = -\infty$ *when* $\lim_{x \to \bar{x}} (-f)(x) = \infty$.

EXERCISE 2.6.6. *Suppose that* $X = \mathbb{R}$ *and* $f(x) = x + a$, *for some* $a \in \mathbb{R}$. *What is* $\lim_{x \to 0} f(x)$*?*

EXERCISE 2.6.7. *Suppose that* $X = \mathbb{R}$ *and* $f : X \to \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} 1/x, & \textit{if } x \neq 0; \\ 0, & \textit{otherwise.} \end{cases}$$

What is $\lim_{x \to 5} f(x)$*? What is* $\lim_{x \to 0} f(x)$*?*

EXAMPLE 2.6.1. *Let* $X = \mathbb{R} \setminus \{0\}$ *and* $f : X \to \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} 1, & \textit{if } x > 0; \\ -1, & \textit{otherwise.} \end{cases}$$

In this case, we claim that $\lim_{x \to 0} f(x)$ *does not exist. To see why, fix* $0 < \varepsilon < 1$, *and notice that for all* $\delta > 0$, *there are* $x_1, x_2 \in B_{\delta}(0)$ *such that* $f(x_1) = 1$ *and* $f(x_2) = -1$, *and, hence,* $|f(x_1) - f(x_2)| = 2 > 2\varepsilon$. *Because of triangle inequality, it is thus impossible that for some* $\bar{y} \in \mathbb{R}$, *we have* $|f(x_1) - \bar{y}| < \varepsilon$ *and* $|f(x_2) - \bar{y}| < \varepsilon$. *Also, it is obvious that* $\lim_{x \to 0} f(x) = \infty$ *and* $\lim_{x \to 0} f(x) = -\infty$ *are both impossible.*

There exists a tight relationship between limits of functions and limits of sequences, which is explored in the following theorem.

THEOREM 2.6.6. *Consider a function* $f : X \to \mathbb{R}$, *where* $X \subseteq \mathbb{R}^K$. *Suppose that* $\bar{x} \in \mathbb{R}^K$ *is a limit point of* X *and that* $\bar{y} \in \mathbb{R}$. *Then,* $\lim_{x \to \bar{x}} f(x) = \bar{y}$ *if, and only if, for every sequence* $(x_n)_{n=1}^{\infty}$ *such that* $x_n \in X \setminus \{\bar{x}\}$, *for all* $n \in \mathbb{N}$, *and that* $\lim_{n \to \infty} x_n = \bar{x}$, *one has that* $\lim_{n \to \infty} f(x_n) = \bar{y}$.

Proof. We argue sufficiency by contradiction: suppose that for every sequence $(x_n)_{n=1}^{\infty}$ such that all $x_n \in X \setminus \{\bar{x}\}$ and that $\lim_{n \to \infty} x_n = \bar{x}$, we have that $\lim_{n \to \infty} f(x_n) = \bar{y}$, but, still, it is *not* true that $\lim_{x \to \bar{x}} f(x) = \bar{y}$. Then, there must exist some $\varepsilon > 0$ such that, for all $\delta > 0$,

$$\exists x \in B'_{\delta}(\bar{x}) \cap X : |f(x) - \bar{y}| \geqslant \varepsilon.$$

By assumption, for all $n \in \mathbb{N}$, there is $x_n \in B'_{1/n}(\bar{x}) \cap X$ for which $|f(x_n) - \bar{y}| \geqslant \varepsilon$. Construct the sequence $(x_n)_{n=1}^{\infty}$. By construction, $x_n^* \in X \setminus \{\bar{x}\}$ for all $n \in \mathbb{N}$, and, since $1/n \to 0$, we have that $\lim_{n \to \infty} x_n = \bar{x}$. However, by assumption, it is not true that $\lim_{n \to \infty} f(x_n) = \bar{y}$, which contradicts the initial hypothesis.

For necessity, consider any sequence $(x_n)_{n=1}^{\infty}$ such that all $x_n \in X \setminus \{\bar{x}\}$ and that $x_n \to \bar{x}$. Fix $\varepsilon > 0$. Since $\lim_{x \to \bar{x}} f(x) = \bar{y} \in \mathbb{R}$, then, there is some $\delta > 0$ such that for all $x \in B'_\delta(\bar{x}) \cap X$ it is true that $|f(x) - \bar{y}| < \varepsilon$. Since $\lim_{n\to\infty} x_n = \bar{x}$, there is $n^* \in \mathbb{N}$ for which $x_n \in B_\delta(\bar{x})$ for all $n \geqslant n^*$. Moreover, since each $x_n \in X \setminus \{\bar{x}\}$, we have that, when $n \geqslant n^*$, $x_n \in B'_\delta(\bar{x}) \cap X$ and, therefore, $|f(x_n) - \bar{y}| < \varepsilon$. Since $\varepsilon > 0$ was arbitrarily chosen, this implies that $\lim_{n\to\infty} f(x_n) = \bar{y}$. □

2.6.3 Properties of limits

THEOREM 2.6.7. *Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$. Let $\bar{x}$ be a limit point of X. Suppose that for number $\bar{y}_1, \bar{y}_2 \in \mathbb{R}$ one has that $\lim_{x\to\bar{x}} f(x) = \bar{y}_1$ and $\lim_{x\to\bar{x}} g(x) = \bar{y}_2$. Then,*[14]

1. $\lim_{x\to\bar{x}}(f+g)(x) = \bar{y}_1 + \bar{y}_2$;
2. $\lim_{x\to\bar{x}}(\alpha f)(x) = \alpha\bar{y}_1$, *for all* $\alpha \in \mathbb{R}$;
3. $\lim_{x\to\bar{x}}(f.g)(x) = \bar{y}_1.\bar{y}_2$; *and*
4. *if* $\bar{y}_2 \neq 0$, *then* $\lim_{x\to\bar{x}} \frac{f}{g}(x) = \bar{y}_1/\bar{y}_2$.

[14] The following notation is introduced. We define

$$(f+g) : X \to \mathbb{R}$$

by

$$(f+g)(x) = f(x) + g(x).$$

We define $(f.g)$ and (αf), for $\alpha \in \mathbb{R}$, accordingly. Now, define

$$X_g^* = \{x \in X \mid g(x) \neq 0\}$$

Then, we define

$$\frac{f}{g} : X_g^* \to \mathbb{R}$$

by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}.$$

Proof. Let us prove only the first two statements of the theorem. For the first statement, we have that for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|f(x) - \bar{y}_1| < \varepsilon/2$ for all $x \in B'_{\delta_1}(\bar{x}) \cap X$, and $|g(x) - \bar{y}_2| < \varepsilon/2$ for all $x \in B'_{\delta_2}(\bar{x}) \cap X$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then, by construction, for all $x \in B'_\delta(\bar{x}) \cap X$ we have that $|f(x) - \bar{y}_1| < \varepsilon/2$ and $|g(x) - \bar{y}_2| < \varepsilon/2$, which implies, by triangle inequality, that

$$|(f+g)(x) - (\bar{y}_1 + \bar{y}_2)| \leqslant |f(x) - \bar{y}_1| + |g(x) - \bar{y}_2| < \varepsilon.$$

For the second statement, note first that if $\alpha = 0$ the proof is trivial. Then, consider $\alpha \neq 0$. Since $\lim_{x\to\bar{x}} f(x) = \bar{y}_1 \in \mathbb{R}$, then for all $\varepsilon > 0$, there is some $\delta > 0$ such that, for all $x \in B'_\delta(\bar{x}) \cap X$, $|f(x) - \bar{y}_1| < \varepsilon/|\alpha|$. This implies that

$$|(\alpha f)(x) - \alpha\bar{y}_1| = |\alpha(f(x) - \bar{y}_1)| = |\alpha||f(x) - \bar{y}_1| < \varepsilon,$$

and, therefore, that $\lim_{x\to\bar{x}}(\alpha f)(x) = \alpha\bar{y}_1$. □

Given the relationship found in Theorem 2.6.6 , it comes as no surprise that a theorem analogous to the previous one holds for sequences.

THEOREM 2.6.8. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences in $\mathbb{R}$. Suppose that for numbers $a, b \in \mathbb{R}$, we have that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then,*

1. $\lim_{n\to\infty}(a_n + b_n) = a + b$;
2. $\lim_{n\to\infty}(\alpha a_n) = \alpha a$, *for all* $\alpha \in \mathbb{R}$;
3. $\lim_{n\to\infty}(a_n.b_n) = a.b$; *and*
4. *if* $b \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty}(a_n/b_n) = a/b$.

The proof of the first two parts is left as an exercise. The following theorem is also very useful:

THEOREM 2.6.9. *For sequences* $(a_n)_{n=1}^{\infty}$ *(in* $\mathbb{R}$*) such that* $a_n > 0$ *for all* $n \in \mathbb{N}$*, the following equivalence holds:*

$$\lim_{n\to\infty} a_n = \infty \Leftrightarrow \lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Proof. Let us prove the sufficiency statement, leaving necessity as an exercise. Suppose that $\lim_{n\to\infty}(1/a_n) = 0$ and fix $\Delta > 0$. Then, for some $n^* \in \mathbb{N}$ one has that $|1/a_n - 0| < 1/\Delta$ when $n \geqslant n^*$; since each $a_n > 0$, it follows that $a_n > \Delta$. □

EXERCISE 2.6.8. *Repeat the last part of Exercise 2.6.1, using the previous theorem. Is it easier? Show that*

$$\lim_{n\to\infty} \left(\frac{15n^5 + 73n^4 - 118n^2 - 98}{30n^5 + 19n^3} \right) = \frac{1}{2}.$$

A very useful property of limits (for both sequences and functions) is that they preserve weak inequalities. This is the content of the following theorem, whose proof is left as an exercise.

THEOREM 2.6.10. *Consider a sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$ *and a number* $\alpha \in \mathbb{R}$*. If* $a_n \leqslant \alpha$*, for all* $n \in \mathbb{N}$*, and* $\lim_{n\to\infty} a_n = a$*, then* $a \leqslant \alpha$*. Similarly, if* $a_n \geqslant \alpha$*, for all* $n \in \mathbb{N}$*, and* $\lim_{n\to\infty} a_n = a$*, then* $a \geqslant \alpha$*.*

EXERCISE 2.6.9. *Can we strengthen our results to say:* "Consider a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ and a number $\alpha \in \mathbb{R}$. If $a_n < \alpha$, for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = a$, then $a < \alpha$."*?*

The next result is the counterpart for limits of functions; again, the proof is left as an exercise.

THEOREM 2.6.11. *Consider* $f : X \to \mathbb{R}$ *and* $\bar{y} \in \mathbb{R}$*, and let* $\bar{x} \in \mathbb{R}^K$ *be a limit point of* X*. If* $f(x) \leqslant \gamma$ *for all* $x \in X$*, and* $\lim_{x\to\bar{x}} f(x) = \bar{y}$*, then* $\bar{y} \leqslant \gamma$*. Similarly, if* $f(x) \geqslant \gamma$ *for all* $x \in X$*, and* $\lim_{x\to\bar{x}} f(x) = \bar{y}$*, then* $\bar{y} \geqslant \gamma$*.*

EXERCISE 2.6.10. *The previous theorem can be proved by two different arguments. Can you give them both? (Hint: one argument is by contradiction; the other one uses Theorem 2.6.10 directly.)*

COROLLARY 2.6.1. *Consider* $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$*, let* $\bar{y}_1, \bar{y}_2 \in \mathbb{R}$*, and let* $\bar{x} \in \mathbb{R}^K$ *be a limit point of* X*. If* $f(x) \geqslant g(x)$*, for all* $x \in X$*,* $\lim_{x\to\bar{x}} f(x) = \bar{y}_1$ *and* $\lim_{x\to\bar{x}} g(x) = \bar{y}_2$*, then* $\bar{y}_1 \geqslant \bar{y}_2$*.*

Obviously, a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^K$ is nothing but an array of K sequences in $\mathbb{R}$: sequence $(a_{k,n})_{n=1}^{\infty}$ for each $k = 1, \ldots, K$. So, it should not come as no surprise that some relations exist between these objects.

THEOREM 2.6.12. *Sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}^K$ *is bounded if, and only if, for each* $k = 1, \ldots, K$*, sequence* $(a_{k,n})_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is bounded.*

THEOREM 2.6.13. *Sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}^K$ *converges to* a *if, and only if, for each* $k = 1, \ldots, K$*, sequence* $(a_{k,n})_{n=1}^{\infty}$ *in* $\mathbb{R}$ *converges to* a_k*.*

Proof. Let us prove sufficiency first. Given any $\epsilon > 0$, for each k there is some $n_k^* \in \mathbb{N}$ such that $|a_{k,n} - a_k| < \epsilon/\sqrt{n}$ whenever $n \geqslant n_k^*$. Letting $n^* = \max\{n_1^*, \ldots, n_K^*\} \in \mathbb{N}$

and $n \geqslant n^*$, by construction,

$$\|a_n - a\| = \left(\sum_k (a_{k,n} - a_k)^2 \right)^{1/2} < \left(\sum_k \frac{\epsilon^2}{K} \right)^{1/2} = \epsilon.$$

For necessity, fix k and let $\epsilon > 0$. By assumption, there is $n^* \in \mathbb{N}$ after which $\|a_n - a\| < \epsilon$, which suffices to imply that $|a_{k,n} - a_k| < \epsilon$. □

2.7 Euler's Number and the Natural Logarithm

One of the must important numbers in mathematics is *Euler's Number*, which is

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

It can be shown (later you will) that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!},$$

and, numerically, $e = 2.71828182\ldots$. For $x \in \mathbb{R}_{++}$, we define the *natural logarithm* of x, denoted $\ln(x)$, as the number y such that $e^y = x$.

2.8 Application: The Value of a Perpetuity

A perpetuity is an asset that promises to pay a constant return, which we denote by R, periodically, *ad infinitum*. Suppose that the interest rate is $r > 0$ at all future periods. Then, the *present value* of the perpetuity is

$$V = R + \frac{R}{1+r} + \frac{R}{(1+r)^2} + \frac{R}{(1+r)^3} + \ldots,$$

where the sum in the right-hand side contains the term $R/(1+r)^T$, for all $T \in \mathbb{N}$.[15] Factoring out the constant R, we write

[15] Indeed, it contains infinitely many non-trivial summands!

$$V = R \sum_{t=0}^{\infty} \frac{1}{(1+r)^t},$$

where the *infinite sum*

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^t}$$

is defined as the limit of the sequence

$$\left(\sum_{t=0}^{T} \frac{1}{(1+r)^t} \right)_{T=1}^{\infty},$$

provided that this limit exists.

Denoting, for simplicity, $\delta = 1/(1+r)$, we can simply write $V = R \sum_{t=0}^{\infty} \delta^t$. Also, we can write $\nu_T = \sum_{t=0}^{T} \delta^t$, so that $V = R \times \lim_{T\to\infty} \nu_T$, if this limit exists.

EXERCISE 2.8.1. *Argue that, if* $|\delta| < 1$, $\lim_{T\to\infty} \delta^T = 0$.

Note that

$$\nu_T(1-\delta) = \sum_{t=0}^{T} \delta^t - \delta \sum_{t=0}^{T} \delta^t = \sum_{t=0}^{T} \delta^t - \sum_{t=1}^{T+1} \delta^t = 1 - \delta^{T+1},$$

so, if $\delta \neq 1$,[16] we have that

$$\nu_T = \frac{1}{1-\delta} - \frac{\delta^{T+1}}{1-\delta}.$$

And it now follows from Exercise 2.8.1 that if we further assume that $|\delta| < 1$,[17] then $\nu_T \to 1/(1-\delta)$ and $V = R/(1-\delta) = R(1+r)/r$.

[16] Namely if $r \neq 0$.

[17] Which is the case, since $r > 0$.

2.9 Application: Choice under Uncertainty

CONSIDER THE CASE IN WHICH the consequences of a decision-maker's choices are not fully determined by her, and are subject to uncertainty. For this purpose, it is useful to endow the choice problem with a probabilistic framework. A *state of the world* is a comprehensive description of the state of all the contingencies that can affect a decision-maker.[18] Let $\mathcal{S}$ be the set of states of the world. A *random variable* is a function whose domain is $\mathcal{S}$. If the codomain of a random variable is the set Y, we say that it is a random variable over Y.

Let $\mathcal{X} \neq \varnothing$ be a set of *outcomes* (or *prospects*),[19] and let Δ be the space of all *probability distributions* over $\mathcal{X}$. For example, if $\mathcal{X}$ is finite, we can write $\mathcal{X} = \{1, \dots, X\}$, and, then, $\Delta = \{p \in \mathbb{R}^X_+ \mid \sum_x p_x = 1\}$.

In general, a *lottery* is defined as a random variable over Δ: it is a function $L : \mathcal{S} \to \Delta$. That is, a lottery is a device that assigns to each state of the world, s, a probability distribution over the set of prospects, $p^s = L(s)$; under this device and given that state, the probability of prospect x is $p^s_x = L(s)(x)$, and $\int_{\mathcal{X}} L(s)(x)\, dx = 1$.

A lottery fixes the probability of a prospect, given a state, but it does not determine the probability of that state. It is commonly interpreted that the probabilities of states are exogenous to economic models, and are usually taken to be subjective to the decision-maker, whereas the probabilities induced by lotteries, given a state, are objective. A full theory that handles both types of probability is possible, but here, for simplicity, we will deal with only one of the two types of uncertainty at a time.[20]

[18] In the words of K. Arrow (1971, *Essays on the Theory of Risk Bearing*, p. 45.), it is «a description of the world so complete that, if true and known, the consequences of every action would be known.»

[19] This could be an abstract set, or, if you would like more definiteness, a set $\mathcal{X} \subseteq \mathbb{R}$, of monetary values.

[20] In fact, a richer theory where the decision maker is unsure of what subjective probabilities to assign to states of the world is possible. Often, people reserve the term «uncertainty» for the latter phenomenon, and use «risk» for the randomness that remains even when probabilities (subjective and objective) are fixed. Here, we won't need this distinction.

2.9.1 Preferences over lotteries

For simplicity, let us assume that there is only one state of the world, so that we can ignore the set $\mathcal{S}$ and can refer to Δ itself as the space of lotteries: in the language of the general setting introduced above, we will now study a problem where the choice space $\mathcal{D}$ is the set of lotteries Δ. Henceforth, we assume that $\succsim$ is rational, and define $\succ$ and $\sim$ as before.

Conceptually, while it may seem to follow naturally that the individual's preferences, $\succsim$, are a binary relation over Δ, it should be noted that when we define preferences in this way, we are imposing the condition that the individual cares about the risk (randomness) she faces, and not about the process that ultimately determines that risk; this condition is known as «consequentialism.»

2.9.2 Further properties of preferences

We say that $\succsim$ satisfies *monotonicity* if given two lotteries, p and p' such that $p \succ p'$, the following statement is true: $\alpha p + (1-\alpha)p' \succ \beta p + (1-\beta)p'$ if, and only if, $\alpha > \beta$. In words, a decision-maker with monotonic preferences prefers more of a better lottery to more of a worse lottery.

Another condition imposes that the decision-maker values the outcomes of the lotteries for themselves and then, independently, the randomness induced over them by the lottery: we say that $\succsim$ satisfies *independence* if given two lotteries p and p', the following statements are true:

1. if $p \succsim p'$, then for any number $0 \leqslant \alpha \leqslant 1$ and any lottery p'' we have that
$$\alpha p + (1-\alpha)p'' \succsim \alpha p' + (1-\alpha)p'';$$
2. if for some number $0 < \alpha < 1$ and some lottery p'' we have that
$$\alpha p + (1-\alpha)p'' \succsim \alpha p' + (1-\alpha)p'',$$
then $p \succsim p'$.

The latter property is controversial, and we will come back to it later. The following exercises relate the two properties.

EXERCISE 2.9.1. *Argue that independence of $\succsim$ implies the following property: for any pair of lotteries p and p':*

1. *if $p \sim p'$, then for any $0 \leqslant \alpha \leqslant 1$ and any lottery p'',*
$$\alpha p + (1-\alpha)p'' \sim \alpha p' + (1-\alpha)p'';$$
2. *if for some $0 < \alpha \leqslant 1$ and some lottery p'' we have that*
$$\alpha p + (1-\alpha)p'' \sim \alpha p' + (1-\alpha)p'',$$
then $p \sim p'$.
3. *if $p \succ p'$, then for any $0 < \alpha < 1$ and any lottery p'',*
$$\alpha p + (1-\alpha)p'' \succ \alpha p' + (1-\alpha)p'';$$
4. *if for some $0 < \alpha < 1$ and some lottery p'' we have that*
$$\alpha p + (1-\alpha)p'' \succ \alpha p' + (1-\alpha)p'',$$
then $p \succ p'$; and
5. *if $p \succ p'$ and $0 < \alpha < 1$, then $p \succ \alpha p + (1-\alpha)p'$ and $\alpha p + (1-\alpha)p' \succ p'$.*

EXERCISE 2.9.2. *Argue that independence of $\succsim$ implies its monotonicity.*[21]

[21] This exercise is a tiny bit more complicated than the others. Hint 1: suppose that you want to write
$$\beta p + (1-\beta)p'$$
as
$$\gamma p + (1-\gamma)(\alpha p + (1-\alpha)p'),$$
given that $\beta > \alpha$; what value must γ have? Hint 2: now, notice the last property of Exercise 2.9.1.

2.9.3 Expected-utility representability

We again ask the question of when $\succsim$ can be represented by a utility function. But in the current setting of uncertainly, we may want to have special properties on the utility function that represents the person's preferences: we say that $\succsim$ has an *expected-utility representation* if there exists a function $u : \mathcal{X} \to \mathbb{R}$ such that for any pair of lotteries p and p', we have that $p \succsim p'$ if, and only if, $E_p(u) \geqslant E_{p'}(u)$. In this case, we can define the utility function over lotteries $U(p) = E_p(u)$, and it is immediate that U represents $\succsim$.[22]

[22] A couple of words on jargon are in order, for sometimes different things are given the same name in economics: some people refer to u as «Bernoulli utility function» and to U as «von Neumann-Morgenstern utility function,» while some other people refer to u itself as the «von Neumann-Morgenstern utility function,» and some other people use both names for u and leave U nameless. This can be problematic, as the two functions are not the same thing: u measures utility over outcomes, while U does it over lotteries. Here, we will refer to U as the utility function and to u as the utility index.

EXERCISE 2.9.3. *Argue that if $\succsim$ has an expected-utility representation then it satisfies independence.*

EXERCISE 2.9.4. *Consider a decision-maker who faces uncertainty over a finite set of possible outcomes, $\mathcal{X} = \{1, 2, \ldots, X\}$*

1. *Suppose that there are only three possible outcomes and the individual's preferences over lotteries are that $p \succsim p'$, if, and only if,*

$$p_1 > p_1' \text{ or } (p_1 = p_1' \text{ and } p_2 \geqslant p_2').$$

Argue that $p \sim p'$ if, and only if, $p = p'$.

2. *Suppose, alternatively, that the individual's preferences are represented by the following function:*

$$U(p) = \begin{cases} 1, & \text{if } p_x = 1/X \text{ for all } x; \\ 0, & \text{otherwise.} \end{cases}$$

Argue that the individual's preferences do not *satisfy the following property: for all p and p' and all $\alpha \in [0, 1]$, if $p \sim p'$, then $\alpha p + (1 - \alpha)p' \sim p'$.*

3. *Suppose now that the individual has the following preferences: for all p and p', it is true that $p \succsim p'$. Find a Bernoulli index for the expected-utility representation of these preferences.*

4. *Argue that, in any case, if the individual's preferences have an expected-utility representation with Bernoulli index $u(x)$, then the index $\tilde{u}(x) = au(x) + b$, for any numbers $a > 0$ and b, also represents $\succsim$: it is true that $p \succsim p'$ if, and only if,*

$$\sum_{x=1}^{X} p_x \tilde{u}(x) \geqslant \sum_{x=1}^{X} p_x' \tilde{u}(x).$$

A seminal result is decision theory is the following:

THEOREM (The von Neumann–Morgenstern Theorem). *Suppose that $\succsim$ satisfies the following* continuity *assumption: for any $x, x', x'' \in \mathcal{X}$ such that $x \succsim x' \succsim x''$, we can find a number $0 \leqslant p \leqslant 1$ such that $(p, x, x'') \sim x'$. If $\succsim$ satisfies independence, then it has an expected-utility representation (with continuous index u).*

Here, we give an informal argument for why the von Neumann-Morgenstern theorem is true. For simplicity, we concentrate only on a small subclass of lotteries, rather than on the whole space Δ.

We say that a lottery is *simple* if it gives positive probability to at most two outcomes in $\mathcal{X}$.[23] For simplicity, then, we can denote a simple lottery as a triple consisting of a number and two outcomes, $L = (p, x, x')$, with $0 \leqslant p \leqslant 1$ and $x, x' \in X$, and with

[23] The term «simple» is normally used for lotteries that pay in outcomes and not in other lotteries; here, I am using it is that sense, but making it stronger to require that they pay in only one or two outcomes.

the interpretation that the lottery gives outcome x with probability p, and outcome x' with probability $1-p$. Let $\mathcal{L}_1$ be the space of simple lotteries, $\mathcal{L}_1 = [0,1] \times \mathcal{X} \times \mathcal{X}$.

A *compound lottery* is a device that gives other lottery or lotteries as prizes. We will concentrate on compound lotteries that give positive probability to at most two simple lotteries,[24] and denote them by (p, L, L'), a number and two simple lotteries, $L, L' \in \mathcal{L}_1$. Let $\mathcal{L}_2$ be the space of compound lotteries, $\mathcal{L}_2 = [0,1] \times \mathcal{L}_1 \times \mathcal{L}_1$.

[24] As before, the term «compound» is normally used for lotteries that pay in other lotteries; here, I am using it is that sense, but making it stronger to require that they pay in only one or two lotteries.

For our argument, we consider only degenerate, simple and our simplified definition of compound lotteries, so we take $\succsim$ as defined over $\mathcal{L} = \mathcal{X} \cup \mathcal{L}_1 \cup \mathcal{L}_2$. In order to keep consistency with the analysis above, we need consider an individual who cares about outcomes, and not about how these outcomes are presented, so we impose the following «consequentialist» assumptions on $\succsim$: for all $p, p' \in [0,1]$ and for all $x, x' \in X$,

1. $(p, x, x') \sim (1-p, x', x)$;

2. $(1, x, x') \sim x$;

3. $(p, (p', x, x'), x') \sim (pp', x, x')$.

For simplicity, suppose also that we can find $x_*, x^* \in X$ such that for every outcome $x \in X$ we have that $x \succsim x_*$ and $x^* \succsim x$.

Exercise 2.9.5. *Argue that:*

1. $x \succ x'$ *and* $0 \leqslant p' < p \leqslant 1$ *imply that* $(p, x, x') \succ (p', x, x')$*;*

2. $L \succ L'$ *and* $0 \leqslant p' < p \leqslant 1$ *imply that* $(p, L, L') \succ (p', L, L')$*;*

3. *if* $x \succsim x'$*, then for any* x'' *and any* $0 \leqslant p \leqslant 1$ *it is true that* $(p, x, x'') \succsim (p, x', x'')$*;*

4. *if for some* x'' *and some* $0 \leqslant p \leqslant 1$ *it is true that* $(p, x, x'') \succsim (p, x', x'')$*, then* $x \succsim x'$*;*

5. *if* $L \succsim L'$*, then for any* L'' *and any* $0 \leqslant p \leqslant 1$ *it is true that* $(p, L, L'') \succsim (p, L', L'')$*;*

6. *if for some* L'' *and some* $0 \leqslant p \leqslant 1$ *it is true that* $(p, L, L'') \succsim (p, L', L'')$*, then* $L \succsim L'$*.*

Since $\succsim$ satisfies continuity and monotonicity, it is relatively easy to construct a utility function representing it over the space of simple lotteries: by continuity, for any lottery in $\mathcal{L}$, we can find $p \in [0,1]$ such that $L \sim (p, x^*, x_*)$; by monotonicity, such $p \in [0,1]$ has to be unique; then, just let $U : \mathcal{L} \to \mathbb{R}$ be defined by letting $U(L)$ be the unique number $p \in [0,1]$ such that $L \sim (p, x^*, x_*)$.

Since $\mathcal{L}$ includes degenerate lotteries, we can define $u : \mathcal{X} \to \mathbb{R}$ by letting $u(x) = U((1, x, x))$. Now, we just want to show that the expected utility property is satisfied in the following sense: for every simple lottery (p, x, x'),

$$U((p, x, x')) = pu(x) + (1-p)u(x').$$

Notice that, by construction,

$$(p, x, x') \sim (U((p, x, x')), x^*, x_*),$$

whereas, by independence,

$$(p, x, x') \sim (p, (u(x), x^*, x_*), (u(x'), x^*, x_*)).$$

By direct computation, it follows that

$$(p, x, x') \sim (pu(x) + (1-p)u(x'), x^*, x_*),$$

which implies, by monotonicity, that $U((p, x, x')) = pu(x) + (1-p)u(x')$, as we wanted.

EXERCISE 2.9.6. *Argue the following: If* U *is an expected-utility representation of* $\succsim$*, then it satisfies the following «linearity» property: for any pair of lotteries* p *and* p' *and any number* $0 \leqslant \alpha \leqslant 1$*,*

$$U(\alpha p + (1-\alpha)p') = \alpha U(p) + (1-\alpha)U(p').$$

THEOREM 2.9.1. *Suppose that* U *and* $\tilde{U}$ *are expected-utility representations of* $\succsim$*. Let* u *and* $\tilde{u}$ *be their respective utility indices. There exist numbers* α *and* $\beta > 0$ *such that* $\tilde{u}(x) = \alpha + \beta u(x)$ *for every* x*.*

The previous proposition is important: only positive *affine* transformations of a utility index preserve the expected-utility representation of $\succsim$;[25] this means that u itself is a cardinal object.

[25] Don't get confused: any monotone transformation of U will represent $\succsim$ as well; but the transformation need not preserve the expected-utility property, which is what requires affinity.

3
Topology of $\mathbb{R}^K$

FROM NOW ON, WE DEAL only with subsets of $\mathbb{R}^K$, for a finite number K; that is, whenever we introduce sets X or Y, we assume that $X, Y \subseteq \mathbb{R}^K$ and use all the algebraic structure of $\mathbb{R}^K$. We also use the structure induced in $\mathbb{R}^K$ by the Euclidean norm. Whenever we take complements, they are relative to $\mathbb{R}^K$.

3.1 Open and Closed Sets

THE TWO KEY CONCEPTS IN topology are those of open and closed sets. Intuitively, a set is open if, at any point in the set, one is allowed to «move freely». Intuitively, a set is closed if one has to «jump» in order to get out of it.

3.1.1 Open sets

DEFINITION 3.1.1. *Set X is* open *if for all $x \in X$, there is some $\varepsilon > 0$ for which $B_\varepsilon(x) \subseteq X$.*

EXAMPLE 3.1.1 (Open intervals are open sets in $\mathbb{R}$). *We define an open interval, denoted (a, b),[1] where $a, b \in \mathbb{R}$, as $\{x \in \mathbb{R} \mid a < x < b\}$. To see that these are open sets (in $\mathbb{R}$), take $x \in (a, b)$, and define $\varepsilon = \min\{x - a, b - x\}/2 > 0$. By construction, $B_\varepsilon(x) \subseteq X$. As a consequence, notice that open balls are open sets in $\mathbb{R}$. The same is true in $\mathbb{R}^K$, for any K.*

[1] Sometimes open intervals are denoted by $]a, b[$ rather than (a, b) in order to distinguish them from ordered pairs in $\mathbb{R}^2$. We will, however, follow the more standard notation.

It is easy to see that if we extend the definition of the open interval (a, b) to the case where $a, b \in \mathbb{R} \cup \{\infty, -\infty\}$, then it continuous to be true that open intervals are open sets. The following theorem is a specific instance of a more general principle: in any space, the empty set and the universe are open sets.

THEOREM 3.1.1. *The empty set and $\mathbb{R}^K$ are open.*

Proof. A set X fails to be open if one can find $x \in X$ such that for all $\varepsilon > 0$ one has that $B_\varepsilon(x) \cap X^c \neq \varnothing$. Clearly, $\varnothing$ cannot exhibit such property. The argument that $\mathbb{R}^K$ is open is left as an exercise. □

THEOREM 3.1.2. *The union of any collection of open sets is an open set. The intersection of any finite collection of open sets is an open set.*

Proof. For the first statement, suppose that Z is the union of a given collection of open sets,[2] and suppose that $x \in Z$. By definition, then, there exists a member X of the collection of sets such that $x \in X$. By assumption, X is open, so that for some $\varepsilon > 0$ one has that $B_\varepsilon(x) \subseteq X$, and it follows, then, that $B_\varepsilon(x) \subseteq Z$.

For the second part, suppose that Z is the intersection of a finite collection of open sets, say $\{X_1, X_2, \ldots, X_{n^*}\}$, and suppose that $x \in Z$. By definition, then, for each $n = 1, 2, \ldots, n^*$, it is true that $x \in X_n$. By assumption, each X_n is open, so that there exists $\varepsilon_n > 0$ such that $B_{\varepsilon_n}(x) \subseteq X_n$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n^*}\} > 0$. By construction, for each n, we have that $B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq X_n$ and therefore $B_\varepsilon(x) \subseteq Z$. □

[2] Whether finite or infinite.

We say that point x is *an interior point of the set* X, if there is some $\varepsilon > 0$ for which $B_\varepsilon(x) \subseteq X$. The set of all the interior points of X is called *the interior of* X, and is usually denoted $\mathrm{int}(X)$.[3] Note that $\mathrm{int}(X) \subseteq X$.

[3] Alternative, but also usual notation is X^o.

EXERCISE 3.1.1. *Show that for every* X, $\mathrm{int}(X)$ *is open and that* X *is open if, and only if,* $\mathrm{int}(X) = X$.

EXERCISE 3.1.2. *Prove that if* $x \in \mathrm{int}(X)$*, then* x *is a limit point of* X.

EXERCISE 3.1.3. *Did we really need finiteness in the second part of Theorem 3.1.2? Consider the following infinite collection of open intervals: for all* $n \in \mathbb{N}$*, define* $I_n = (-1/n, 1/n)$*. Find the intersection of all those intervals, denoted* $\cap_{n=1}^\infty I_n$*. Is it an open set?*

3.1.2 Closed sets

DEFINITION 3.1.2. *Set* X *is* closed *if for every sequence* $(x_n)_{n=1}^\infty$ *in* $\mathbb{R}^K$ *that satisfies that* $x_n \in X$*, at all* $n \in \mathbb{N}$*, and for which there is some* $\bar{x} \in \mathbb{R}^K$ *to which it converges, we have that* $\bar{x} \in X$.

Given a set $X \subseteq \mathbb{R}^K$, we define its *closure*, denoted by $\mathrm{cl}(X)$, as the set[4]

$$\mathrm{cl}(X) = \{x \in \mathbb{R}^K \mid \forall \varepsilon > 0, B_\varepsilon(x) \cap X \neq \varnothing\}.$$

[4] Alternative notation is $\bar{X}$.

As before, the empty set and the universe are closed sets.

THEOREM 3.1.3. *The empty set and* $\mathbb{R}^K$ *are closed.*

Proof. In order for set X to fail to be closed, there has to exist $(x_n)_{n=1}^\infty$ satisfying that all $x_n \in X$, and that $x_n \to \bar{x}$, yet $\bar{x} \notin X$. Clearly, one cannot find such sequence if $X = \varnothing$. The argument for $\mathbb{R}^K$ is left as an exercise. □

Recall that $\varnothing$ and $\mathbb{R}^K$ are also open. In $\mathbb{R}^K$ these are the only two sets that have both properties;[5] we will prove this result later, but state it here as a theorem for $K = 1$:

[5] But this principle does not generalize to other spaces.

THEOREM 3.1.4 (Conectedness of $\mathbb{R}$). *Let* $A, B \subseteq \mathbb{R}$ *be open and disjoint. If* $A \cup B = \mathbb{R}$*, then either* $A = \varnothing$ *or* $B = \varnothing$.

EXERCISE 3.1.4. *If* $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$*,* $a < b$*, is* (a, b) *closed? We define the* half-closed *interval* $(a, b]$*, where* $a \in \mathbb{R} \cup \{-\infty\}$*,* $b \in \mathbb{R}$*,* $a < b$*, as* $(a, b] = \{x \in \mathbb{R} \mid a < x \leqslant b\}$*. Similarly, we define the half-closed interval* $[a, b)$ *where* $a \in \mathbb{R}$*,* $b \in \mathbb{R} \cup \{\infty\}$*,* $a < b$*, as* $[a, b) = \{x \in \mathbb{R} \mid a \leqslant x < b\}$*. Are half-closed intervals closed sets? Are they open? If* $x \in \mathbb{R}^K$*, is* $\{x\}$ *an open set, a closed set or neither?*

THEOREM 3.1.5. *A set X is closed if, and only if, X^c is open.*

Proof. Suppose that X^c is open, and consider any sequence $(x_n)_{n=1}^{\infty}$ satisfying that all $x_n \in X$ and converging to some $\bar{x}$; we need to show that $\bar{x} \in X$. In order to argue by contradiction, suppose that $\bar{x} \in X^c$. Since X^c is open, there is some $\varepsilon > 0$ for which $B_\varepsilon(\bar{x}) \subseteq X^c$. Since $x_n \to \bar{x}$, there is $n^* \in \mathbb{N}$ such that $\|x_n - \bar{x}\| < \varepsilon$ when $n \geqslant n^*$. Then, for any $n \geqslant n^*$, we have that $x_n \in B_\varepsilon(\bar{x}) \subseteq X^c$, which is impossible.

Suppose now that X is closed, and fix $x \in X^c$. We need to show that for some $\varepsilon > 0$ one has that $B_\varepsilon(x) \subseteq X^c$. Again, suppose not: for all $\varepsilon > 0$, it is true that $B_\varepsilon(x) \cap X \neq \varnothing$. Clearly, then, for all $n \in \mathbb{N}$ we can pick $x_n \in B_{1/n}(x) \cap X$. Construct a sequence $(x_n)_{n=1}^{\infty}$ of such elements. Since $1/n \to 0$ it follows that $x_n \to x$, and all $x_n \in X$ and X is closed, then $x \in X$, contradicting the fact that $x \in X^c$. □

THEOREM 3.1.6. *The intersection of* any *collection of closed sets is closed. The union of any finite collection of closed sets is closed.*

Proof. This argument is left as an exercise. (Hopefully, you can use the generalized version of DeMorgan's laws that you proved in Exercise 1.2.1.) □

EXERCISE 3.1.5. *Prove that, given a set $X \subseteq \mathbb{R}^K$, $x \in \mathrm{cl}(X)$ if, and only if, there exists a sequence $(x_n)_{n=1}^{\infty}$ in X such that $x_n \to x$.*

EXERCISE 3.1.6. *Prove that for every set $X \subseteq \mathbb{R}^K$, $X \subseteq \mathrm{cl}(X)$, and X is closed if, and only if, $X = \mathrm{cl}(X)$.*

EXAMPLE 3.1.2. *Closed intervals are closed sets. We define an closed interval, denoted $[a, b]$, where $a, b \in \mathbb{R}$ and $a \leqslant b$ as $\{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$. To see that these are closed sets, notice that $[a, b]^c = (-\infty, a) \cup (b, \infty)$, and conclude based on previous results.*

EXERCISE 3.1.7. *Did we really need finiteness in the second part of Theorem 3.1.6? Consider the following infinite collection of closed intervals: for all $n \in \mathbb{N}$, define $J_n = [1 + 1/n, 3 - 1/n]$. Find the union of all those intervals, denoted $\cup_{n=1}^{\infty} J_n$. Is it a closed set?*

EXERCISE 3.1.8. *A point $x \in \mathbb{R}^K$ is said to be* in the boundary *of set $X \subseteq \mathbb{R}^K$, if for all $\varepsilon > 0$, $B_\varepsilon(x) \cap X \neq \varnothing$ and $B_\varepsilon(x) \cap X^c \neq \varnothing$. Let $\mathrm{bd}(X)$ be the set of all points in the boundary of X.*[6] *Argue that $\mathrm{bd}(X) = \mathrm{cl}(X) \setminus \mathrm{int}(X)$.*

[6] Alternative notation is X^∂.

3.2 Compact Sets

A SET $X \subseteq \mathbb{R}^K$ IS SAID to be *bounded above* if there exists $\alpha \in \mathbb{R}^K$ such that $x \leqslant \alpha$ for all $x \in X$; it is said to be *bounded below* if for some $\beta \in \mathbb{R}^K$ one has that $x \geqslant \beta$ is true for all $x \in X$; and it is said to be *bounded* if it is bounded above and below.

EXERCISE 3.2.1. *Show that a set X is bounded if and only if there is some $\alpha \in \mathbb{R}_+$ for which one has that $\|x\| \leqslant \alpha$ for all $x \in X$.*

DEFINITION. *A set $X \subseteq \mathbb{R}^K$ is said to be* compact *if it is closed and bounded.*

The previous definition is fine for subsets of $\mathbb{R}^K$ but this is not true for more general spaces. The general concept (in topology) is that a set is compact if, whenever you can "cover" it with a collection of open sets, then you can also do it with finitely many of those sets (people usually say: "if every open cover has a finite subcover").

EXERCISE 3.2.2. *Prove the following statement: if $(x_n)_{n=1}^{\infty}$ is a sequence defined on a compact set X, then it has a subsequence that converges to a point in X.*

3.3 Infimum and Supremum

FIX A NONEMPTY SET $X \subseteq \mathbb{R}$. A number $\alpha \in \mathbb{R}$ is said to be *an upper bound of* X if $x \leqslant \alpha$ for all $x \in X$, and is said to be *a lower bound of* X if the opposite inequality holds. Number $\alpha \in \mathbb{R}$ is said to be *the least upper bound of* X, denoted $\alpha = \sup X$, if: (i) α is an upper bound of X; and (ii) $\gamma \geqslant \alpha$ for any other upper bound γ of X. Analogously, number $\beta \in \mathbb{R}$ is said to be *the greatest lower bound of* X, denoted $\beta = \inf X$, if: (i) β is a lower bound of X; and (ii) if γ is a lower bound of X, then $\gamma \leqslant \beta$.

THEOREM 3.3.1. *Let $X \neq \emptyset$, $X \subseteq \mathbb{R}$. $\alpha = \sup X$ if, and only if, for all $\varepsilon > 0$ it is true that (i) for all $x \in X$, one has that $x < \alpha + \varepsilon$; and (ii) for some $x \in X$ one has that $\alpha - \varepsilon < x$.*

The proof of the previous theorem is left as an exercise. Now, remember that in Chapter 1, we announced one axiom of $\mathbb{R}$ that is not satisfied by the Rationals? It is the following:

AXIOM 3.3.1 (Axiom of Completeness). *Let $X \subseteq \mathbb{R}$ be nonempty. If X is bounded above, then it has a least upper bound.*

The Axiom of Completeness gives us the tool for a formal proof of the Bolzano-Weierstrass Theorem (2.6.5), as well as the proof of Theorem 3.1.4.

Proof of Theorem 2.6.5: By Theorems 2.6.12 and 2.6.13, it suffices that we consider just the case $K = 1$. Since $(a_n)_{n=1}^{\infty}$ is bounded, there exists $a \in \mathbb{R}$ such that, $-a < a_n < a$ for all n. Define the set

$$X = \{x \in \mathbb{R} \mid a_n \geqslant x \text{ for infinitely many terms of } (a_n)_{n=1}^{\infty}\}$$

Since $-a \in X$ and for all $x \in X$, $x \leqslant a$, it follows from Axiom 3.3.1 that X has a least upper bound. Let $\alpha = \sup X$. Fix $\varepsilon > 0$ and $\tilde{n} \in \mathbb{N}$. By definition, $\alpha + \varepsilon/2 \notin X$, which means that for some $n^* \in \mathbb{N}$ one has that $a_n < \alpha + \varepsilon$ for all $n \geqslant n^*$. Now, if there is $\hat{n} \in \mathbb{N}$ that satisfies that, for all $n \geqslant \hat{n}$, inequality $a_n < \alpha - \varepsilon/2$ holds, then, it follows that if $x \in X$, then $x \leqslant \alpha - \varepsilon/2$, which contradicts the fact that $\alpha = \sup X$. So, it must be that for all $\hat{n} \in \mathbb{N}$, one can find some $n \geqslant \hat{n}$ for which $a_n \geqslant \alpha - \varepsilon/2$ is true. It follows, then, that there is $n > \tilde{n}$, such that $\alpha - \varepsilon < a_n < \alpha + \varepsilon$.

Then, we can define a sequence $(n_m)_{m=1}^{\infty}$, as follows:

$$n_1 = \min\{n \in \mathbb{N} \mid \alpha - 1 < a_n < \alpha + 1\},$$

and, recursively,

$$n_m = \min\{n \in \mathbb{N} \mid n > n_{m-1} \wedge \alpha - 1/m < a_n < \alpha + 1/m\}.$$

It is straightforward that $(a_{n_m})_{m=1}^{\infty}$ is a convergent subsequence of $(a_n)_{n=1}^{\infty}$. $\square$

Proof of Theorem 3.1.4: Suppose otherwise: let $A, B \subseteq \mathbb{R}$ be such that $A \cap B = \varnothing$, $A \cup B = \mathbb{R}$, $A \neq \varnothing$ and $B \neq \varnothing$. Fix $a \in A$ and $b \in B$, assuming, with no loss of generality, that $a < b$.

Let $X = A \cap [a, b]$ and $Y = B \cap [A, B]$. Note that these sets are disjoint and satisfy that $X \cup Y = [a, b]$. Importantly, they satisfy the following properties:

$$\forall x \in X, \exists \varepsilon > 0 : [x, x + \varepsilon) \subseteq X, \tag{$*$}$$

and

$$\forall y \in Y, \exists \varepsilon > 0 : (y - \varepsilon, y] \subseteq Y \tag{$**$}$$

Let $\bar{x} = \sup X$, which exists by Axiom 3.3.1. By construction, $\bar{x} \in [a, b]$. If $\bar{x} \in Y$, by $(**)$, $(\bar{x} - \varepsilon, \bar{x}] \subseteq Y$ for some $\varepsilon > 0$. This implies that $\bar{x} - \varepsilon/2$ is an upper bound for X, which is impossible since $\bar{x}$ is its least upper bound. If $\bar{x} \in X$, by $(*)$, $[\bar{x}, \bar{x} + \varepsilon) \subseteq X$, which is impossible since $\bar{x}$ is an upper bound of X. It follows that $\bar{x} \notin X \cup Y$, which contradicts the fact that $X \cup Y = [a, b]$. $\square$

3.4 Application: Consumption Budgets

Consider the problem of a consumer who must choose a bundle of $L \in \mathbb{N}$ perfectly divisible commodities. Assume that this individual can only consume non-negative amounts of these goods, so that her consumption space is $\mathbb{R}_+^L$. Let $p \in \mathbb{R}_{++}^L$ denote the prices that are in place. If she has a nominal wealth m that constrains her purchases of commodities, then her budget set is $B(p, m) = \{x \in \mathbb{R}_+^L \mid p \cdot x \leqslant m\}$. We want to observe that this set is compact.

First, let us argue that it is bounded. This is easy to see, for if $x \in B(p, m)$, then for all commodity $l = 1, \ldots, L$, one has that $x_l \leqslant m/\bar{p}$, where $\bar{p} = \min\{p_1, \ldots, p_L\} > 0$, given the assumption that no commodity can be consumed in a negative amount.

Now, suppose that $x \notin B(p, m)$. This can happen for one of two (nonexclusive) reasons. First, it could be that for some l one has $x_l < 0$; in this case, letting $\varepsilon = |x_l|/2 > 0$, one immediately has that $B_\varepsilon(x) \subseteq B(p, m)^c$. The other possibility is that $p \cdot x > m$; in this case, one can let

$$\varepsilon = \frac{p \cdot x - m}{L \max_l\{p_l\}} > 0,$$

and note that, if $\tilde{x} \in B_\varepsilon(x)$, then

$$p \cdot \tilde{x} > \sum_l p_l(x_l - \varepsilon) \geqslant p \cdot x - L \max_l\{p_l\}\varepsilon = m,$$

so that $\tilde{x} \notin B(p, m)$. In any case, for some $\varepsilon > 0$ we have that $B_\varepsilon(x) \subseteq B(p, m)^c$, so we conclude that $B(p, m)^c$ is open and, hence, by Theorem 3.1.5, that $B(p, m)$ is closed.

4 Continuity

THROUGHOUT THIS CHAPTER, MAINTAIN the assumption that X is a subset of the finite-dimensional space $\mathbb{R}^K$.

4.1 Continuous Functions

DEFINITION 4.1.1. *Function* $f : X \to \mathbb{R}$ *is* continuous at $\bar{x} \in X$ *if for all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $|f(x) - f(\bar{x})| < \varepsilon$ *for all* $x \in B_\delta(\bar{x}) \cap X$*. It is* continuous *if it is continuous at all* $\bar{x} \in X$*.*

Note that continuity at $\bar{x}$ is a local concept. Second, note that $\bar{x}$ in the definition may but need not be a limit point of X. Therefore, two points are worth noticing: if $\bar{x}$ is not a limit point of X, then any $f : X \to \mathbb{R}$ is continuous at $\bar{x}$ (why?); and if, on the other hand, $\bar{x}$ is a limit point of X, then $f : X \to \mathbb{R}$ is continuous at $\bar{x}$ if, and only if, $\lim_{x\to\bar{x}} f(x) = f(\bar{x})$. Intuitively, this occurs when a function is such that in order to get arbitrarily close to $f(\bar{x})$ in the range, all we need to do is to get close enough to $\bar{x}$ in the domain. By Theorem 2.6.6, it follows that when $\bar{x} \in X$ is a limit point of X, f is continuous at $\bar{x}$ if, and only if, whenever we take a sequence of points in the domain that converges to $\bar{x}$, the sequence formed by their images converges to $f(\bar{x})$ (that in this case the concept is not vacuous follows from Exercise 2.6.5).

EXERCISE 4.1.1. *Consider the function introduced in Exercise 2.6.7. Is it continuous?*

EXERCISE 4.1.2. *Consider the function introduced in Example 2.6.1. Is it continuous? What if we change the function, slightly, as follows:* $f : \mathbb{R} \to \mathbb{R}$*, defined as*

$$f(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$$

Is it continuous?

4.2 Images and Pre-Images under Continuous Functions

THE FOLLOWING RESULT OFFERS a characterization of the definition of continuity. This result has been useful in economics.

THEOREM 4.2.1. *Function* $f : \mathbb{R}^K \to \mathbb{R}$ *is continuous if, and only if, for all open set* $U \subseteq \mathbb{R}$, *one has that* $f^{-1}[U]$ *is open.*

Proof. Fix $\bar{x} \in \mathbb{R}^K$ and $\varepsilon > 0$. By Example 3.1.1, we know that $B_\varepsilon(f(\bar{x}))$ is open and, therefore, so is $f^{-1}[B_\varepsilon(f(\bar{x}))]$. Since $\bar{x} \in f^{-1}[B_\varepsilon(f(\bar{x}))]$, we have that there exists some $\delta > 0$ for which $B_\delta(\bar{x}) \subseteq f^{-1}[B_\varepsilon(f(\bar{x}))]$. For such δ, the latter means that that for all $x \in B_\delta(\bar{x})$ one has that $|f(x) - f(\bar{x})| < \varepsilon$.

Now, let $U \subseteq \mathbb{R}$ be an open set, and let $\bar{x} \in f^{-1}[U]$. By definition, $f(\bar{x}) \in U$, and since U is open, there is some $\varepsilon > 0$ for which $B_\varepsilon(f(\bar{x})) \subseteq U$. Since f is continuous, there exists $\delta > 0$ such that $|f(x) - f(\bar{x})| < \varepsilon$ for all $x \in B_\delta(\bar{x})$. The latter implies that $B_\delta(\bar{x}) \subseteq f^{-1}[U]$. □

I stated the previous theorem in a weaker form than it really has to be. Actually, we don't really need the domain of f to be $\mathbb{R}^K$. If the domain is $X \subseteq \mathbb{R}^K$, the result continues to hold, but we need to qualify the definition of open set, to make it relative to the set X. We implicitly do that in the following theorem, which we leave without proof.

THEOREM 4.2.2. *Function* $f : X \to \mathbb{R}$ *is continuous if, and only if, for every open set* $U \subseteq \mathbb{R}$, *there exists an open set* $O \subseteq \mathbb{R}^K$ *such that* $f^{-1}[U] = O \cap X$.

Of course, analogous results for the pre-images of closed sets follow. Importantly, continuous functions also preserve some properties on the images of sets.

THEOREM 4.2.3. *If function* $f : X \to \mathbb{R}$ *is continuous and set* $C \subseteq X$ *is compact, then set* $f[C]$ *is compact too.*

Proof. Let $(y_n)_{n=1}^\infty$ be a sequence in $f[C]$ and such that $y_n \to y$. Fix $(x_n)_{n=1}^\infty$ in C such that $f(x_n) = y_n$. Since C is bounded, by Theorem 2.6.5 there exists a subsequence $(x_{n_m})_{m=1}^\infty$ that converges to some x, with $x \in C$ because C is closed. By continuity, $y = \lim_{m\to\infty} y_{n_m} = \lim_{m\to\infty} f(x_{n_m}) = f(x)$, so $y \in f[C]$.

Now, suppose that for all $\Delta \in \mathbb{R}$, there is $y \in f[C]$ such that $|y| \geqslant \Delta$. Then, for all $n \in \mathbb{N}$, there is $x_n \in C$ for which $|f(x_n)| \geqslant n$. Since C is compact, as before, there exists a subsequence $(x_{n_m})_{m=1}^\infty$ that converges to some $x \in C$. By continuity, $|f(x)| = \lim_{m\to\infty} |f(x_{n_m})| = \infty$, which is impossible. □

It is important to note that the result does not hold for sets that are only closed, or only bounded.

4.3 Properties and the Intermediate Value Theorem

THE FOLLOWING PROPERTIES OF continuous functions are derived from the properties of limits.

THEOREM 4.3.1. *Suppose that* $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$ *are continuous at* $\bar{x} \in X$, *and let* $\alpha \in \mathbb{R}$. *Then, the functions* $f + g$, αf *and* $f.g$ *are continuous at* $\bar{x}$. *Moreover, if* $g(\bar{x}) \neq 0$, *then* $\frac{f}{g}$ *is continuous at* $\bar{x}$.

Proof. This theorem follows from Theorem 2.6.7. For example, if $\bar{x}$ is a limit point of X, then, continuity of f and g at $\bar{x}$ implies that $\lim_{x\to\bar{x}} f(x) = f(\bar{x})$ and $\lim_{x\to\bar{x}} g(x) =$

$g(\bar{x})$. Since $f(\bar{x}) \in \mathbb{R}$ and $g(\bar{x}) \in \mathbb{R}$, it follows from Theorem 2.6.7 that

$$\lim_{x \to \bar{x}} (f+g)(x) = f(\bar{x}) + g(\bar{x}) = (f+g)(\bar{x}),$$

so that $(f+g)$ is continuous at $\bar{x}$. The proof of the rest of the theorem is similar. $\square$

The following result is very intuitive:

THEOREM 4.3.2 (The Intermediate Value Theorem in $\mathbb{R}$). *If function $f : [a, b] \to \mathbb{R}$ is continuous, then for every number γ between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ for which $f(x) = \gamma$.*[1]

[1] It does not matter whether $f(a) \geqslant f(b)$ or $f(a) < f(b)$ – we could simply have written that $\gamma \in [f(a), f(b)] \cup [f(b), f(a)]$.

Proof. If $\gamma = f(a)$ or $\gamma = f(b)$, the result is obvious. Assume that $f(a) < \gamma < f(b)$, and denote $S = f^{-1}[(-\infty, \gamma)]$. Since $a \in S$, it follows that $S \neq \varnothing$. Since $S \subseteq [a, b]$, it follows that S is bounded. Then, by the Axiom of Completeness (Axiom 3.3.1) we have that $\bar{x} = \sup S$ exists. By definition, for all $n \in \mathbb{N}$, the number $\bar{x} - \frac{1}{n}$ is not an upper bound of S (see Theorem 3.3.1). Hence, for all n there must exist $x_n \in S$ for which $\bar{x} - \frac{1}{n} < x_n \leqslant \bar{x}$. Construct such sequence $(x_n)_{n=1}^{\infty}$. By construction, $f(x_n) < \gamma$, whereas, since $1/n \to 0$, we have that $x_n \to \bar{x}$. And since f is continuous, we have that

$$\lim_{n \to \infty} f(x_n) = f(\bar{x}) \leqslant \gamma,$$

where the inequality follows from Theorem 2.6.10.

Now, define $\tilde{x}_n = \min\{b, \bar{x} + 1/n\}$. Consider any n. If $\tilde{x}_n = b$, then $\tilde{x}_n \notin S$. Else, $\tilde{x}_n = \bar{x} + 1/n > \bar{x}$, from where if $\tilde{x}_n \in S$, we have that for some $x \in S$ it is true that $x > \sup S$ which is a contradiction. It must then be that every $\tilde{x}_n \notin S$, which implies that $f(\tilde{x}_n) \geqslant \gamma$. Again, since $\tilde{x}_n \to \bar{x}$,[2] we have that

[2] Can you see why?

$$\lim_{n \to \infty} f(\tilde{x}_n) = f(\bar{x}) \geqslant \gamma$$

by continuity of f and Corollary 2.6.10.

The argument when $f(b) < \gamma < f(a)$ is similar. $\square$

It should be clear that if we consider f defined on $X \subseteq \mathbb{R}^K$, even with $a, b \in X$, the object $[a, b]$ is not well defined. The (line) segment connecting a and b, however, is:

$$\{x \in \mathbb{R}^K \mid \exists \vartheta \in [0,1] : \vartheta a + (1-\vartheta) b \in X\}.$$

The following result generalizes the previous theorem:

THEOREM 4.3.3. *Let $a, b \in X$ be such that the segment connecting them is contained in X.*[3] *If function $f : X \to \mathbb{R}$ is continuous, then for every number γ lying between $f(a)$ and $f(b)$, there is some $\bar{x}$ in the segment connecting a and b for which $f(\bar{x}) = \gamma$.*

[3] That is, a and b are such that the set

$$\{x \in \mathbb{R}^K \mid \exists \vartheta \in [0,1] : \vartheta a + (1-\vartheta) b \in X\}$$

is a subset of X

Proof. Define the function $\varphi : [0,1] \to \mathbb{R}$ by $\varphi(\vartheta) = f(\vartheta a + (1-\vartheta) b)$, which we can do because

$$\{x \in \mathbb{R}^K \mid \exists \vartheta \in [0,1] : \vartheta a + (1-\vartheta) b \in X\} \subseteq X.$$

By construction, $\varphi(1) = f(a)$ and $\varphi(0) = f(b)$. By a later result, Theorem 4.6.1,[4] we have that φ is continuous, so it follows that for some $\bar{\vartheta} \in [0,1]$ it is true that $\varphi(\bar{\vartheta}) = \gamma$. Let $\bar{x} = \bar{\vartheta} a + (1-\bar{\vartheta}) b$. $\square$

[4] Which does *not* require the theorem we are now showing!

4.4 Left- and Right-Continuity

WHEN WE ARE DEALING with functions defined on $X \subseteq \mathbb{R}$, we can easily identify, for each $\bar{x} \in X$, which part of the domain is *above* $\bar{x}$ and which one is *below*. This property allows us to study how the function behaves as we approach $\bar{x}$ from above (the right) or below (the left).

Consider a function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$. Suppose that $\bar{x}$ is a limit point of X, and let $\ell \in \mathbb{R}$. One says that

$$\lim_{x \searrow \bar{x}} f(x) = \ell,$$

when for every $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ whenever $x \in X \cap B_\delta(\bar{x})$ and $x > \bar{x}$. In such case, ℓ is said to be *the limit of function* f *as* x *tends to* $\bar{x}$ *from above.* Similarly,

$$\lim_{x \nearrow \bar{x}} f(x) = \ell,$$

when for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in X \cap B_\delta(\bar{x})$ satisfying that $x > \bar{x}$. In this case, ℓ is said to be *the limit of function* f *as* x *tends to* $\bar{x}$ *from below.*

Function $f : X \to \mathbb{R}$ is *right-continuous at* $\bar{x} \in X$, where $\bar{x}$ is a limit point of X, if

$$\lim_{x \searrow \bar{x}} f(x) = f(\bar{x}).$$

It is *right-continuous* if it is right-countinuous at every $\bar{x} \in X$ that is a limit point of X. Similarly, $f : X \to \mathbb{R}$ is *left-continuous at* $\bar{x}$ if

$$\lim_{x \nearrow \bar{x}} f(x) = f(\bar{x}),$$

and one says that f is left-continuous if it is left-continuous at all limit point $\bar{x} \in X$.

EXERCISE 4.4.1. *Consider the function introduced in Exercise 4.1.2. Is it right-continuous? Left-continuous? What if, keeping the rest of the function unchanged, we redefine* $f(0) = -1$*? Is it left- or right-continuous? What if* $f(0) = 1$.

4.5 Application: Preferences and Utility Functions

CONSIDER AGAIN THE SITUATION of Section 3.4. The individual's preferences are a complete pre-order $\succsim$ on $X = \mathbb{R}^L_+$, as in Section 1.5, with $\succ$ and $\sim$ defined as there.

In this setting, $\succsim$ is said to be *strictly monotone* if $x > x'$ implies $x \succ x'$, and *strongly convex* if for any bundle x, any bundle $x' \neq x$ such that $x \succsim x'$, and any scalar $0 < \alpha < 1$, it is true that $\alpha x + (1 - \alpha)x' \succ x'$. It is *continuous* if for every pair of convergent sequences $(x_n)_{n=1}^\infty$ and $(x'_n)_{n=1}^\infty$ defined in $\mathbb{R}^L_+$ and satisfying that $x_n \succsim x'_n$ at all $n \in \mathbb{N}$, one has that

$$\lim_{n\to\infty} x_n \succsim \lim_{n\to\infty} x'_n.$$

Finally, relation $\succsim$ is *represented by function* $u : \mathbb{R}^L_+ \to \mathbb{R}$ if $u(x) \geqslant u(x')$ occurs if, and only if, $x \succsim x'$. The function u that represents $\succsim$ is called a *utility function.*[5]

[5] Notice that if a preference relation is representable, then there are infinitely many different utility functions that represent it.

THEOREM 4.5.1. *Suppose that* $\succsim$ *is strictly monotone, strongly convex and continuous. Then, it can be represented by a continuous utility function.*

Proof. The argument is constructive: for each $x \in \mathbb{R}^L_+$, define $u(x)$ as the number for which $u(x)e \sim x$, where $e = (1, \ldots, 1)$, provided that such number exists and is uniquely defined. For this assignment to constitute a function, we must show that such number does indeed exist and, moreover, is unique. So, fix x, and define the sets

$$B = \{t \in \mathbb{R}_+ \mid te \succsim x\} \text{ and } W = \{t \in \mathbb{R}_+ \mid x \succsim te\},$$

both of which are closed, since $\succsim$ is continuous. Also, since $\succsim$ is strictly monotone, by the Axiom of Completeness there exist numbers $\bar{t}$ and $\underline{t}$ such that $B = [\underline{t}, \infty)$ and $W = [0, \bar{t}]$. Given that $\succsim$ is complete, we further have that $\mathbb{R}_+ = B \cup W$, from where $\underline{t} \leqslant \bar{t}$, and, therefore, $B \cap W \neq \varnothing$, which implies that at least one number that can be assigned as $u(x)$ exists. By monotonicity, such number must be unique.

Now, suppose that $x \succsim x'$. By transitivity of $\succsim$, it must be that $u(x)e \succsim u(x')e$. By monotonicity, then, $u(x) \geqslant u(x')$. On the other hand, if $u(x) \geqslant u(x')$, by monotonicity we would have that $u(x)e \succsim u(x')e$ and by transitivity $x \succsim x'$. This proves that u represents $\succsim$.

To complete the proof, we need to argue that u is continuous. By Theorem 4.2.2, it suffices that we show that for all pair of numbers $a, b \in \mathbb{R}_+$, set $u^{-1}[(a,b)]$ is open, as a subset of $\mathbb{R}^L_+$. By monotonicity and transitivity,

$$u^{-1}[(a,b)] = \{x \in \mathbb{R}^L_+ \mid be \succ u(x)e \succ ae\} = \{x \in \mathbb{R}^L_+ \mid be \succ x \succ ae\}.$$

That is,

$$\begin{aligned} u^{-1}[(a,b)] &= \{x \in \mathbb{R}^L_+ \mid be \succ x\} \cap \{x \in \mathbb{R}^L_+ \mid x \succ ae\} \\ &= \{x \in \mathbb{R}^L_+ \mid x \succsim be\}^c \cap \{x \in \mathbb{R}^L_+ \mid ae \succsim x\}^c \end{aligned}$$

By continuity, each of the sets on the right-hand side of the expression is open (as a subset of $\mathbb{R}^L_+$), since their complements are closed. It follows from Theorem 3.1.2 that $u^{-1}[(a,b)]$ is open too. □

4.6 Continuity of Composite Functions

FIX THE SETS $X \subseteq \mathbb{R}^K$, for a finite dimension K, and $Y \subseteq \mathbb{R}$. Given the functions $f : X \to Y$ and $g : Y \to \mathbb{R}$, we define the *composite function* $g \circ f : X \to \mathbb{R}$ by letting $(g \circ f)(x) = g(f(x))$, for all $x \in X$.

THEOREM 4.6.1. *Suppose that* $f : X \to Y$ *is continuous at* $\bar{x} \in X$, *and* $g : Y \to \mathbb{R}$ *is continuous at* $f(\bar{x})$. *Then,* $g \circ f$ *is continuous at* $\bar{x}$.

Proof. Fix $\varepsilon > 0$. Since g is continuous at $f(\bar{x})$, there is some $\gamma > 0$ such that $|g(y) - g(f(\bar{x}))| < \varepsilon$ for all $y \in B_\gamma(f(\bar{x})) \cap Y$. Since f is continuous at $\bar{x}$ and $\gamma > 0$, there also is some $\delta > 0$ such that $f(x) \in B_\gamma(f(\bar{x}))$ for all $x \in B_\delta(\bar{x}) \cap X$. And since $f(x) \in Y$, we then have that $f(x) \in B_\gamma(f(\bar{x})) \cap Y$, and therefore $|g(f(x)) - g(f(\bar{x}))| < \varepsilon$ for all $x \in B_\delta(\bar{x}) \cap X$, which proves the claim. □

COROLLARY 4.6.1. *If* $f : X \to Y$ *and* $g : Y \to \mathbb{R}$ *are continuous,* $g \circ f$ *is continuous.*

5
Differentiability

For simplicity of presentation, we first consider functions defined on $\mathbb{R}$, and study higher-dimensional spaces later.

5.1 Functions on $\mathbb{R}$

Throughout this section, we maintain the assumption that $X \subseteq \mathbb{R}$ is open.

Suppose that we have a function $f : X \to \mathbb{R}$, fix $x \in X$, and define the function

$$H_x = \{h \in \mathbb{R} \setminus \{0\} \mid x + h \in X\}.$$

Now, for all $h \in H_x$, evaluate the expression

$$\frac{f(x+h) - f(x)}{h}.$$

Since x is fixed, the expression depends on (is a function of) h only, on the nonempty (why?) domain H_x. Moreover, since 0 is a limit point of H_x, we can use definition 2.6.1, to study the object

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

5.1.1 Differentiability

Definition 5.1.1. *Function* $f : X \to \mathbb{R}$ *is* differentiable at $x \in X$ *if for some* $\ell \in \mathbb{R}$ *it is true that*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \ell.$$

It is differentiable *if it is differentiable at all* $x \in X$.

Notice that the definition does require the limit to be a real number. Besides, since we only define $\lim_{x \to \bar{x}} g(x)$ when $\bar{x}$ is a limit point of the domain of function g, our definition of differentiability implicitly requires x to be a limit point of $X \setminus \{x\}$ and, therefore, of X. But it follows from Exercises 3.1.1 and 3.1.2 that this is always the case since X is open.[1]

[1] One can study differentiability in a slightly more general context by not restricting X and only defining the concept at limit points of the domain. In $\mathbb{R}$, applying our definition to non-interior limit points will encompass the concepts of left- and right-differentiability, which we won't cover here.

Suppose that $f : X \to \mathbb{R}$ is differentiable at $x \in X$, then we define *the derivative of*

f *at* x, denoted $f'(x)$, to be the number

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

EXAMPLE 5.1.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = x^2$. *We want to know whether the function is differentiable. Fix* $x \in \mathbb{R}$. *Now, for any* $h \neq 0$,

$$\frac{f(x+h) - f(x)}{h} = 2x + h,$$

so, by Exercise 2.6.6, we know that f *is indeed differentiable and* $f'(x) = 2x$ *at all* $x \in \mathbb{R}$.

EXAMPLE 5.1.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = |x|$. *We want to know whether the function is differentiable at* 0. *Fix* $x = 0$ *and evaluate, for* $h \neq 0$,

$$\frac{f(x+h) - f(x)}{h} = \begin{cases} 1, & \textit{if } h > 0; \\ -1, & \textit{if } h < 0. \end{cases}$$

By Example 2.6.1, we know that in this case

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

does not exist, so f *is not differentiable at* 0.

A useful characterization of differentiability is the following:

THEOREM 5.1.1. *A function* $f : X \to \mathbb{R}$ *is differentiable, and has derivative* $f'(x) = \ell$, *at* x *if, and only if, for some* $\varepsilon > 0$ *and some function* $\varphi : B_\varepsilon(0) \to \mathbb{R}$ *it is true that*

$$\lim_{h \to 0} \frac{\varphi(h)}{h} = 0,$$

while $f(x+h) = f(x) + \ell h + \varphi(h)$ *for all* $h \in B_\varepsilon(0)$.

Proof. Suppose first that there are ε and φ that obey the mentioned properties. Then, by direct computation,

$$\frac{f(x+h) - f(x)}{h} - \ell = \frac{\varphi(h)}{h},$$

from where the fact that

$$\lim_{h \to 0} \frac{\varphi(h)}{h} = 0$$

implies the sufficiency claim.

Now, suppose that f is differentiable at x. Since x is open, there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X$. Define $\varphi : B_\varepsilon(0) \to \mathbb{R}$ by $\varphi(h) = f(x+h) - f(x) - \ell h$, which is well defined since, $x + h \in B_\varepsilon(x) \subseteq X$. Then,

$$\frac{\varphi(h)}{h} = \frac{f(x+h) - f(x)}{h} - \ell,$$

from where, by definition,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \ell$$

implies necessity. □

5.1.2 *Continuity and differentiability*

For simplicity of notation, notice that if $h \in H_{\bar{x}}$, then there exists $x \in X$ for which $x = \bar{x} + h$. Thus, it follows that

$$\lim_{h \to 0} \frac{f(\bar{x}+h) - f(\bar{x})}{h} = \lim_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{x - \bar{x}}.$$

A very important relation is the following:

THEOREM 5.1.2. *If a function $f : X \to \mathbb{R}$, is differentiable at $\bar{x} \in X$, then it is continuous at $\bar{x} \in X$.*

Proof. Since f is differentiable at $\bar{x} \in X$, $\lim_{x \to \bar{x}}(f(x) - f(\bar{x}))/(x - \bar{x}) = \ell$ for some $\ell \in \mathbb{R}$. By Theorem 2.6.7, then

$$\lim_{x \to \bar{x}} f(x) = \lim_{x \to \bar{x}} \left\{ f(\bar{x}) + \left[\frac{f(x) - f(\bar{x})}{x - \bar{x}} \right] (x - \bar{x}) \right\} = f(\bar{x}),$$

so that f is continuous at $\bar{x}$. □

5.1.3 *Computing derivatives*

There are some very well known rules to compute derivatives.

THEOREM 5.1.3. *Suppose that functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are differentiable. Then for all $x \in X$ and all $k \in \mathbb{R}$, we have that*

1. $(f + g)'(x) = f'(x) + g'(x)$;

2. $(k.f)'(x) = k.f'(x)$;

3. $(f.g)'(x) = g(x)f'(x) + f(x)g'(x)$;

4. *if* $g(x) \neq 0$, *then*
$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2};$$

5. $(f^k)'(x) = k \cdot f(x)^{k-1} \cdot f'(x)$.

Proof. The first two parts follow straightforwardly from the properties of limits. These two parts are left as exercise. Once the first part has been proven, the fifth part can be proven for the case $k \in \mathbb{N}$ using the Principle of Mathematical Induction (over k). The general case is more complicated and we will not attempt to prove it. We now prove the third and fourth parts.

For the third part, notice that for $x \in X$, and $h \in \mathbb{R} \setminus \{0\}$ such that $x + h \in X$,

$$\frac{(f.g)(x+h) - (f.g)(x)}{h} = g(x)\frac{f(x+h) - f(x)}{h} + f(x+h)\frac{g(x+h) - g(x)}{h},$$

so the claim follows frm Theorems 5.1.1 and 2.6.7.

For the fourth part, notice that, if $g(x + h) \neq 0$ and $g(x) \neq 0$,

$$\frac{(f/g)(x+h) - (f/g)(x)}{h} = \frac{(f(x+h) - f(x))g(x)}{hg(x+h)g(x)} - \frac{(g(x+h) - g(x))f(x)}{hg(x+h)g(x)}$$

so the result follows, again, from Theorems 5.1.1 and 2.6.7. □

EXERCISE 5.1.1. *Find the derivatives of* $f(x) = \sqrt{x}/(1+x)$, $f(x) = x/(1+\sqrt{x})$ *and* $f(x) = 1/(1+\sqrt{x})$ *at* $x \in \mathbb{R}_{++}$.[2]

EXERCISE 5.1.2. *Let* $\bar{n} \in \mathbb{N}$. *Suppose that each function in the collection* $f_n : X \to \mathbb{R}$, $n = 1, \ldots, \bar{n}$ *is differentiable, and define* $g : X \to \mathbb{R}$ *by* $g(x) = f_1(x) \times f_2(x) \times \ldots \times f_{\bar{n}}(x)$. *Argue that* g *is differentiable, compute* $g'(x)$ *and show that*[3,4]

$$\frac{g'(x)}{g(x)} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \ldots + \frac{f_{\bar{n}}'(x)}{f_{\bar{n}}(x)}.$$

Another important result is, whose proof we will not attempt here, even though it is not complicated, is the following.

THEOREM 5.1.4. *If* $x \in \mathbb{R}$, $y \in \mathbb{R}_{++}$, *then* $(e^x)' = e^x$ *and* $\ln(y)' = 1/y$.

[2] If this has not been (boring) enough for you, also solve Exercise 2.11 (except for part k), in page 29 of Simon and Blume.

[3] Hint: use the Principle of Mathematical Induction.

[4] Shorthand notation for the last two expressions would respectively be $g(x) = \prod_{n=1}^{\bar{n}} f_n(x)$ and $g'(x)/g(x) = \sum_{n=1}^{\bar{n}} f_n'(x)/f_n(x)$.

5.2 Differentiability of Composite Functions: the Chain Rule

AS IN SECTION 4.6, fix $X \subseteq \mathbb{R}$, $Y \subseteq \mathbb{R}$, and $f : X \to Y$ and $g : Y \to \mathbb{R}$.

THEOREM 5.2.1 (The Chain Rule). *Suppose that* $f : X \to Y$ *is differentiable at* $x \in X$, *and* $g : Y \to \mathbb{R}$ *is differentiable at* $f(x)$. *Then,* $g \circ f$ *is differentiable at* x, *and*

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

Proof. By definition of derivative, we are interested in

$$\frac{(g \circ f)(x+h) - (g \circ f)(x)}{h} = \frac{g(f(x+h)) - g(f(x))}{h},$$

for any number $h \neq 0$ for which $x + h \in X$. For simplicity of notation, let $y = f(x)$, denote the set $\Omega = \{k \in \mathbb{R} \mid (y+k) \in Y\}$ and define the function $\varphi : \Omega \to \mathbb{R}$ by

$$\varphi(k) = \begin{cases} \frac{g(y+k)-g(y)}{k} - g'(y), & \text{if } k \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $\lim_{k \to 0} \varphi(k) = 0$, which, together with the fact that g is continuous (by Theorem 5.1.2), suffices to ensure that φ is continuous, by Theorem 4.3.1. Also, for all $k \in \Omega$ it is true that $g(y+k) - g(y) = k(\varphi(k) + g'(y))$.

Denoting $k_h = f(x+h) - f(x)$,[5] we have that, for $h \neq 0$,

$$\frac{g(f(x+h)) - g(f(x))}{h} = \frac{k_h}{h}(\varphi(k_h) + g'(y)).$$

Now, notice that, by definition,

$$\lim_{h \to 0} \frac{k_h}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \in \mathbb{R},$$

whereas[6] $\lim_{h \to 0} \varphi(k_h) = \lim_{k \to 0} \varphi(k) = 0$ and $g'(y) = g'(f(x)) \in \mathbb{R}$. By Theorem 2.6.7,

$$\lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{h} = g'(f(x))f'(x) \in \mathbb{R},$$

as claimed. □

The following is now immediate.

[5] We keep the subscript h to remark that k will change as h does. Formally, we are introducing a function $k : \{h \in \mathbb{R} \mid (x+h) \in X\} \to \Omega$. (Notice that this function is continuous.) For simplicity, we are denoting the function using just the subscript: we are writing k_h rather that $k(h)$.

[6] By construction,

$$\lim_{h \to 0} \varphi(k(h)) = \lim_{h \to 0} (\varphi \circ k)(h).$$

Now, by Corollary 4.6.1, $\varphi \circ k$ is continuous, so $\lim_{h \to 0}(\varphi \circ k)(h) = (\varphi \circ k)(0) = 0$.

COROLLARY 5.2.1. *Suppose that* $f : X \to Y$ *and* $g : Y \to \mathbb{R}$ *are differentiable. Then,* $(g \circ f) : X \to \mathbb{R}$ *is differentiable.*

5.2.1 Higher order derivatives

Suppose that function $f : X \to \mathbb{R}$ is differentiable. Then, for all $x \in X$, there is a unique number $f'(x) \in \mathbb{R}$. In other words, this means that $f' : X \to \mathbb{R}$, for the derivative itself assigns a real number $f'(x)$ to each $x \in X$.

If $f : X \to \mathbb{R}$, is differentiable and $f' : X \to \mathbb{R}$ is continuous, we say that f is continuously differentiable. In such case, we say that $f \in C^1$, and refer to C^1 as the class of continuously differentiable functions.

Now, suppose that $f \in C^1$. If f' is differentiable at $x \in X$, we say that f *is twice differentiable at* x, and define the second-order derivative of f at x, denoted $f''(x)$, to be the derivative of f' at x. In other words, we define $f''(x)$ to be $(f')'(x)$, whenever f' is differentiable at x. This means that for some $\ell \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \ell,$$

and we let $f''(x) = \ell$. Consequently, function $f : X \to \mathbb{R}$ is said to be twice differentiable (on X) if it is twice differentiable at all $x \in X$.

As before, if $f : X \to \mathbb{R}$ is twice differentiable, then $f'' : X \to \mathbb{R}$. In this case, if $f'' : X \to \mathbb{R}$ is continuous, we say that f is *twice continuously differentiable* and that $f \in C^2$, where C^2 is the class of twice continuously differentiable functions. It follows then that $C^2 \subseteq C^1$.

We can continue to define higher-order levels of differentiability in a recursive manner. Fix $k \in \mathbb{N}$, and denote by C^{k-1} the class of $(k-1)$ times continuously differentiable functions. For $f \in C^{k-1}$, denote by $f^{[k-1]}(x)$ the $(k-1)$-order derivative of f at $x \in X$. Then, if $f^{[k-1]}$ is differentiable at $x \in X$, we say that f is k times differentiable at x, and define the k-order derivative of f at x, denoted $f^{[k]}(x)$, to be the derivative of $f^{[k-1]}$ at x. Then, f is said to be k times differentiable if it is k times differentiable at all $x \in X$.

And once again, if f is k times differentiable, then $f^{[k]} : X \to \mathbb{R}$, and if $f^{[k]} : X \to \mathbb{R}$ is continuous, we say that f is k times continuously differentiable and that $f \in C^k$, where C^k is the class of k times continuously differentiable functions. It follows, then, that $C^k \subseteq C^{k-1} \subseteq \ldots \subseteq C^2 \subseteq C^1$.

Finally, if for all $k \in \mathbb{N}$, we have that $f \in C^k$, then we say that f is infinitely differentiable (or "smooth") and that $f \in C^\infty$, where C^∞ is the class of infinitely differentiable functions.

EXAMPLE 5.2.1. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$*, defined by*

$$f(x) = \begin{cases} \frac{x^3}{3}, & \textit{if } x \geqslant 0; \\ -\frac{x^3}{3}, & \textit{if } x < 0. \end{cases}$$

It is easy to see that f *is differentiable and*

$$f'(x) = \begin{cases} x^2, & \textit{if } x \geqslant 0; \\ -x^2, & \textit{if } x < 0, \end{cases}$$

which is continuous, so that $f \in C^1$. *From* f', *we further conclude that* f *is twice* *differentiable and* $f''(x) = 2|x|$, *which is continuous, so that* $f \in C^2$. *However, we know from Example 5.1.2 that* f'' *is not differentiable at* 0. *Thus, we conclude that for all* $k \geqslant 3$ *it is true that* $f \notin C^k$, *or thatf* $\in C^2 \setminus C^3$.

EXERCISE 5.2.1 (Polynomials). *A* polynomial of degree n, *defined on a domain* X, *is a function* $P : X \to R$, *defined by* $P(x) = \sum_{j=0}^{n} a_j x^j$, *where for each* $j \in \{0, 1, 2, \dots, n\}$, $a_j \in \mathbb{R}$, *and* $a_n \neq 0$. *Show that any polynomial is of class* C^∞. *(Hint: Probably the easiest way to do this is by mathematical induction on the order of the polynomial.)*

5.2.2 Derivatives and limits

A very useful result is the following theorem, whose proof we will not give here.

THEOREM 5.2.2 (L'Hopital's Rule). *Suppose that* $f : (a, b) \to \mathbb{R}$ *and* $g : (a, b) \to \mathbb{R} \setminus \{0\}$, *where* $-\infty \leqslant a < b \leqslant \infty$, *are differentiable. Let* $\bar{x} \in [a, b]$, *suppose that* $\lim_{x \to \bar{x}} f'(x)/g'(x) = \ell$, *for* $\ell \in \mathbb{R} \cup \{-\infty, \infty\}$. *If* $\lim_{x \to \bar{x}} f(x) = 0$ *and* $\lim_{x \to \bar{x}} g(x) = 0$, *then* $\lim_{x \to \bar{x}} (f/g)(x) = \ell$.

5.3 Functions on $\mathbb{R}^K$

5.3.1 Partial differentiability

We now consider a general, higher-dimensional set $X \subseteq \mathbb{R}^K$, open. Suppose that $x \in X$ and fix $k \in \{1, \dots, K\}$. It is easy to see that the set

$$\Delta = \{\delta \in \mathbb{R} \mid (x_1, \dots, x_{k-1}, x_k + \delta, x_{k+1}, \dots, x_K) \in X\}$$

is an open subset of $\mathbb{R}$, and that the function $\varphi : \Delta \to \mathbb{R}$, defined by

$$\varphi(\delta) = f(x_1, \dots, x_{k-1}, x_k + \delta, x_{k+1}, \dots, x_K)$$

is well defined. So, we can directly apply all the ideas of the previous section of this chapter to function φ and study the differentiability of function f when, starting from the point x, we change the k-th argument of the function, while maintaining the other arguments fixed at $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_K)$. When φ is differentiable at zero, we say that f *is partially differentiable with respect to* x_k *at* x, and say that the partial derivative of f with respect to x_k at x,

$$\frac{\partial f}{\partial x_k}(x) = \varphi'(0).$$

When f is partially differentiable with respect to x_k at every $x \in X$, we say that f *is partially differentiable with respect to* x_k. When f is partially differentiable with respect to x_k for every $k \in \{1, \dots, K\}$, we say that f *is partially differentiable.*

Notice that each

$$\frac{\partial f}{\partial x_k}(x)$$

depends on x and not just on x_k (obviously, exceptions exist). Importantly, we can study continuity of these functions and talk of continuous partial differentiability, and we can also study differentiability of the functions and introduce higher-order partial differentiability. For example, if the function $\partial f / \partial x_k$ is differentiable with respect to

x_j at x, then we say that the second derivative of f with respect to x_j and x_k is

$$\frac{\partial^2 f}{\partial x_j \partial x_k}(x) = \frac{\partial(\frac{\partial f}{\partial x_k})}{\partial x_j}(x).$$

When possible, we define *the Hessian of* f *at* x as the double-array ($K \times K$ matrix)

$$Hf(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_K}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_K \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_K \partial x_1}(x) \end{pmatrix}.$$

A crucial result in calculus, whose proof we will omit, is the following theorem.

THEOREM 5.3.1 (Young's Theorem). *If* f *is twice partially differentiable and all its second partial derivatives are continuous, then* $Hf(x)$ *is symmetric.*

5.3.2 Differentiability

The concept of partial differentiability is important (and straightforward) but, at least in principle, limited: it only considers perturbations in the "canonical" directions: one keeps all but one of the arguments fixed. A stronger concept is needed for more general perturbations, which we can do by analogy to Theorem 5.1.1.

DEFINITION 5.3.1. *A function* $f : X \to \mathbb{R}$ is differentiable, and has gradient $Df(x) = \Delta$, at $x \in X$ *if there are a number* $\varepsilon > 0$ *and a function* $\varphi : B_\varepsilon(0) \to \mathbb{R}$ *such that*

$$\lim_{h \to 0} \frac{\varphi(h)}{\|h\|} = 0$$

and that $f(x + h) = f(x) + \Delta h + \varphi(h)$ *for all* $h \in B_\varepsilon(0)$.

For reasons that will later be obvious, the gradient of a function point x is the direction in which the function increases most rapidly, when perturbed infinitesimally in its domain. As before, we can study continuity of Df as a function of f and talk of continuous differentiability.

The definition is not easy to verify, but the following result bridges the gap between differentiability and the more operational concept of partial differentiability

THEOREM 5.3.2. *If* f *is continuously partially differentiable at* x*, then it is differentiable at* x*, and*

$$Df(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_K}(x) \right).$$

If, on the other hand, f *is continuously differentiable at* x*, then it is continuously partially differentiable at* x*, and*

$$\left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_K}(x) \right) = Df(x).$$

Higher-order differentiability is also possible. If $f : X \to \mathbb{R}$ is differentiable at x and there exist $\varepsilon > 0$ and $\varphi : B_\varepsilon(0) \to \mathbb{R}$ such that

$$\lim_{h \to 0} \|h\|^{-2} \varphi(h) = 0$$

and $f(x+h) = f(x) + Df(x)h + \frac{1}{2}h^\top \Delta h + \varphi(h)$, for all $h \in B_\varepsilon(0)$, for some real-valued $K \times K$ matrix Δ, then we say that f is twice differentiable at x and $D^2 f(x) = \Delta$. It follows from the previous theorem that if f is twice continuously partially differentiable at x, then it is twice differentiable at x, and $D^2 f(x) = Hf(x)$, whereas, on the other hand, if f is twice continuously differentiable at x, then it is twice continuously partially differentiable at x, and $Hf(x) = Df(x)$.

5.4 *Taylor Approximations*

FOR SIMPLICITY OF EXPOSITION, throughout this section, we maintain fixed an open set $X \subseteq \mathbb{R}$.

5.4.1 *Approximations by polynomials*

Suppose that we have a function $f : X \to \mathbb{R}$, $f \in C^m$, and that $0 \in X$. Suppose also that for some integer $n \leqslant m$, we want to construct an n-degree polynomial, $P_n : X \to \mathbb{R}$, such that the values of f and its first n derivatives evaluated at 0 are the same as the values of P_n and its first n derivatives at that same point.

We know that for all $(a_j)_{j=0}^n$, a polynomial has the form

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

and we know that any $P_n \in C^\infty$. Also, since[7]

$$\begin{aligned}
P_n'(x) &= a_1 + 2a_2 x + \ldots + n a_n x^{n-1}, \\
P_n''(x) &= 2a_2 + (3 \times 2) a_3 x + \ldots + n(n-1) a_n x^{n-2}, \\
P_n'''(x) &= (3 \times 2) a_3 + (4 \times 3 \times 2) a_4 x + \ldots + n(n-1)(n-2) a_n x^{n-3}, \\
&\vdots \\
P_n^{[n]}(x) &= n(n-1)(n-2)\ldots(2)(1) a_n.
\end{aligned}$$

Immediately, $P_n(0) = a_0$, $P_n'(0) = a_1$, $P_n''(0) = 2a_2$, $P_n'''(0) = (3 \times 2)a_3 = 3!a_3$, and so on, up to

$$P_n^{[n]}(0) = n(n-1)(n-2)\ldots(2)(1)a_n = n!a_n.$$

Now, since what we want is to find $(a_j)_{j=0}^n$ such that $P_n(0) = f(0)$, $P_n'(0) = f'(0)$, $P_n''(0) = f''(0)$, and so on, until, $P_n^{[n]}(0) = f^{[n]}(0)$, it is immediate that the (only) array $(a_j)_{j=0}^n$ that satisfies such equalities is given by $a_0 = f(0)$ and $a_n = \frac{1}{k!} f^{[k]}(0)$, for each $k = 1, \ldots, n$. Put another way, our desired polynomial is

$$P_n(x) = f(0) + f'(0)x + \frac{1}{2} f''(0) x^2 + \ldots + \frac{1}{n!} f^{[n]}(0) x^n.$$

Since the restrictions we imposed are such that f and P_n are very close to each other and have the same derivatives when x is close to 0, we say that P_n is an n-th-order approximation to f about 0. Usually, this fact is expressed by saying that

$$f(x) \approx f(0) + f'(0)x + \frac{1}{2} f''(0) x^2 + \ldots + \frac{1}{n!} f^{[n]}(0) x^n.$$

One important remark is in order. Notice first that for all array $\{a_j\}_{j=0}^n$, all $x \in X$

[7] In what follows, we are going to use an extensive notation. If you want to keep notation short, although complicated, notice that for each $k \leqslant n$,

$$P_n^{[k]}(x) = \sum_{j=k}^n \frac{j!}{(j-k)!} a_j x^{j-k}.$$

and all $k > n$ one necessarily has that

$$P_n^{[k]}(x) = 0.$$

This implies that we should not try to equate more than the first n derivatives of the polynomial to the ones of the function. However, it is true that the higher the degree of the polynomial (subject to the differentiability of f), the better the approximation to the function. We will later come back to this point, but the following example may be illustrative:

EXAMPLE 5.4.1. *Let* $f : [-0.5, 0.5] \to \mathbb{R}$ *be defined by*

$$f(x) = \frac{1}{(1+x)^2}.$$

One can check that $f \in C^\infty$. *Also, we have that*

$$f'(x) = \frac{-2}{(1+x)^3}$$

and

$$f''(x) = \frac{6}{(1+x)^4}.$$

In particular, $f(0) = 1$, $f'(0) = -2$ *and* $f''(0) = 6$. *Thus, we have that* $P_1(x) = 1 - 2x$ *and* $P_2(x) = 1 - 2x + 3x^2$. *The graph of function* f *and these two polynomial approximations is Figure 5.1. Notice how well the second-order polynomial (green curve) approximates the function (blue curve). It certainly does better than the first-order polynomial (red line)!*

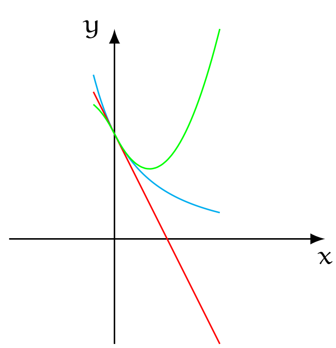

Figure 5.1: Accuracy of Taylor approximations

EXERCISE 5.4.1. *Develop first- and second-order polynomial approximations to* $f : [-1, 1] \to \mathbb{R}$, *defined by* $f(x) = e^x$.

EXERCISE 5.4.2. *Argue that* $e = \sum_{n=0}^{\infty}(1/n!)$.

5.4.2 Taylor approximations

The method that we used in the previous section is limited in that it requires that $0 \in X$, and in that we can only approximate the function about 0. The natural way to generalize this particular case is the use of Taylor polynomials.

Let $\bar{x} \in \mathbb{R}$. An n-degree polynomial about $\bar{x}$ is a function $P_{n,\bar{x}} : \mathbb{R} \to \mathbb{R}$ of the form

$$P_{n,\bar{x}}(x) = a_0 + a_1(x - \bar{x}) + a_2(x - \bar{x})^2 + \ldots + a_n(x - \bar{x})^n,$$

where $a_j \in \mathbb{R}$, for $j = 0, \dots, n$, and $a_n \neq 0$. It is easy to show that every such function is of class C^∞.

Now, suppose that we have a function $f : X \to \mathbb{R}$ of class C^m, and that for some $n \leqslant m$ and some $\bar{x} \in X$, we want to construct an n-degree Taylor polynomial such that the values of f and its first n derivatives evaluated at $\bar{x}$ are the same as the values of $P_{n,\bar{x}}$ and its first n derivatives at that same point. Then, we only have to repeat the procedure we already used: we have that

$$\begin{aligned}
P'_{n,\bar{x}}(x) &= a_1 + 2a_2(x-\bar{x}) + \dots + na_n(x-\bar{x})^{n-1}, \\
P''_{n,\bar{x}}(x) &= 2a_2 + (3\times 2)a_3(x-\bar{x}) + \dots + n(n-1)a_n(x-\bar{x})^{n-2}, \\
P'''_{n,\bar{x}}(x) &= (3\times 2)a_3 + (4\times 3\times 2)a_4(x-\bar{x}) + \dots + n(n-1)(n-2)a_n(x-\bar{x})^{n-3}, \\
&\vdots \\
P^{[n]}_{n,\bar{x}}(x) &= n(n-1)(n-2)\dots(2)(1)a_n,
\end{aligned}$$

so that $P_{n,\bar{x}}(\bar{x}) = a_0$, $P'_{n,\bar{x}}(\bar{x}) = a_1$, $P''_{n,\bar{x}}(\bar{x}) = 2a_2$, $P'''_{n,\bar{x}}(\bar{x}) = 3!a_3$, and so on, up to

$$P^{[n]}_{n,\bar{x}}(\bar{x}) = n!a_n;$$

and, since what we want is to find $\{a_j\}_{j=0}^n$ such that $P_{n,\bar{x}}(\bar{x}) = f(\bar{x})$, $P'_{n,\bar{x}}(\bar{x}) = f'(\bar{x})$, $P''_{n,\bar{x}}(\bar{x}) = f''(\bar{x})$, and so on, until

$$P^{[n]}_{n,\bar{x}}(\bar{x}) = f^{[n]}(\bar{x}),$$

it is easy to see that the (only) sequence $(a_j)_{j=0}^n$ that satisfies such equalities is to let $a_0 = f(\bar{x})$ while

$$a_n = \frac{1}{k!} f^{[k]}(\bar{x})$$

for all $k = 1, \dots, n$.

When we use these particular values of $(a_j)_{j=0}^n$ we obtain the n-th-degree Taylor polynomial approximation to f about $\bar{x}$. We denote this function by $T_{f,n,\bar{x}} : X \to \mathbb{R}$, and define it as

$$\begin{aligned}
T_{f,n,\bar{x}}(x) &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{1}{2}f''(\bar{x})(x-\bar{x})^2 + \dots + \frac{1}{n!}f^{[n]}(\bar{x})(x-\bar{x})^n \\
&= f(\bar{x}) + \sum_{j=1}^{n} \frac{1}{j!} f^{[j]}(\bar{x})(x-\bar{x})^j.
\end{aligned}$$

Again, in order to highlight that this is an approximation to f, it is usually written that

$$f(x) \approx f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{1}{2}f''(\bar{x})(x-\bar{x})^2 + \dots + \frac{1}{n!}f^{[n]}(\bar{x})(x-\bar{x})^n.$$

EXERCISE 5.4.3. *Argue that*

$$e^x = \sum_{n=0}^{\infty} (1/n!)x^n.$$

EXERCISE 5.4.4. *Develop first- and second-order Taylor approximations around* 1 *to* $f : \mathbb{R}_{++} \to \mathbb{R}$, *defined by* $f(x) = \ln(x)$.

5.4.3 The remainder

In this section, we maintain the assumption that we have $f : X \to \mathbb{R}$, $f \in C^m$, and that for some $n \in \mathbb{N}$, $n \leqslant m$, and some $\bar{x} \in X$, $T_{f,n,\bar{x}} : X \to \mathbb{R}$ is the n^{th}-degree Taylor approximation to f about $\bar{x}$.

Definition 5.4.1. *We define the remainder of the n-degree Taylor polynomial approximation to f about $\bar{x}$, denoted $R_{f,n,\bar{x}}$, by the function $R_{f,n,\bar{x}} : X \to \mathbb{R}$, where, for all $x \in X$,*

$$R_{f,n,\bar{x}}(x) = f(x) - T_{f,n,\bar{x}}(x)$$

The remainder measures (locally) the error that we are making when approximating the function by the n^{th}-degree Taylor polynomial. It follows then, by definition, that

$$f(x) = T_{f,n,\bar{x}}(x) + R_{f,n,\bar{x}}(x)$$

and that

$$R_{f,n,\bar{x}}(\bar{x}) = 0$$

but these properties are of no particular interest, since they are imposed by construction.

5.4.4 Mean value and Taylor's theorems

Before we introduce the most important property of the remainder, the following result is interesting:

Theorem 5.4.1 (The Mean Value Theorem). *Suppose that we have $f : X \to \mathbb{R}$, f differentiable. If $x, \bar{x} \in X$ are such that $[x, \bar{x}] \subseteq X$, $[x, \bar{x}] \neq \varnothing$, then $\exists x^* \in [x, \bar{x}]$ such that*

$$f(x) = f(\bar{x}) + f'(x^*)(x - \bar{x})$$

Similarly, if $x, \bar{x} \in X$ are such that $[\bar{x}, x] \subseteq X$, $[\bar{x}, x] \neq \varnothing$, then $\exists x^* \in [\bar{x}, x]$ such that

$$f(x) = f(\bar{x}) + f'(x^*)(x - \bar{x})$$

Notice the similarity between the expression resulting from the mean value theorem and a first-degree polynomial approximation about $\bar{x}$. The only difference is that the derivative is not (necessarily) evaluated at $\bar{x}$, but (maybe) at some other point in the interval between $\bar{x}$ and x. The importance of the result is that, with just that little change, our result is no longer an approximation: it is exact!

The mean value theorem allows to prove the following result. Recall that we maintain the assumptions introduced at the beginning of this section:

Theorem 5.4.2 (Taylor's Theorem). *Suppose that f is $(n+1)$ times differentiable (i.e. $f^{[n+1]} : X \to R$ exists). If $x, \bar{x} \in X$ are such that $[x, \bar{x}] \subseteq X$, $[x, \bar{x}] \neq \varnothing$, then $\exists x^* \in [x, \bar{x}]$:*

$$R_{f,n,\bar{x}}(x) = \frac{1}{(n+1)!} f^{[n+1]}(x^*)(x - \bar{x})^{n+1}$$

Similarly, if $x, \bar{x} \in X$ are such that $[\bar{x}, x] \subseteq X$, $[\bar{x}, x] \neq \varnothing$, then $\exists x^* \in [\bar{x}, x]$:

$$R_{f,n,\bar{x}}(x) = \frac{1}{(n+1)!} f^{[n+1]}(x^*)(x - \bar{x})^{n+1}$$

Two remarks are in order. Notice first that $n < m$ suffices as hypothesis for the theorem. But it is also important to notice that we do not require $f^{[n+1]}$ to be continuous, only to exist.

Second, notice again how similar this expression for the remainder is to each one of the terms of the Taylor polynomial. Again, the only difference is that the derivative in the remainder is computed at some point in the interval between $\bar{x}$ and x (an interval that must, obviously, be part of the domain), rather than at $\bar{x}$. The importance comes then from the fact that, if we have $f : X \to \mathbb{R}$, $f \in C^n$, and $f^{[n+1]}$ exists, then, for some in the x^* interval between $\bar{x}$ and x, the expression

$$\begin{aligned} f(x) &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{1}{2}f''(\bar{x})(x-\bar{x})^2 + \dots \\ &\quad + \frac{1}{n!}f^{[n]}(\bar{x})(x-\bar{x})^n + \frac{1}{(n+1)!}f^{[n+1]}(x^*)(x-\bar{x})^{n+1} \end{aligned}$$

is not an approximation. It is exact.

Unfortunately, we do not (yet) have the necessary elements to prove the mean value theorem (and, therefore, Taylor's theorem). However, we should be able to convince ourselves that the mean value theorem is intuitively clear.

5.4.5 Local accuracy of Taylor approximations

Although the mean value and Taylor's theorems are very important, in many cases one doesn't have the possibility to find which particular point in the interval between $\bar{x}$ and x will make our expressions exact. In those cases, we must stick to our n^{th}-order approximation. In this section, we claim that Taylor approximations are very good whenever x and $\bar{x}$ are close to each other, and that the higher n the better (locally) is the approximation. The exact sense in which this is true is that we will argue that as $x \to \bar{x}$, we have that $R_{f,n,\bar{x}}(x) \to 0$ faster than $(x-\bar{x})^n$. In other words, we claim that

$$\lim_{x\to\bar{x}} \left(\frac{R_{f,n,\bar{x}}(x)}{(x-\bar{x})^n} \right) = 0$$

Notice that if $n < m$, then, by Taylor's theorem, the result is straightforward. However, since we didn't prove the theorem, you may still be doubting about the result. We now offer a heuristic argument, for $n = 1$.

In such a case, it is clear that

$$R_{f,1,\bar{x}}(x) = f(x) - f(\bar{x}) - f'(\bar{x})(x-\bar{x})$$

so that for all $x \neq \bar{x}$,

$$\frac{R_{f,1,\bar{x}}(x)}{(x-\bar{x})} = \frac{f(x)-f(\bar{x})}{(x-\bar{x})} - f'(\bar{x})$$

and, by definition,

$$\begin{aligned} \lim_{x\to\bar{x}} \frac{R_{f,1,\bar{x}}(x)}{(x-\bar{x})} &= \lim_{x\to\bar{x}} \left(\frac{f(x)-f(\bar{x})}{(x-\bar{x})} - f'(\bar{x}) \right) \\ &= f'(\bar{x}) - f'(\bar{x}) \\ &= 0 \end{aligned}$$

meaning that $R_{f,1,\bar{x}}(x)$ goes to 0 faster than $x - \bar{x}$.

EXERCISE 5.4.5 (L'Hopital's rule is really useful, I). *Remember L'Hopital's rule? If so, you can further convince yourself of our claim for* $n \geqslant 2$. *For example, if* $n = 2$, *notice that*

$$R_{f,2,\bar{x}}(x) = R_{f,1,\bar{x}}(x) - \frac{1}{2} f''(\bar{x})(x - \bar{x})^2$$

Now, you can use L'Hopital's rule (after showing that it applies, of course,) to argue that

$$\lim_{x \to \bar{x}} \frac{R_{f,1,\bar{x}}(x)}{(x - \bar{x})^2} = \frac{1}{2} f''(\bar{x})$$

from where it follows that

$$\lim_{x \to \bar{x}} \frac{R_{f,2,\bar{x}}(x)}{(x - \bar{x})^2} = 0$$

Moreover, if you do this, you can prove the general result by mathematical induction.

6
Linear Algebra

WE NOW REVIEW, VERY briefly, the theory of linear spaces and linear operations on Real spaces. For the purposes of this chapter, fix numbers $K, J, L, N \in \mathbb{N}$.

6.1 Matrices

IT IS OFTEN USEFUL to generalize the idea of vector to richer arrays of numbers. A $K \times J$ matrix is an array of $J \in \mathbb{N}$ vectors in $\mathbb{R}^K$, each of which is taken as a column. If $K = J$ the matrix is said to be *square*.

EXAMPLE 6.1.1. *The identity matrix of size* K, *denoted* $\mathbb{I}$, *is* $\mathbb{I} = (e_1, e_2, \dots, e_K)$.

Given a $K \times J$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,J} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,J} \\ \vdots & \vdots & \ddots & \vdots \\ a_{K,1} & a_{K,2} & \cdots & a_{K,J} \end{pmatrix},$$

the transpose of A, denoted by $A^\top$, is the $J \times K$ matrix

$$A^\top = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{K,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{K,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,J} & a_{2,J} & \cdots & a_{K,J} \end{pmatrix}.$$

EXERCISE 6.1.1. *Prove the following: if* A *is a* $K \times J$ *matrix and* B *is a* $J \times L$ *matrix, then* $(AB)^\top = B^\top A^\top$.

A matrix A is said to be symmetric if $A^\top = A$, which obviously requires it to be square.

6.2 Linear Functions

A VERY SIMPLE, YET extremely useful kind of functions is defined by restricting the images they give to sums of vectors and to products of vectors and scalars. A function

$f : \mathbb{R}^K \to \mathbb{R}^J$, where $J, K \in \mathbb{N}$, is said to be *linear* if:

1. for all $x, x' \in \mathbb{R}^K$, $f(x + x') = f(x) + f(x')$;
2. for all $x \in \mathbb{R}^K$ and for all $\vartheta \in \mathbb{R}$, $f(\vartheta x) = \vartheta f(x)$.

THEOREM 6.2.1. *A function $f : \mathbb{R}^K \to \mathbb{R}^J$ is linear if, and only if, there exists a $J \times K$ matrix A such that for all $x \in \mathbb{R}^K$, $f(x) = Ax$.*

Proof. Sufficiency is obvious. For necessity, consider the following $(J \times K)$ matrix $A = (f(e_1), f(e_2), \cdots, f(e_K))$. □

EXERCISE 6.2.1. *Show that if $f : \mathbb{R}^K \to \mathbb{R}^J$ and $g : \mathbb{R}^J \to \mathbb{R}^L$ are linear, then $g \circ f$ is linear.*

Given a finite sequence $(x_n)_{n=1}^N$ in $\mathbb{R}^K$, we define the *span of* $(x_n)_{n=1}^N$, denoted $\mathrm{Sp}(x_n)_{n=1}^N$, as

$$\mathrm{Sp}(x_n)_{n=1}^N = \left\{ x \in \mathbb{R}^K \mid \exists (\vartheta_n)_{n=1}^N \in \mathbb{R} : \sum_{n=1}^N \vartheta_n x_n = x \right\}.$$

That is, the span of a sequence of vectors is the set of all its possible linear combinations.

Let $f : \mathbb{R}^K \to \mathbb{R}^J$ be linear, and fix the $J \times K$ matrix A such that $f(x) = Ax$ for all $x \in \mathbb{R}^K$. The *range of* f is simply the span of A. The set

$$\ker(A) = \{x \in \mathbb{R}^K \mid Ax = 0\}$$

is known as the *kernel*, or *nullspace*, of A.

6.3 Determinants

GIVEN A 2×2 MATRIX A, its is the number *determinant*

$$\det(A) = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}.$$

The absolute value of this number is the are of the parallelogram defined by the origin, the two rows of the matrix and their sum.

For any $K \times K$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,K} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ a_{K,1} & a_{K,2} & \cdots & a_{K,K} \end{pmatrix}$$

denote by $A^{\neg(j,\ell)}$ the $(K-1) \times (K-1)$ matrix resulting from deleting the j^{th} row and the ℓ^{th} column. The *determinant of* A, denoted by $\det(A)$, is recursively defined by[1]

$$\det(A) = \sum_{k=1}^K (-1)^{1+k} a_{1,k} \det(A^{\neg(1,k)})$$

[1] The latter definition is known as cofactor expansion along the first row.

It is an important fact that the determinant of a matrix can be found using any one of the rows or columns for the cofactor expansion, which implies that $\det(A) = \det(A^\top)$.

EXERCISE 6.3.1. *Prove the following results:*

1. $\det(\mathbb{I}) = 1$

2. *For any pair of* $K \times K$ *matrices* A *and* B, $\det(AB) = \det(A)\det(B)$.

Important and easily verifiable facts about determinants are the following: let $(a_k)_{k=1}^{K}$ be a finite sequence in $\mathbb{R}^K$,

1. for all $x \in \mathbb{R}^K$

$$\det(a_1 + x, a_2, \cdots, a_K) = \det(a_1, a_2, \cdots, a_K) + \det(x, a_2, \cdots, a_K);$$

2. for all $\vartheta \in \mathbb{R}$,

$$\det(\vartheta a_1, a_2, \cdots, a_K) = \vartheta \det(a_1, a_2, \cdots, a_K);$$

3. $\det(a_1 + a_2, a_2, \cdots, a_K) = \det(a_1, a_2, \cdots, a_K)$; and

4. $\det(a_2, a_1, \cdots, a_K) = -\det(a_1, a_2, \cdots, a_K)$.

6.4 *Linear Independence, Dimension and Rank*

THE CANONICAL VECTORS IN $\mathbb{R}^K$ are the sequence $(e_k)_{k=1}^{K}$ in $\mathbb{R}^K$, where, for all $k = 1, \dots, K$, $e_{k,\ell} = 1$ if $k = \ell$, and $e_{k,\ell} = 0$ otherwise. Evidently, $\text{Sp}(e_k)_{k=1}^{K} = \mathbb{R}^K$ and

$$\sum_{k=1}^{K} \vartheta_k e_k = (0)_{k=1}^{K} \Rightarrow (\vartheta_k)_{k=1}^{K} = (0)_{k=1}^{K}.$$

These two properties are important, as they imply that the vectors suffice to "generate" the whole of $\mathbb{R}^K$.

In general, a finite sequence $(x_n)_{n=1}^{N}$ in $\mathbb{R}^K$ is said to be *linearly independent* if

$$\sum_{n=1}^{N} \vartheta_n x_n = (0)_{k=1}^{K} \Rightarrow (\vartheta_n)_{n=1}^{N} = (0)_{n=1}^{N}.$$

EXERCISE 6.4.1. *Prove the following:* $(x_n)_{n=1}^{N}$ *is linearly dependent if, and only if, for some* $n^* \in \{1, \dots, N\}$ *and some* $(\vartheta_n)_{n \in \{1,\dots,N\} \setminus \{n^*\}}$,

$$x_{n^*} = \sum_{n \neq n^*} \vartheta_n x_n$$

A set $\mathcal{L} \subseteq \mathbb{R}^K$ is said to be a *linear subspace* of $\mathbb{R}^K$, denoted by $\mathcal{L} \sqsubseteq \mathbb{R}^K$, if: (i) it is closed under addition: for all $x, y \in \mathcal{L}$, $x + y \in \mathcal{L}$; and (ii) it is closed under scalar multiplication: for all $x \in \mathcal{L}$ and all $\vartheta \in \mathbb{R}$, we have that $\vartheta x \in \mathcal{L}$.

EXERCISE 6.4.2. *Prove the following: for every* $(x_n)_{n=1}^{N}$ *in* $\mathbb{R}^K$, $\text{Sp}(x_n)_{n=1}^{N} \sqsubseteq \mathbb{R}^K$.

Any sequence $(x_n)_{n=1}^{K}$ in $\mathbb{R}^K$ such that $\text{Sp}(x_n)_{n=1}^{K} = \mathbb{R}^K$ is called a *basis for* $\mathbb{R}^K$. In particular, $(e_k)_{k=1}^{K}$ is called *the canonical basis for* $\mathbb{R}^K$. In general, sequence $(x_n)_{n=1}^{N}$ is a *basis* for a linear subspace $\mathcal{L}$ of $\mathbb{R}^K$ if it is linearly independent and $\text{Sp}(x_n)_{n=1}^{N} = \mathcal{L}$.

Any two bases of a linear subspace $\mathcal{L}$ have the same number of vectors. This common number is known as the *dimension* of $\mathcal{L}$, denoted $\dim(\mathcal{L})$. The dimension of $\mathbb{R}^K$ is K, while the dimension of any linear subspace of $\mathbb{R}^K$ is at most K.

Using our previous notation, given a $J \times K$ matrix $A = (a_1, a_2, \cdots, a_J)$, the *column span* of A, denoted by $\mathrm{Sp}(A)$, is $\mathrm{Sp}(a_j)_{j=1}^J$. The *rank* of A, denoted by $\mathrm{rank}(A)$, is the dimension of $\mathrm{Sp}(A)$. The *nullity* of A, denoted by $\mathrm{nul}(A)$, is the dimension of $\ker(A)$.

THEOREM 6.4.1 (The Fundamental Theorem of Linear Algebra, I). *Given a $J \times K$ matrix A,*

1. $\mathrm{rank}(A) + \mathrm{nul}(A) = K$*;*

2. $\{x \in \mathbb{R}^K \mid \forall y \in \mathrm{Sp}(A^\top), x \cdot y = 0\} = \ker(A)$.

Proof. For the first part, let $\{u_1, \dots, u_L\}$ be a basis for $\ker(A)$,[2] and let $\{v_1, \dots, v_{K-L}\}$ be such that

$$\{u_1, \dots, u_L, v_1, \dots, v_{K-L}\}$$

is a basis for $\mathbb{R}^K$. It suffices for us to show that $\{Av_1, \dots, Av_{K-L}\}$ is a basis for $\mathrm{Sp}(A)$.

To see this, let $x \in \mathbb{R}^K$ and find (unique) scalars such that

$$\sum_{\ell=1}^{L} \alpha_\ell u_\ell + \sum_{j=1}^{K-L} \beta_j v_j = x.$$

Then,

$$Ax = \sum_{\ell=1}^{L} \alpha_\ell A u_\ell + \sum_{j=1}^{K-L} \beta_j A v_j = \sum_{j=1}^{K-L} \beta_j A v_j.$$

Since x was arbitrary, it follows that $\mathrm{Sp}\{Av_1, \dots, Av_{K-L}\} = \mathrm{Sp}(A)$. To complete the argument, it remains to show that $\{Av_1, \dots, Av_{K-L}\}$ is linearly independent. For this, suppose that $\sum_{j=1}^{K-L} \gamma_j A v_j = 0$. Then, $A \sum_{j=1}^{K-L} \gamma_j v_j = 0$, which means that $\sum_{j=1}^{K-L} \gamma_j v_j \in \ker(A)$. We can then find scalars such that

$$\sum_{j=1}^{K-L} \gamma_j v_j = \sum_{\ell=1}^{L} \delta_\ell u_\ell,$$

or

$$\sum_{j=1}^{K-L} \gamma_j v_j + \sum_{\ell=1}^{L} (-\delta_\ell) u_\ell = 0.$$

Since $\{u_1, \dots, u_L, v_1, \dots, v_{K-L}\}$ is a basis for $\mathbb{R}^K$, it follows that

$$\gamma_1 = \ldots = \gamma_L = \delta_1 = \ldots = \delta_{K-L} = 0,$$

as needed.

The second part is left as an exercise. $\square$

[2] So $L = \mathrm{nul}(A)$.

Note that, necessarily, $\mathrm{rank}(A) \leqslant \min(K, J)$. Intuitively, the rank is the largest number of linearly independent rows or columns of the matrix.

6.5 Inverse Matrix

A $K \times K$ MATRIX A IS SAID to be *invertible* if there exists another $K \times K$ matrix B such that $BA = AB = \mathbb{I}$. In this case, matrix B is said to be *the inverse*[3] of A and is denoted by A^{-1}.

[3] Notice that we are saying *the*, and not *an* inverse. The reason is that a matrix can have at most one inverse: Suppose that

$$BA = AB = \mathbb{I} = AC = CA.$$

Then,

$$(B - C)A = 0,$$

so

$$(B - C)AB = 0,$$

and

$$(B - C)\mathbb{I} = 0.$$

Obviously, this means that $B = C$. It follows that if B and C are inverses of A, then $B = C$.

EXERCISE 6.5.1. *Prove that if A is invertible, then so is A^{-1}, and $(A^{-1})^{-1} = A$.*

EXERCISE 6.5.2. *Prove that if A and B are invertible $K \times K$ matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

The second part of the Fundamental Theorem of Linear Algebra says the following:

THEOREM 6.5.1 (The Fundamental Theorem of Linear Algebra, II). *Given a $K \times K$ matrix A, the following statements are equivalent:*

1. *A is invertible.*
2. *$\det(A) \neq 0$.*
3. *$\operatorname{rank}(A) = K$.*
4. *the columns of A, $(a_k)_{k=1}^K$, are linearly independent.*

A quick way to see that the first two claims of this theorem are equivalent is to verify, by direct computation, that given invertible A,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

where

$$\operatorname{adj}(A) = \begin{pmatrix} \bar{a}_{1,1} & \bar{a}_{1,2} & \cdots & \bar{a}_{1,K} \\ \bar{a}_{2,1} & \bar{a}_{2,2} & \cdots & \bar{a}_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{K,1} & \bar{a}_{K,2} & \cdots & \bar{a}_{K,K} \end{pmatrix}^{\top}$$

and

$$\bar{a}_{j,l} = [(-1)^{j+l} \det(A^{-(j,l)})]$$

The number $\bar{a}_{j,\ell}$ is known as the (j,ℓ) *co-factor of* A. Matrix $\operatorname{adj}(A)$ is called the *adjugate matrix* of A, and its transpose is the *co-factor matrix*.

An important implication of the theorem is that a linear function $f : \mathbb{R}^J \to \mathbb{R}^K$ is bijective if, and only if, the matrix A that generates it is invertible. In such case, the inverse function is $f^{-1}(y) = A^{-1}y$, which is linear too.

6.6 Eigenvalues and Eigenvectors

LET A $K \times K$ MATRIX A BE FIXED. An *eigenvector* of A is $x \in \mathbb{R}^K \setminus \{0\}$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{R}$. The λ associated to an eigenvector is called *its eigenvalue*.[4] Intuitively, an eigenvector is such that the linear transformation simply re-scales it by its associated eigenvalue. This implies that eigenvectors can only be determined up to scalar multiplication.

[4] Less common names are *characteristic vector* and *characteristic value*. In Latin languages, they are also called "auto" vector and "auto" value, with the same semantic root as the German "eigen". The reason why the German term remains used in English, I imagine, is that Euler was probably the first person to discover these objects. Lagrange discovered them independently, but a few years later.

The system that defines them eigenvectors and eigenvalues of matrix A is

$$(A - \lambda\mathbb{I})x = 0,$$

and is known as the matrix's *characteristic equation*. We are interested in solutions $x \neq 0$, so we can find the eigenvalues by solving the equation

$$\det(A - \lambda\mathbb{I}) = 0,$$

which is known as the matrix's *characteristic function*. Notice that it is a K^{th}-degree polynomial, so it has at most K real roots.[5]

Three key results are the following: denote by $\lambda_1, \dots, \lambda_K$ the eigenvalues of A; then,

1. $\sum_{k=1}^{K} \lambda_k = \sum_{k=1}^{K} a_{k,k}$;[6]
2. $\prod_{k=1}^{K} \lambda_k = \det(A)$; and
3. the number of non-zero eigenvalues of A is $\operatorname{rank}(A)$.

[5] And, in fact, has K roots, whether real or imaginary.

[6] This number $\sum_{k=1}^{K} a_{k,k}$ is known as *the trace of* A and is denoted $\operatorname{Tr}(A)$.

6.7 Quadratic Forms

Let a $K \times K$ matrix A be fixed, once again. The quadratic form associated to A is the function $f_A : \mathbb{R}^K \to \mathbb{R}; f_A(x) = x^\top Ax$.

Definition 6.7.1. *Matrix A is:*

1. negative definite *if for all* $x \in \mathbb{R}^K \setminus \{0\}$, $f_A(x) < 0$;
2. negative semidefinite *if for all* $x \in \mathbb{R}^K$, $f_A(x) \leqslant 0$
3. positive definite *if for all* $x \in \mathbb{R}^K \setminus \{0\}$, $f_A(x) > 0$; *and*
4. *negative semidefinite if for all* $x \in \mathbb{R}^K$, $f_A(x) \geqslant 0$.

Exercise 6.7.1. *Show that $\mathbb{I}$ is positive definite and that matrix A is positive (semi-)definite if, and only if, $-A$ is negative (semi-)definite.*

Exercise 6.7.2. *Show that A is negative semidefinite if, and only if, f_A satisfies the following property: for all $x, x' \in \mathbb{R}^K$, for all $\vartheta \in [0, 1]$,*

$$f_A(\vartheta x + (1-\vartheta)x') \geqslant \vartheta f_A(x) + (1-\vartheta) f_A(x').$$

A very useful result is the following: A is positive definite if all its eigenvalues are (strictly) positive, and negative definite if all its eigenvalues are (strictly) negative; moreover, it is positive semidefinite, but not definite, if all its eigenvalues are non-negative and at least one is zero, and negative semidefinite, but not definite, if all its eigenvalues are nonpositive and at least one is zero.

The last observations and the second result about eigenvalues make the following theorems very intuitive. Given a $K \times K$ matrix A, define its *principal minor of order* $k = 1, \dots, K$ as the number

$$\det \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k} \end{pmatrix}$$

that is, the determinant of the sub-matrix resulting from keeping the first k rows and columns, dropping the rest. Also, the determinant of *any* sub matrix resulting from keeping k rows and the corresponding columns, and dropping the rest of the matrix, is called *a minor of* A *of order* k.

THEOREM 6.7.1. A *is negative definite if, and only if, each principal minor of order* k *of* A *has sign* $(-1)^k$. *It is positive definite if, and only if, all principal minors are positive.*

THEOREM 6.7.2. A *is negative semidefinite if, and only if,* all minors *of order* k *of* A *have sign* $(-1)^k$ *or* 0. *It is positive semidefinite if, and only if, all minors are nonnegative.*

6.8 Linear Systems of Inequalities

A VERY IMPORTANT RESULT in linear analysis, known as *The Theorem of the Alternative* or the *Minkowski–Farkas Lemma*, studies the existence of solutions to linear systems. It says:[7]

THEOREM 6.8.1 (Minkowski–Farkas). *Let* A *be a given* $K \times J$ *matrix. One, and only one, of the following two statements is true:*

1. *there exists column vector* $x \in \mathbb{R}^J$ *such that* $Ax > 0$ *(resp.* $Ax \gg 0$*); or,*
2. *there exists a row vector* $y \in \mathbb{R}^K_{++}$ *(resp.* $y \in \mathbb{R}^K_+ \setminus \{0\}$*) such that* $yA = 0$.

This theorem is proved in an exercise of the following chapter. The next section offers an important application of it in economics.

6.9 Application: Testing Consumer Theory

THE STANDARD FOR WHAT is to be considered scientific knowledge has been a prominent topic of debate in epistemology. Karl Popper argued that scientists should actively try to prove their theories wrong, rather than merely attempt to verify them through inductive reasoning. The Popperian postulate thus states that a scientific discovery ought to distinguish the theory from its empirical implications and that the empirical implications should be contrasted to reality, in order for the theory to be corroborated (however, not verified) or refuted. If a theory fails a test, and there exists no reasonable excuse that can itself be tested, then the theory should be abandoned.

This "empiricist" position, often referred to as "falsificationism," had been previously exposed by Poincaré who, in 1908, wrote that

> [W]hen a theory has been established, we have first to look for cases in which the rule stands the best chance of being found at fault.

This principle was introduced to economics by Paul Samuelson, for whom "meaningful theorems" are hypotheses about "empirical data which could conceivably be refuted.[8] It seems desirable to obtain testable implications from the equilibrium concepts in economics, even if one considers the views of Popper to be an extreme. We now consider this problem for the case of a consumer in the context of Section 4.5.

[7] Note that ϑ is taken as a column vector, while Π is a row vector.

[8] P. Samuelson (1947, *Foundations of Economic Analysis*, p.4).

Suppose that we have observed a data set $D = \{(x_t, p_t, m_t)\}_{t=1}^T$, with $T \in \mathbb{N}$, where for all t, $p_t \in \mathbb{R}^L_{++}$, $m_t \in \mathbb{R}_+$, $x_t \in B(p_t, m_t)$, and $p_t \cdot x_t = m_t$.

We shall say that (utility) function $u : \mathbb{R}^L_+ \to \mathbb{R}$ *rationalizes data set* D, if

$$\operatorname{argmax}_{x \in B(p_t, m_t)} u(x) = \{x_t\}$$

for all t. We shall also say that D satisfies the *Strong Axiom of Reveaked Preferences, SARP,* if for any finite sequence $(x^k, p^k, m^k)_{k=1}^K$ definied in D we have that if $p^k \cdot x^{k+1} \leqslant m^k$, for all $k = 1, \dots, K-1$, and $x^1 \neq x^K$, then $p^K \cdot x^1 > m^1$.

THEOREM 6.9.1. *Let* $D = \{(x_t, p_t, m_t)\}_{t=1}^T$ *be a data set such that* $x_t \neq x_{t'}$ *whenever* $t \neq t'$.[9] *The next statements are equivalent:*

[9] This assumption is made only for simplicity of the argument that follows.

1. *There exists a function* $u : \mathbb{R}^L_+ \to \mathbb{R}$ *that rationalizes* D.

2. D *satisfies SARP.*

3. *There exist numbers* $(\lambda_t, \mu_t) \in \mathbb{R}_{++} \times \mathbb{R}$, *for each* $t \in \{1, \dots, T\}$, *such that,*

$$\mu_t < \mu_{t'} + \lambda_{t'} p_{t'} \cdot (x_t - x_{t'}),$$

   whenever $t \neq t'$.

4. *There exist a continuous, strictly concave, strictly monotonic function* $u : \mathbb{R}^L_+ \to \mathbb{R}$ *that rationalizes* D.

Proof: Note that it suffices for us to show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

Proof that $1 \Rightarrow 2$*:* Let $(x^k, p^k, m^k)_{k=1}^K$ be a finite sequence defined in D, such that the following conditions are satisfied:

(i) for each $k = 1, \dots, K-1$, $p^k \cdot x^{k+1} \leqslant m^k$;

(ii) $x^1 \neq x^K$; and

(iii) $p^K \cdot x^1 \leqslant m^1$.

Let function $u : \mathbb{R}^L_+ \to \mathbb{R}$ rationalize D. Condition (i) implies that $u(x^{k+1}) \leqslant u(x^k)$ for all $k = 1, \dots, K-1$. It immediately follows that $u(x^K) \leqslant u(x^1)$. Since, by (iii), $p^K \cdot x^1 \leqslant m^1$ and

$$x^K \in \operatorname{argmax}_{x \in B(p^K, m^K)} u(x),$$

we further have that

$$x^1 \in \operatorname{argmax}_{x \in B(p^K, m^K)} u(x).$$

But, then, (ii) contradicts the fact that

$$\operatorname{argmax}_{x \in B(p^K, m^K)} u(x) = \{x^K\}.$$

Proof that $2 \Rightarrow 3$*:* Define the $T(T-1) \times 2T$ matrix

$$W = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & p_1 \cdot (x_2 - x_1) & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 & p_1 \cdot (x_3 - x_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -1 & p_1 \cdot (x_T - x_1) & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & p_2 \cdot (x_1 - x_2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & p_T \cdot (x_{T-1} - x_T) \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

and suppose that there is no vector

$$\vartheta = (\mu_1, \mu_2, \mu_3, \dots, \mu_T, \lambda_1, \lambda_2, \dots, \lambda_T)^T$$

such that $W\vartheta \gg 0$. Then, by Theorem 6.8.1,[10] there exist two arrays of non-negative numbers $\{\pi_{t,t'} \mid t = 1, \dots, T \text{ and } t \neq t'\}$ and $\{\gamma_t \mid t = 1, \dots, T\}$, at least one of which is strictly positive, such that:

(i) For all t, $\sum_{t' \neq t} \pi_{t,t'} = \sum_{t' \neq t} \pi_{t',t}$; and

(ii) For all t, $\sum_{t' \neq t} \pi_{t,t'} p_t \cdot (x_{t'} - x_t) + \gamma_t = 0$.

If $\pi_{t,t'} = 0$ for all t and t', condition (ii) implies that $\gamma_t = 0$ for all t, which is impossible. It follows that for some t_1, there exists $t_2 \neq t_1$ for which $\pi_{t_1,t_2} > 0$. Since $\gamma_{t_1} \geqslant 0$, condition (ii) implies that we can fix t_2 such that $\pi_{t_1,t_2} > 0$ and $p_{t_1} \cdot (x_{t_2} - x_{t_1}) \leqslant 0$.

Now, by condition (i), there exists $t_3 \neq t_2$ such that $\pi_{t_2,t_3} > 0$. As before, $\gamma_{t_2} \geqslant 0$ and condition (ii) imply that we can fix t_3 such that $\pi_{t_2,t_3} > 0$ and $p_{t_2} \cdot (x_{t_3} - x_{t_2}) \leqslant 0$. If $t_3 = t_1$, we have a contradiction of WARP, so it must be that $t_3 \neq t_1$.

Using condition (i) again, as before there must exist $t_4 \neq t_3$ such that $\pi_{t_3,t_4} > 0$ and $p_{t_3} \cdot (x_{t_4} - x_{t_3}) \leqslant 0$. If $t_3 = t_1$ or $t_3 = t_2$, we again have a contradiction of WARP. To avoid this contradiction it must be that $t_4 \neq t_1$ and $t_4 \neq t_2$, and we can proceed as before. But since $T \in \mathbb{N}$, after at most T steps the contradiction of WARP cannot be avoided.

Proof that $3 \Rightarrow 4$*:* Define function $h : \mathbb{R}^L \to \mathbb{R}$, by

$$h(x) = \sqrt{\|x\|^2 + 1} - 1.$$

This function is differentiable, strictly concave and satisfies that

(i) $h(x) = 0$ if, and only if, $x = 0$;

(ii) $h(x) > 0$ for all $x \neq 0$; and

(iii) for all $l \in \{1, \dots, L\}$, and all x,

$$\frac{\partial h}{\partial x_l}(x) \in [0, 1)$$

[10] For simplicity, we rephrase under the current notation:

> *Let* W *be a* $T(T-1) \times 2T$ *matrix. One, and only one, of the following statements is true:*
>
> 1. *there exists* $\vartheta \in \mathbb{R}^{2T}$ *such that* $W\vartheta \gg 0$*; or,*
> 2. *there exists* $\Pi \in \mathbb{R}_+^{T(T-1)}$, $\Pi \neq 0$*, such that* $\Pi W = 0$.

Since $T \in \mathbb{N}$, there exists $\varepsilon \in \mathbb{R}_{++}$ such that for all t, t',

$$\mu_{t'} < \mu_t + \lambda_t p_t \cdot (x_{t'} - x_t) - \varepsilon h(x_{t'} - x_t).$$

Now, for each t, define function $\phi_t : \mathbb{R}^L \to \mathbb{R}$ por

$$\phi_t(x) = \mu_t + \lambda_t p_t \cdot (x - x_t) - \varepsilon h(x - x_t),$$

which is strictly concave. Note also that for all l, la cual es estríctamente cóncava. Además, note que

$$\frac{\partial \phi_t}{\partial x_l}(x) = \lambda_t p_{t,l} - \varepsilon \frac{\partial h}{\partial x_l}(x - x_t) > \lambda_t p_{t,l} - \varepsilon,$$

so one can take $\varepsilon > 0$ small enough that ϕ_t is strictly monotonic.[11]

Now, define the utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$ by

$$u(x) = \min_{t \in \{1,\dots,T\}} \{\phi_t(x)\}.$$

This function is continuous, strictly concave and strictly monotone,[12] so all that remains to show is that

$$\operatorname{argmax}_{x \in B(p_t, m_t)} u(x) = \{x_t\},$$

for all t. By definition of D, $x_t \in B(p_t, m_t)$. Also, note that $u(x_t) \geqslant \mu_t$, since otherwise there would be t' such that

$$\phi_{t'}(x_t) = \mu_{t'} + \lambda_{t'} p_{t'} \cdot (x_t - x_{t'}) - \varepsilon h(x_t - x_{t'}) < \mu_t,$$

which would contradict the definition of ε. Also, by definition there exists t' for which

$$u(x_t) = \phi_{t'}(x_t) \leqslant \phi_t(x_t) = \mu_t.$$

These last two results imply that $u(x_t) = \mu_t$.

Now, suppose that $x \in \mathbb{R}^L_+ \setminus \{x_t\}$ satisties that $p_t \cdot x \leqslant m_t$. Then,

$$u(x) = \min_{t' \in \{1,\dots,T\}} \{\phi_{t'}(x)\} \leqslant \phi_t(x) = \mu_t + \lambda_t p_t \cdot (x - x_t) - \varepsilon h(x - x_t).$$

Then, as

$$\lambda_t p_t \cdot x \leqslant \lambda_t m_t = \lambda_t p_t \cdot x_t$$

we have that $\lambda_t p_t \cdot (x - x_t) \leqslant 0$. And since $\varepsilon h(x - x_t) > 0$, moreover $u(x) < \mu_t = u(x_t)$, as needed.

Proof that $4 \Rightarrow 1$*:* This is obvious. *Q.E.D.*

[11] Of course, we can do this since $L \in \mathbb{N}$ and $T \in \mathbb{N}$.

[12] It is also differentiable, except at T points. These kinks can be smoothed using a technique called a convolution, which is beyond these notes.

7
Convex Analysis

THROUGHOUT THIS CHAPTER, WE maintain the assumption that $X \subseteq \mathbb{R}^K$, $K \in \mathbb{N}$ and $X \neq \varnothing$.

7.1 *Convex Sets and Separation*

SET X IS SAID to be *convex* if for all $x, y \in X$ and for all $\vartheta \in [0,1]$, we have that $\vartheta x + (1-\vartheta)y \in X$.

EXERCISE 7.1.1. *Show that $X = \{x\}$ is convex. Show that for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, $B_\varepsilon(x)$ is convex. Is $\mathbb{N}$ convex? Is $\varnothing$ convex? Is $\mathbb{R}$ convex? Show that if X and Y are convex, $X \cap Y$ is convex. Is $X \cup Y$ convex in that same case?*

A key result in mathematical economics is the following theorem.

THEOREM 7.1.1 (The separating hyperplane theorem). *If $Q, Q' \subseteq \mathbb{R}^A$ are disjoint and convex, there exist $\vartheta \in \mathbb{R}^A \setminus \{0\}$ and $k \in \mathbb{R}$ such that $q\vartheta \leqslant k$ for all $q \in Q$, and $q\vartheta \geqslant k$ for all $q \in Q'$.*

The reason why this result is called the *separating hyperplane* theorem is that $\vartheta \in \mathbb{R}^A \setminus \{0\}$ and $k \in \mathbb{R}$ define the set

$$\{q \in \mathbb{R}^A \mid q\vartheta = k\},$$

is indeed a plane if $A = 3$ For general A, this set is known as a *hyperplane*. Vector ϑ is called the hyperplane *normal* vector, since it is orthogonal to any vector of the form $(q - q')$, for q, q' in the set. Scalar k determines the position of the hyperplane. Once the hyperplane is defined, $\mathbb{R}^A$ is split in two *half-spaces*, one on each side of the hyperplane. The theorem says that any two disjoint convex sets can be separated, in the sense that we can find a hyperplane that leaves each set on a different hyperplane.

For the sake of simplicity, we shall give the proof of a weaker version of the theorem, based on the following result.

LEMMA 7.1.1. *If $Q \subseteq \mathbb{R}^A$ is nonempty, closed and convex, and $\bar{q} \in \mathbb{R}^A \setminus Q$, there exist $q^* \in Q$, $\vartheta \in \mathbb{R}^A \setminus \{0\}$ and $k \in \mathbb{R}$ such that $\bar{q}\vartheta < k$, $q^*\vartheta = k$ and $q\vartheta \geqslant k$ for all $q \in Q$.*

We have to defer to the next chapter the proof of this lemma. A simple consequence of it is the following:

Proof of a weak version of Theorem 7.1.1: For a simple argument, let us prove the following statement:

> *If* $Q, Q' \subseteq \mathbb{R}^A$ *are disjoint, closed and convex, there exists* $\vartheta \in \mathbb{R}^A \setminus \{0\}$ *such that* $q\vartheta \leqslant q\vartheta$ *for all* $q \in Q$ *and all* $q' \in Q'$.

For this, define the set

$$\Delta = Q - Q' = \{\delta \in \mathbb{R}^A \mid \exists (q, q') \in Q \times Q' : q - q' = \delta\}.$$

This set is nonempty, closed and and convex. Since $Q \cap Q' = \varnothing$, $0 \notin \Delta$. By Lemma 7.1.1, there exist $\delta^* \in \Delta$, $\vartheta \in \mathbb{R}^A \setminus \{0\}$ and $\kappa > 0$, such that $\delta^* \vartheta = \kappa$ and $\delta\vartheta \geqslant \kappa$ for all $\delta \in \Delta$. Now, for all $q \in Q$ and all $q' \in Q'$, $(q - q') \in \Delta$, so $(q - q')\vartheta \geqslant \kappa > 0$. □

A stronger version of the theorem of separation of convex sets is the following:

THEOREM 7.1.2. *Let* $Q, Q' \subseteq \mathbb{R}^T$ *be disjoint and convex. If* Q *is closed and* Q' *is compact, then there exist* $\vartheta \in \mathbb{R}^T \setminus \{0\}$ *and* $k \in \mathbb{R}$ *such that for all* $q \in Q$, $q\vartheta < k$, *and for all* $q \in Q'$, $q\vartheta > k$.

EXERCISE 7.1.2. *In this exercise you are going to prove Theorem 6.8.1. First, define the sets*

$$\langle W \rangle = \{\rho \in \mathbb{R}^T \mid \exists \vartheta \in \mathbb{R}^A : W\vartheta = \rho\},$$

which is known as the column span of W*; and the* $(T-1)$*-dimensional simplex,*

$$\Delta = \{\rho \in \mathbb{R}^T_+ \mid \rho \cdot (1, \dots, 1) = 1\}.$$

The following steps will give you the desired proof.

1. *Argue that if* $\langle W \rangle \cap \Delta = \varnothing$, *then there exists some* $\Pi \in \mathbb{R}^T \setminus \{0\}$ *such that for all* $\rho \in \langle W \rangle$ *and all* $\rho' \in \Delta$, $\Pi\rho < \Pi\rho'$.

2. *Fix the* $\Pi \in \mathbb{R}^T$ *found in the previous step. Noting that* $0 \in \langle W \rangle$, *argue that for any* $\rho' \in \Delta$, $0 < \Pi\rho'$.

3. *Argue that the previous result implies that* $\Pi \gg 0$.

4. *Noting that* $(1/T, \dots, 1/T) \in \Delta$, *argue that, for all* $\vartheta \in \mathbb{R}^A$,

$$(\Pi W)\vartheta < \frac{1}{T}\sum_{t=1}^{T} \Pi_t.$$

5. *Argue that the previous result implies that* $\Pi W = 0$.

6. *With respect to the two statements in the Theorem of the Alternative, argue now that if the first statement is* not *true, then the second statement must be true.*

7. *Again with respect to the two statements in the theorem, argue that they cannot be true at the same time.*

7.2 Application: No-Arbitrage Pricing in Financial Economics

CONSIDER AN ECONOMY IN WHICH there are only two periods: present and future. A *state of nature* is a comprehensive description of the world. In the future one of S states of nature can occur.[1] Denote by $\mathcal{S} = \{1, \ldots, S\}$ the set of dates – technically, the *sample space*. For simplicity of notation, we will refer to the present as state $s = 0$.

[1] In the words of K. Arrow (1971, *Essays on the Theory of Risk Bearing*, p. 45), it is "a description of the world so complete that, if true and known, the consequences of every action would be known."

An *asset* is a promise to pay a certain return, which may depend on the state of the world: it is a random variable $r : \mathcal{S} \to \mathbb{R}$, with $r(s)$ being the return in state s. In this simple setting, we can simply identify an asset r with a (column) vector in $\mathbb{R}^S$, namely with

$$r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_S \end{pmatrix} = \begin{pmatrix} r(1) \\ r(2) \\ \vdots \\ r(S) \end{pmatrix}.$$

We assume that there are A assets, which we index by $a \in \mathcal{A} = \{1, \ldots, A\}$ and denote by $(r^1, r^2, \ldots, r^A)$. We will follow the convention of using superscripts to denote assets and subscripts to denote states of nature:

$$r^a = \begin{pmatrix} r_1^a \\ r_2^a \\ \vdots \\ r_S^a \end{pmatrix}.$$

The return matrix, or *financial market*, is $R = (r^1, r^2, \ldots, r^A)$, an $S \times A$ matrix. We also denote

$$R = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_S \end{pmatrix},$$

so that $r_s \in \mathbb{R}^A$ is the vector of returns of all assets in state s, taken, by convention, as a row.

The following conditions are assumed:

CONDITION 1. *There exists a positive asset:* $r^1 > 0$.

CONDITION 2. *There are no redundant assets:* $\operatorname{rank}(R) = A$.

The first condition implies little loss of relevance: cash, or sovereign debt, are considered to be positive. The second condition implies some loss of generality, but the theory is well equipped to deal with this, as we will later see. The condition implicitly requires that $A \leqslant S$.

The column span of the return matrix, $\langle R \rangle$, is called the *space of feasible revenue transfers*, or the space of admissible claims. For a given revenue transfer $\rho \in \langle R \rangle$, the (column) vector $\vartheta \in \mathbb{R}^A$ such that $\rho = R\vartheta$ is a portfolio with return ρ, or an investment strategy that delivers ρ.

For each asset $a \in \mathcal{A}$, we denote by $q^a \in \mathbb{R}$ its price in the present market. The vector of asset prices is $q = (q^1, q^2, \ldots, q^A)$, which we take as a row.

7.2.1 No arbitrage

We say that *an arbitrage opportunity exists at prices* q if one can make money for sure and for free: there is a portfolio ϑ such that $R\vartheta > 0$ and $q\vartheta \leqslant 0$ – that is, there is a portfolio that makes no losses, and does win money in a certain state, and which people can get for free, or even be paid to get it.

EXERCISE 7.2.1. *Show that if there exists ϑ such that $R\vartheta \geqslant 0$ and $q\vartheta < 0$, then an arbitrage opportunity exists at prices q.*

DEFINITION 7.2.1. *We say that* q allows no arbitrage opportunities *if $R\vartheta > 0$ implies $q\vartheta > 0$.*

The basic theory of asset pricing is that markets in equilibrium cannot allow for arbitrage opportunities. We, therefore, concentrate on the set of prices that allow no arbitrage opportunities:

$$Q = \{q \in \mathbb{R}^A \mid R\vartheta > 0 \implies q\vartheta > 0\}.$$

EXERCISE 7.2.2. *Show that Q_1 is nonempty and convex, is a (positive) cone*[2] *and satisfies that for all $q \in Q_1$, $q^1 \in \mathbb{R}_{++}$.*

[2] That is, for all $q \in Q_1$, for all $\alpha \in \mathbb{R}_{++}$, $\alpha q \in Q_1$

EXAMPLE 7.2.1. *Suppose that there are two assets and two states in the economy, so that $S = A = 2$. Let the financial market be*

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Figure 7.1 shows the area of portfolios with positive returns. Two portfolios there characterize this area: $\bar{\vartheta} = (0, 1)$ and $\hat{\vartheta} = (-1, 1)$. These two portfolios have to be priced strictly positively, for otherwise they would be arbitrage opportunities. This, then, requires that $q\bar{\vartheta} = q^2 > 0$, and that $q\hat{\vartheta} = -q^1 + q^2 > 0$. It is immediate that

$$Q_1 = \{q \in \mathbb{R}^2 \mid q^2 > q^1 \text{ and } q^2 > 0\}.$$

To verify this result, it suffices to notice that if $q^2 \leqslant q^1$, then $\vartheta = (-1, 1)^\top$ is an arbitrage opportunity; while if $q^2 \leqslant 0$, then $\vartheta = (0, 1)^\top$ is an arbitrage opportunity.

EXAMPLE 7.2.2. *Suppose now that there are three states and two assets (namely, $S = 3$ and $A = 2$), and let*

$$R = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}$$

be the financial market.

Figure 7.2 shows the set of portfolios with positive return. Using some reference portfolios, as in Example 7.2.1, we get that

$$Q_1 = \left\{q \in \mathbb{R}^2 \mid \tfrac{1}{3}q^1 < q^2 < 3q^1\right\} \subseteq \mathbb{R}^2_{++}.$$

Once again, a good sanity check is to find arbitrage opportunities for typical prices outside set Q_1. Notice that if $q^2 \leqslant \frac{1}{3}q^1$, then $\vartheta = (-1, 3)^\top$ is an arbitrage opportunity; if $q^2 \geqslant 3q^1$, then so is $\vartheta = (3, -1)^\top$; if $q^1 \leqslant 0$, then so is $\vartheta = (1, 0)^\top$; and if $q^2 \leqslant 0$, then $\vartheta = (0, 1)^\top$ is an arbitrage opportunity.

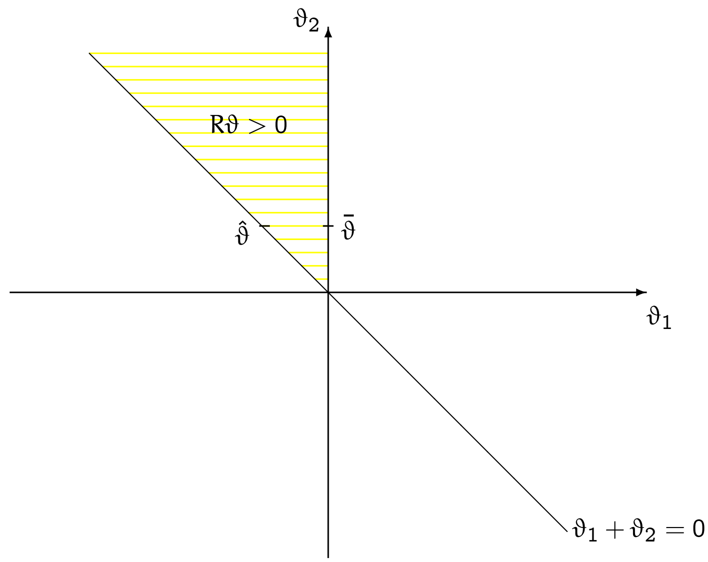

Figure 7.1: Space of positive returns in Example 7.2.1.

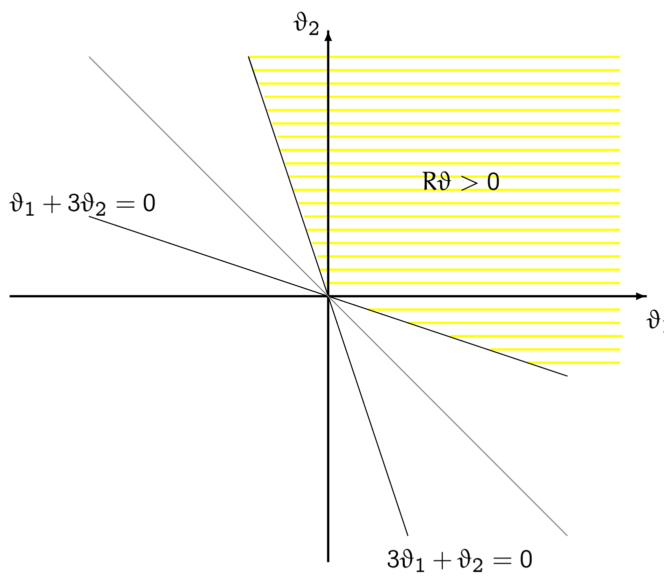

Figure 7.2: Space of positive returns in Example 7.2.2.

It turns out that the set of no-arbitrage prices is closely related to another set of asset prices: those that can be explained as discounted expected returns of the assets. For reasons that will be clear below, we define as *rationalizable* all vectors of asset prices in set

$$Q_2 = \{q \in \mathbb{R}^A \mid \exists \pi \in \mathbb{R}^S_{++} : \pi R = q\}.$$

It turns out that Q_1 is closely related to another set of asset prices: those that can be explained as discounted expected returns of the assets. Let

$$Q_2 = \{q \in \mathbb{R}^A \mid \exists \pi \in \mathbb{R}^S_{++} : \pi R = q\},$$

where π is a measure (a generalization of a probability) and is taken as a row vector.

EXERCISE 7.2.3. *Show that* Q_2 *is nonempty and convex, is a (positive) cone and satisfies that for all* $q \in Q_2$, $q^1 \in \mathbb{R}_{++}$.

EXAMPLE 7.2.3. *Consider the case of Example 7.2.1 above. It follows by direct computation that*

$$Q_2 = \{q \in \mathbb{R}^2 \mid q^2 > q^1 \text{ and } q^2 > 0\}.$$

Similarly, in the case of Example 7.2.2, notice that

$$Q_2 = \left\{q \in \mathbb{R}^2 \mid \tfrac{1}{3}q^1 < q^2 < 3q^1\right\}.$$

EXERCISE 7.2.4. *Let* $S = A = 2$, *and*

$$R = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}.$$

1. *Find* Q_1 *and* Q_2.
2. *Find an arbitrage opportunity for each of the following prices:* $(1,1)$, $(-1,1)$, $(-1,-1)$, *and* $(1,-1)$.

EXERCISE 7.2.5. *Let* $S = 3$ *and* $A = 2$, *and*

$$R = \begin{pmatrix} 1 & 2 \\ -2 & 3 \\ 6 & 1 \end{pmatrix}.$$

1. *Find* Q_1 *and* Q_2.
2. *Let* $q \notin Q_1$ *and find an arbitrage opportunity.*
3. *For the same* q *as in 3, find a portfolio with nonnegative returns in both states and strictly negative cost, and show that this portfolio can be used to define an arbitrage opportunity.*

7.2.2 State prices

Given the previous examples and exercises, it should not be surprising that Q^1 and Q^2 are closely related.

THEOREM 7.2.1. *Under Conditions 1 and 2,* $Q_1 = Q_2$.

Proof. Suppose first that $\bar{q} \in Q_1 \setminus Q_2$. Let $Q = \{\bar{q}\}$ and $Q' = Q_2$. By the separating hyperplane theorem, there exist $\vartheta \in \mathbb{R}^A \setminus \{0\}$ and $k \in \mathbb{R}$ such that $\vartheta\bar{q} \leqslant k$ and for all $q \in Q_2$, while $\vartheta q \geqslant k$. Fix one such (ϑ, k), and observe the following:

1. If $k > 0$, then for $\hat{q} \in Q'$, if $\alpha > 0$ is close enough to 0, then $\vartheta(\alpha\hat{q}) = \alpha\vartheta\hat{q} < k$, which is impossible because $\alpha\hat{q} \in Q'$; so, $k \leqslant 0$.
2. If there is $\hat{q} \in Q'$ such that $\vartheta\hat{q} < 0$, then, for $\alpha > 0$ large enough, $\vartheta(\alpha\hat{q}) = \alpha\vartheta\hat{q} < k$, which is impossible because $\alpha\hat{q} \in Q'$. So, we conclude that for all $q \in Q'$, $\vartheta q \geqslant 0$, from where it follows that $\vartheta\bar{q} \leqslant 0$ and for all $q \in Q'$, $\vartheta q \geqslant 0$.
3. If $r_{\hat{s}}\vartheta < 0$, then, by defining $\pi \in \mathbb{R}^S_{++}$ by $\pi_{\hat{s}} = 1 - \epsilon$ and for all $s \in \mathcal{S} \setminus \{\hat{s}\}$, $\pi_s = \epsilon$, with $\epsilon \in (0,1)$, we get that, if ϵ is small enough, $q\vartheta = \pi R\vartheta < 0$, which is impossible because $q = \pi R \in Q'$; so, it follows that $R\vartheta \geqslant 0$.

The latter implies that $R\vartheta > 0$, because $\vartheta \neq 0$ and Condition 2 holds. But, now, $R\vartheta > 0$ and $\bar{q}\vartheta \leqslant 0$, which is impossible because $\bar{q} \in Q_1$. It follows that $Q_1 \subseteq Q_2$.

Now, suppose that $q \in Q_2$. Then, for some $\pi \in \mathbb{R}^S_{++}$ we have that $\pi R = q$. Now, it is immediate that if $R\vartheta > 0$, then $q\vartheta = \pi R\vartheta > 0$, which implies that $q \in Q_1$. Then, $Q_2 \subseteq Q_1$. □

The vector of $\pi \in \mathbb{R}^S_{++}$ such that $\pi R = q$, given q that allows no arbitrage opportunities, is the key variable in asset pricing theory (it is called an equivalent martingale measure). Before we go further, it is important to understand what this variable measures.

Suppose that $r_{\hat{s}}\vartheta = 1$ and $r_s\vartheta = 0$ for every $s \in \S \setminus \{\hat{s}\}$.[3] If q allows no arbitrage, then $\pi R = q$ implies that $\vartheta q = \pi_{\hat{s}}$. This means that π_s is the cost of a portfolio that delivers 1 if state $\hat{s}$ realizes and nothing otherwise: it is the marginal cost of one unit of revenue in state $\hat{s}$. For this reason, π is called the vector of *state prices*.

[3] That is, portfolio ϑ pays 1 if state $\hat{s}$ realizes and nothing otherwise; when that portfolio consists of just one unit of a given asset, such asset is called an Arrow, or elementary, security.

Now, suppose that portfolio ϑ is such that $r_s\vartheta = 1 + i$, for some constant $i \in \mathbb{R}_+$. This means that portfolio ϑ has a risk-less return of $1 + i$. If q allows no arbitrage, then $\pi R = q$ implies that $\vartheta q = \sum_{s=1}^S \pi_s(1+i)$, so if i is taken to be an interest rate paid on an investment of cost 1, without risk, it follows that $q\vartheta = 1$, and so $\sum_{s=1}^S \pi_s = (1+i)^{-1}$. Let $\pi_0 = \sum_{s=1}^S \pi_s$. By our previous result, it follows that π_0 is the discount factor associated to interest rate i.

7.2.3 *The fundamental theorem of asset pricing*

Clearly, state prices need not be well defined probabilities: $\sum_{s=1}^S \pi_s = 1$ is not guaranteed. Importantly, if we now define $p \in \mathbb{R}^S_{++}$ by

$$p = \left(\frac{\pi_1}{\pi_0}, \frac{\pi_2}{\pi_0}, \cdots, \frac{\pi_S}{\pi_0}\right),$$

it follows that p is a legitimate vector of probabilities (it is crucial to notice that we do not imply that these are the real probabilities, or some agent's beliefs).

Before proceeding further, we strengthen Condition 1 to:

Condition 3. *There is a riskless asset:* $r^1 = (1, 1, \cdots, 1)^\top$.

Theorem 7.2.2 (The fundamental theorem of asset pricing). *Let* $q \in Q_1$ *and* $\rho \in \langle R \rangle$. *If* $\vartheta \in \mathbb{R}^S$ *is such that* $R\vartheta = \rho$, *then* $q\vartheta = (1+i)^{-1}E_p(\rho)$, *for* any *probability measure* $p \in \mathbb{R}^S_{++}$ *that can be constructed as*

$$p = \left(\frac{\pi_1}{\sum_{s=1}^S \pi_s}, \frac{\pi_2}{\sum_{s=1}^S \pi_s}, \cdots, \frac{\pi_S}{\sum_{s=1}^S \pi_s}\right),$$

with $\pi \in \mathbb{R}^S_{++}$ *such that* $\sum_{s=1}^S \pi_s = (1+i)^{-1}$ *and* $\pi R = q$.

Proof. Since $q \in Q^1$, $\exists \pi \in \mathbb{R}^S_{++}$ such that $\pi R = q$. Since $r^1 = (1, 1, \ldots, 1)^\top$, it follows that $q^1 = \sum_{s=1}^S \pi_s$. Define

$$i = \frac{1}{\sum_{s=1}^S \pi_s} - 1$$

and

$$p = \left(\frac{\pi_1}{\sum_{s=1}^S \pi_s}, \frac{\pi_2}{\sum_{s=1}^S \pi_s}, \cdots, \frac{\pi_S}{\sum_{s=1}^S \pi_s}\right),$$

That $\sum_{s=1}^{S} \pi_s = (1+i)^{-1}$ is by construction. Now,

$$q\vartheta = \pi R\vartheta = (\sum_{s=1}^{S} \pi_s) pR\vartheta = (\sum_{s=1}^{S} \pi_s) p\rho = (1+i)^{-1} E_p(\rho),$$

which completes the proof. □

It is important to notice that any two π and π' such that $q = \pi R = \pi' R$ will generate the same i. The theorem does not imply that portfolios should be valued as their discounted expected return using one's probabilistic beliefs, or some objective probabilities! What it says is that if there are no arbitrage opportunities in the market, then the value of a portfolio must necessarily be its discounted expected return according to *some* probabilities.

7.3 Concave and Convex Functions

SUPPOSE THAT X IS A convex set. Function $f : X \to \mathbb{R}$ is said to be *concave* if for all $x_1, x_2 \in X$ and all $\vartheta \in [0, 1]$, it is true that

$$f(\vartheta x_1 + (1-\vartheta)x_2) \geqslant \vartheta f(x_1) + (1-\vartheta) f(x_2).$$

It is said to be *convex* if for all $x_1, x_2 \in X$, and $\vartheta \in [0, 1]$,

$$\vartheta f(x_1) + (1-\vartheta) f(x_2) \geqslant f(\vartheta x_1 + (1-\vartheta)x_2)$$

EXERCISE 7.3.1. *1. Let $X = \mathbb{R}$. Is $f(x) = x$ a concave or a convex function?*

2. Let $X = \mathbb{R}$. Is $f(x) = x^2$ a concave or a convex function?

3. Let $X = \mathbb{R}$. Is $f(x) = x^3$ a concave or a convex function? What if $X = \mathbb{R}_+$?

4. Let $X = \mathbb{R}_+$. Is $f(x) = \sqrt{x}$ a concave or a convex function?

5. Let $X = \mathbb{R}$. Is $f(x) = |x|$ a concave or a convex function? What if $X = \mathbb{R}_+$?

6. Let $X = \mathbb{R}^2$. Is $f(x) = x_1 + x_2$ a concave or a convex function?

7. Let $X = \mathbb{R}^2_+$. Is $f(x) = x_1 x_2$ a concave or a convex function?

THEOREM 7.3.1. *A function $f : X \to \mathbb{R}$ is concave if, and only if, the function $(-f) : X \to \mathbb{R}$, defined as $(-f)(x) = -f(x)$, is convex.*

The importance of the previous theorem is that it allows us to derive properties of convex functions straightforwardly from those of the concave functions.

THEOREM 7.3.2. *If $f : X \to \mathbb{R}$ is concave, then for all $k \in \mathbb{N}$, and for any finite sequences $(x_n)_{n=1}^{k}$ and $(\vartheta_n)_{n=1}^{k}$ satisfying that for all $n \in \{1, 2, \dots, k\}$, $x_n \in X$, and $\vartheta_n \in [0, 1]$ and that $\sum_{n=1}^{k} \vartheta_n = 1$, it is true that*[4]

$$f\left(\sum_{n=1}^{k} \vartheta_n x_n\right) \geqslant \sum_{n=1}^{k} \vartheta_n f(x_n)$$

[4] Hint: remember Principle 1.3.1.

7.4 Concavity and Second-Order Derivatives

NOTICE THAT CONCAVITY IS not a differential property. However, when a function is twice differentiable, there exists a tight relationship between concavity, and the sign of the second derivatives of the function.

THEOREM 7.4.1. *Consider* $f : X \to \mathbb{R}$, $f \in C^2$. *Then,* f *is concave if, and only if, for all* $x \in X$, $D^2 f(x)$ *is negative semidefinite.*

Proof. (If:) Let $x_1, x_2 \in X$ and $\vartheta \in [0,1]$. Since X is convex, we have that $\vartheta x_1 + (1-\vartheta)x_2 \in X$, and by Taylor's theorem, we have that

$$\begin{aligned} f(x_1) &= f(\vartheta x_1 + (1-\vartheta)x_2) + (1-\vartheta)Df(\vartheta x_1 + (1-\vartheta)x_2)(x_1 - x_2) \\ &\quad + \frac{(1-\vartheta)^2}{2}(x_1 - x_2)^\top D^2 f(x_1^*)(x_1 - x_2) \end{aligned}$$

and that

$$\begin{aligned} f(x_2) &= f(\vartheta x_1 + (1-\vartheta)x_2) + \vartheta Df(\vartheta x_1 + (1-\vartheta)x_2)(x_2 - x_1) \\ &\quad + \frac{\vartheta^2}{2}(x_2 - x_1)^\top D^2 f(x_2^*)(x_2 - x_1), \end{aligned}$$

where x_1^* lies in the interval between x_1 and $\vartheta x_1 + (1-\vartheta)x_2$, and x_2^* lies in the interval between x_2 and $\vartheta x_1 + (1-\vartheta)x_2$. Since for all $x \in X$, $D^2 f(x$ is negative semidefinite, it follows that

$$\begin{aligned} f(x_1) &\leqslant f(\vartheta x_1 + (1-\vartheta)x_2) + (1-\vartheta)Df(\vartheta x_1 + (1-\vartheta)x_2)(x_1 - x_2) \\ f(x_2) &\leqslant f(\vartheta x_1 + (1-\vartheta)x_2) + \vartheta Df(\vartheta x_1 + (1-\vartheta)x_2)(x_2 - x_1). \end{aligned}$$

Now, multiplying the first equation by ϑ and the second one by $(1-\vartheta)$, both of which are nonnegative, and adding, one gets that

$$\vartheta f(x_1) + (1-\vartheta)f(x_2)(\vartheta x_1 + (1-\vartheta)x_2).$$

(Only if:)[5] For all $\vartheta \in (0,1]$ and for all $x, \bar{x} \in X$, $x \neq \bar{x}$, we have that

$$f(\vartheta x + (1-\vartheta)\bar{x}) \geqslant \vartheta f(x) + (1-\vartheta)f(\bar{x}).$$

Denote $\Delta = x - \bar{x} \neq 0$. This implies that $x = \bar{x} + \Delta$ and that

$$\vartheta x + (1-\vartheta)\bar{x} = \bar{x} + \vartheta\Delta,$$

so that our inequality becomes

$$f(\bar{x} + \vartheta\Delta) \geqslant \vartheta f(\bar{x} + \Delta) + (1-\vartheta)f(\bar{x}),$$

Since $\vartheta \neq 0$, this is

$$f(\bar{x} + \Delta) \leqslant f(\bar{x}) + \left[\frac{f(\bar{x} + \vartheta\Delta) - f(\bar{x})}{\vartheta\Delta}\right]\Delta,$$

and since the latter is true for all $\vartheta \neq 0$ it is also true when we take $\vartheta \to 0$.[6] This implies that[7]

[5] For simplicity, we consider only the case $K = 1$ here, and defer the general case for the Appendix of the Chapter.

[6] By Corollary 2.6.1.

[7] The following inequality is important by itself. If says that when a differentiable function is concave, it first-order Taylor approximation always overestimates it.

$$f(\bar{x}+\Delta) \leqslant f(\bar{x}) + f'(\bar{x})\Delta. \tag{$*$}$$

Now, keeping $\bar{x}$ fixed, fix also $z \in \mathbb{R} \setminus \{0\}$. Consider the function

$$\varphi : \{\vartheta \in [0,1] | (\bar{x}+\vartheta z) \in X\} \to \mathbb{R},$$

defined by $\varphi(\vartheta) = f(\bar{x}+\vartheta z)$. Since $f \in C^2$, we have $\varphi \in C^2$, and by the chain rule

$$\varphi'(\vartheta) = f'(\bar{x}+\vartheta z)z$$

and

$$\varphi''(\vartheta) = f'(\bar{x}+\vartheta z)z^2.$$

Also, for all $\vartheta \in (0,1]$, by Taylor's theorem, 5.4.2, we have that for some $\vartheta^* \in [0,\vartheta]$,

$$\varphi(\vartheta) = \varphi(0) + \varphi'(0)\vartheta + \frac{1}{2}\varphi''(\vartheta^*)\vartheta^2,$$

which means that

$$f(\bar{x}+\vartheta z) = f(\bar{x}) + f'(\bar{x})z\vartheta + \frac{1}{2}f''(\bar{x}+\vartheta^* z)z^2\vartheta^2,$$

so that

$$\frac{1}{2}f''(\bar{x}+\vartheta^* z)z^2\vartheta^2 = f(\bar{x}+\vartheta z) - f(\bar{x}) - f'(\bar{x})z\vartheta.$$

By Equation ($*$) above, we have that the right hand side of the last inequality is non-positive.[8] Moreover, since $\vartheta > 0$ and $z \neq 0$, this implies that $f''(\bar{x}+\vartheta^* z) \leqslant 0$.

[8] Just use $z\vartheta$ instead of Δ.

Finally, since $\vartheta^* \in [0,\vartheta]$ and $f \in C^2$, if we take the limit as $\vartheta \to 0$, we find that $f''(\bar{x}) \leqslant 0$. □

COROLLARY 7.4.1. *Consider $f : X \to \mathbb{R}$, $f \in C^2$. Then, f is convex if, and only if, for all $x \in X$, $D^2 f(x)$ is positive semidefinite.*

Proof. It follows directly from Theorems 7.3.1, 7.4.1 and 5.1.3. □

EXERCISE 7.4.1 (L'Hopital's rule is really useful, II). *We now get another proof of the only if part of Theorem 7.4.1 for the case $K = 1$. Your job is to complete and justify all the steps. We argue by contradiction: given $f : X \to \mathbb{R}$, $f \in C^2$, suppose that f is concave but for some $x \in X$, $f''(x) > 0$. Take $y \in X \setminus \{x\}$ and $\vartheta \in (0,1)$. By concavity, we have that*

$$f(\vartheta x + (1-\vartheta)y) \geqslant \vartheta f(x) + (1-\vartheta)f(y).$$

Now, since $f \in C^2$, we can use the mean value theorem to get that

$$f(\vartheta x + (1-\vartheta)y) \geqslant f(x) + (1-\vartheta)f'(x)(y-x) + (1-\vartheta)\frac{1}{2}f''(x^*)(y-x)^2$$

for some x^* in the interval between x and y. Since $y \in X \setminus \{x\}$, this implies

$$(1-\vartheta)\frac{1}{2}f''(x^*) \leqslant \frac{f(\vartheta x + (1-\vartheta)y) - f(x) - (1-\vartheta)f'(x)(y-x)}{(y-x)^2}.$$

You can use L'Hopital's rule and the chain rule (twice) to show that

$$\lim_{y \to x} \frac{f(\vartheta x + (1-\vartheta)y) - f(x) - (1-\vartheta)f'(x)(y-x)}{(y-x)^2} = (1-\vartheta)^2\frac{1}{2}f''(x),$$

while

$$\lim_{y \to x}((1-\vartheta)\frac{1}{2}f''(x^*)) = (1-\vartheta)\frac{1}{2}f''(x),$$

and, therefore,

$$(1-\vartheta)\frac{1}{2}f''(x) \leqslant (1-\vartheta)^2\frac{1}{2}f''(x).$$

Now, since $f''(x) > 0$, *this implies that* $(1-\vartheta) \leqslant (1-\vartheta)^2$, *contradicting the fact that* $\vartheta \in (0,1)$.

7.5 Quasiconcave and Strongly Concave Functions

AGAIN, SUPPOSE THAT X IS a convex set. A function $f : X \to \mathbb{R}$ is said to be *quasiconcave* if for all $x_1, x_2 \in X$, and all $\vartheta \in [0,1]$, it is true that

$$f(\vartheta x_1 + (1-\vartheta)x_2) \geqslant \min\{f(x_1), f(x_2)\}.$$

It is said to be *strictly concave* if for all $x_1, x_2 \in X$ and $\vartheta \in [0,1]$ it is true that

$$f(\vartheta x_1 + (1-\vartheta)x_2) > \vartheta f(x_1) + (1-\vartheta)f(x_2).$$

Finally, $f : X \to \mathbb{R}$ is said to be *strongly quasiconcave* if for all $x_1, x_2 \in X$ and $\vartheta \in (0,1)$, it is true that

$$f(\vartheta x_1 + (1-\vartheta)x_2) > \min\{f(x_1), f(x_2)\}.$$

As before, we can find relationships between concavity, convexity and second-order derivatives. Again, we assume that X is convex.

THEOREM 7.5.1. *Consider* $f : X \to \mathbb{R} \in C^2$. *If for all* $x \in X$, $Df(x) \neq 0$ *and for all* $\Delta \in \mathbb{R}^K \setminus \{0\}$ *such that* $\Delta \cdot Df(x) = 0$, *it is true that* $\Delta^\top D^2 f(x)\Delta < 0$, *then* f *is strictly concave.*

Proof. The proof is complicated and therefore omitted. □

Notice that the previous result is pretty intuitive from the point of view of a second-order Taylor approximation.

THEOREM 7.5.2. *Consider* $f : X \to \mathbb{R}$, *a twice differentiable function. If for all* $x \in X$, $D^2 f(x)$ *is negative definite, then* f *is strictly concave.*

Notice that in Theorems 7.5.1 and 7.5.2, the condition is sufficient but not necessary (in contrast with Theorem 7.4.1). Of course, definitions and corollaries for convex, quasiconvex and strictly convex functions follow straightforwardly.

7.6 Composition of Functions and Concavity

A KEY POINT TO OBSERVE is that quasiconcavity is an ordinal property, whereas concavity has cardinal character:

THEOREM 7.6.1. *If* $f : X \to \mathbb{R}$ *is quasiconcave and* $g : \mathbb{R} \to \mathbb{R}$ *is increasing (i.e.* $y' \geqslant y \Rightarrow g(y') \geqslant g(y)$*), then* $g \circ f$ *is quasiconcave.*

THEOREM 7.6.2. *If* $f : X \to \mathbb{R}$ *is concave and* $g : \mathbb{R} \to \mathbb{R}$ *is concave and increasing, then* $g \circ f$ *is concave.*

EXERCISE 7.6.1. *Propose and prove a result analogous to the previous one to imply that* $g \circ f$ *is strictly concave.*

Appendix

Our goal is to generalize the necessity part of Theorem 7.4.1 to $K \geqslant 2$. That is, to argue that $f \in C^2$ is concave only if for all $x \in D$, matrix $D^2 f(x)$ is negative semidefinite.

Given $x \in D$ and $\Delta \in \mathbb{R}^K \setminus \{0\}$, define

$$\begin{aligned} D_{x,\Delta} &= \{\delta \in \mathbb{R} \mid x + \delta\Delta \in D\} \subseteq \mathbb{R} \\ \varphi_{x,\Delta} &: D_{x,\Delta} \to \mathbb{R}; \varphi_{x,\Delta}(\delta) = f(x + \delta\Delta) \end{aligned}$$

We use the following lemmas

LEMMA 7.6.1. *Let* $x \in D$ *and* $\Delta \in \mathbb{R}^n \setminus \{0\}$. *If* D *is convex,* $D_{x,\Delta}$ *is convex. If* f *is concave,* $\varphi_{x,\Delta}$ *is concave. If* Δ *is open,* $\Delta_{x,\Delta}$ *is open. If* f *is differentiable, then* $\varphi_{x,\Delta}$ *is differentiable and for all* $\delta \in D_{x,\Delta}$, $\varphi'_{x,\Delta}(\delta) = \Delta \cdot Df(x + \delta\Delta)$. *If* $f \in C^2$, *then* $\varphi_{x,\Delta} \in C^2$ *and for all* $\delta \in D_{x,\Delta}$, $\varphi'_{x,\Delta}(\delta) = \Delta^\top D^2 f(x + \delta\Delta)\Delta$.

Proof. We first show that $D_{x,\Delta}$ is convex: let $\delta, \delta' \in D_{x,\Delta}$ and $\lambda \in [0, 1]$. By definition, $x + \delta\Delta \in D$ and $x + \delta'\Delta \in D$. Since D is convex, $\lambda(x + \delta\Delta) + (1 - \lambda)(x + \delta'\Delta) = x + (\lambda\delta + (1 - \lambda)\delta')\Delta \in D$, so $(\lambda\delta + (1 - \lambda)\delta') \in D_{x,\Delta}$.

We now show that $\varphi_{x,\Delta}$ is concave. Let $\delta, \delta' \in D_{x,\Delta}$ and $\lambda \in [0, 1]$. By definition, $x + \delta\Delta \in D$ and $x + \delta'\Delta \in D$. By concavity,

$$\begin{aligned} \varphi_{x,\Delta}(\lambda\delta + (1 - \lambda)\delta') &= f(x + (\lambda\delta + (1 - \lambda)\delta')\Delta) \\ &= f(\lambda(x + \delta\Delta) + (1 - \lambda)(x + \delta'\Delta)) \\ &\geqslant \lambda f(x + \delta\Delta) + (1 - \lambda)f(x + \delta'\Delta) \\ &= \lambda\varphi_{x,\Delta}(\delta) + (1 - \lambda)\varphi_{x,\Delta}(\delta') \end{aligned}$$

To see that $D_{x,\Delta}$ is open, let $\delta \in D_{x,\Delta}$. By definition, $x + \delta\Delta \in D$ and, since D is open, $\exists\varepsilon' > 0$ such that $B_{\varepsilon'}(x + \delta\Delta) \subseteq D$. Define $\varepsilon = \varepsilon'/\|\Delta\| > 0$ and suppose that $\delta' \in B_\varepsilon(\delta)$. Then, $\|x - (x + \delta'\Delta)\| = |\delta|\|\Delta\| < \varepsilon\|\Delta\| = \varepsilon'$, so $x + \delta'\Delta \in B_{\varepsilon'}(x) \subseteq D$ and, hence, $\delta' \in D_{x,\Delta}$, which implies that $B_\varepsilon(\delta) \subseteq D_{x,\Delta}$.

That if f is differentiable, then $\varphi_{x,\Delta}$ is differentiable and for all $\delta \in D_{x,\Delta}$, $\varphi'_{x,\Delta}(\delta) = \Delta \cdot Df(x + \delta\Delta)$, and if $f \in C^2$, then $\varphi_{x,\Delta} \in C^2$ and $\forall\delta \in D_{x,\Delta}$, $\varphi''_{x,\Delta}(\delta) = \Delta^\top D^2 f(x + \delta\Delta)\Delta$ is left as an exercise. □

LEMMA 7.6.2. *Suppose that there exists* $x \in D$ *such that* $D^2 f(x)$ *is not negative semidefinite. Then, there exists* $\Delta \in \mathbb{R}^n \setminus \{0\}$ *such that* $\varphi_{x,\Delta}$ *is not concave.*

Proof. Since $D^2 f(x)$ is not negative semidefinite, there is $\Delta \in \mathbb{R}^n \setminus \{0\}$ such that

$$\Delta^\top D^2 f(x)\Delta > 0,$$

and by the previous lemma $\varphi_{x,\Delta} \in C^2$ and $\varphi'_{x,\Delta}(0) > 0$. From the case $K = 1$, it follows that $\varphi_{x,\Delta}$ is not concave. □

8
Unconstrained Maximization

THROUGHOUT THIS CHAPTER, WE maintain the assumptions that set $D \subseteq \mathbb{R}^K$, for a finite K, is nonempty.

8.1 Maximizers

WE START WITH A RESULT that illustrates that, in a sense, the supremum of a set may be too weak as a concept of optimality.

THEOREM 8.1.1. *Let* $Y \subseteq \mathbb{R}$ *and let* $b = \sup Y$. *One has that* $b \notin Y$ *if, and only if, for all* $a \in Y$, *there is an* $a' \in Y$ *such that* $a' > a$.

Proof. Sufficiency is left as an exercise. For necessity, note that if there is an $a \in Y$ such that for all $a' \in Y$ it is true that $a' \leqslant a$, then, by definition, $b \leqslant a$, whereas $a \leqslant b$, which implies that $b = a \in Y$, a contradiction. □

It follows that we need a stronger concept of extremum, in particular one that implies that the extremum lies within the set. Thus, a point $b \in \mathbb{R}$ is said to be *the maximum of set* $Y \subseteq \mathbb{R}$, denoted $b = \max A$, if $b \in Y$ and for all $a \in Y$ it is true that $a \leqslant b$. The proofs of the following two results are left as exercises.

THEOREM 8.1.2. *If* $\max Y$ *exists, then it is unique.*

THEOREM 8.1.3. *If* $\max Y$ *exists, then* $\sup Y$ *exists and* $\sup Y = \max Y$. *If* $\sup Y$ *exists and* $\sup Y \in Y$, *then* $\max Y$ *exists and* $\max Y = \sup Y$.

EXERCISE 8.1.1. *Given* $Y, Y' \subseteq \mathbb{R}$, *prove the following:*

1. *If* $Y \neq \varnothing$ *and* $Y' \neq \varnothing$ *are such that for all* $(a, a') \in Y \times Y'$ *one has that* $a \leqslant a'$, *then* $\sup Y$ *and* $\inf Y'$ *exist, and* $\sup Y \leqslant \inf Y'$.

2. *If* $\sup Y$ *and* $\sup Y'$ *exist,* $\lambda, \lambda' \in \mathbb{R}_{++}$ *and*

$$\tilde{Y} = \{\tilde{a} \mid \exists (a, a') \in Y \times Y' : \lambda a + \lambda' a' = \tilde{a}\},$$

   then $\sup \tilde{Y} = \lambda \sup Y + \lambda' \sup Y'$.

3. *If* $\sup Y$ *and* $\sup Y'$ *exist, and for all* $a \in Y$ *there is an* $a' \in Y'$ *such that* $a \leqslant a'$, *then* $\sup Y \leqslant \sup Y'$.

Show also that a strict version of the third statement is not true.

The following theorem will prove useful.

THEOREM 8.1.4. *If* $Y \subseteq \mathbb{R}$ *is closed and* $\sup Y$ *exists, then* $\max Y$ *exists and* $\max Y = \sup Y$.

Proof. Let $\bar{y} = \sup Y$. By Theorem 3.3.1, for all $n \in \mathbb{N}$ there is some $y_n \in Y$ for which $\bar{y} - 1/n < y_n < \bar{y}$. Clearly, $y_n \to \bar{y}$, so, since Y is closed, $\bar{y} \in Y$. □

Now, it typically is of more interest in economics to find extrema of functions, rather than extrema of sets. To a large extent, the distinction is only apparent: what we will be looking for are the extrema of the image of the domain under the function. A point $\bar{x} \in D$ is said to be *a global maximizer of* $f : D \to \mathbb{R}$ if for all $x \in D$ it is true that $f(x) \leqslant f(\bar{x})$. Point $\bar{x} \in D$ is said to be *a local maximizer of* $f : D \to \mathbb{R}$ if there exists some $\varepsilon > 0$ such that for every $x \in B_\varepsilon(\bar{x}) \cap D$ it is true that $f(x) \leqslant f(\bar{x})$.

When $\bar{x} \in D$ is a local (global) maximizer of $f : D \to \mathbb{R}$, the number $f(\bar{x})$ is said to be *a local (the global) maximum of* f. Notice that, in the latter case, $f(\bar{x}) = \max f[D]$, although more standard notation for $\max f[D]$ is $\max_D f$ or $\max_{x \in D} f(x)$.[1] Notice that there is a conceptual difference between maximum and maximizer! Also, notice that a function can have only one global maximum even if it has multiple global maximizers, but the same is not true for the local concept. The set of maximizers of a function is usually denoted by $\text{argmax}_D f$.

By analogy, $b \in \mathbb{R}$ is said to be *the supremum of* $f : D \to \mathbb{R}$, denoted $b = \sup_D f$ or $b = \sup_{x \in D} f(x)$, if $b = \sup f[D]$. Importantly, note that there is no reason why $\exists x \in D$ such that $f(x) = \sup_D f$ even if the supremum is defined.

[1] A point $\bar{x} \in D$ is said to be *a local minimizer of* $f : D \to \mathbb{R}$ if there is an $\varepsilon > 0$ such that for all $x \in B_\varepsilon(\bar{x}) \cap D$ it is true that $f(x) \geqslant f(\bar{x})$. Point $\bar{x} \in D$ is said to be *a global minimizer of* $f : D \to \mathbb{R}$ if for every $x \in D$ it is true that $f(x) \geqslant f(\bar{x})$. From now on, we only deal with maxima, although the minimization problem is obviously covered by analogy.

8.2 Existence

WE NOW PRESENT A weak version of a result that is very useful in economics.

THEOREM 8.2.1 (Weierstrass). *Let* $C \subseteq D$ *be nonempty and compact. If the function* $f : D \to \mathbb{R}$ *is continuous, then there are* $\bar{x}, \underline{x} \in C$ *such that for all* $x \in C$ *it is true that* $f(\underline{x}) \leqslant f(x) \leqslant f(\bar{x})$.

Proof. It follows from Theorem 4.2.3 that $f[C]$ is compact. Now, let $\bar{y} = \sup f[C]$, which exists by Axiom 3.3.1. By Theorem 8.1.4, $\bar{y} \in f[C]$, and it follows that there is $\bar{x} \in C$ such that $f(\bar{x}) = \bar{y}$. By definition, then, for every $x \in C$ it is true that $f(x) \leqslant \bar{y} = f(\bar{x})$.and it follows that there is $\bar{x} \in C$ such that $f(\bar{x}) = \bar{y}$. By definition, then, for every $x \in C$ it is true that $f(x) \leqslant \bar{y} = f(\bar{x})$.

Existence of $\underline{x}$ is left as an exercise. □

The importance of this result is that when the domain of a continuous function is closed and bounded, then the function does attain its maxima and minima within its domain. This allows is to obtain many important results in economics and mathematics. The following argument illustrates this.

Proof of Lemma 7.1.1: Since $Q \neq \varnothing$, we can fix $\tilde{q} \in Q$. Note that set

$$C = \{q \in Q \mid \|q - \bar{q}\| \leqslant \|\tilde{q} - \bar{q}\|\}$$

is nonempty and compact. Since function $\|q - \bar{q}\|$ is continuous, 8.2.1 guarantees that problem $\min_{q \in C} \|q - \bar{q}\|$ has a solution. Any solution to this problem also solves

$$\min_{q \in Q} \|q - \bar{q}\|, \tag{$*$}$$

by construction.

Let q^* solve the latter problem, and construct $\vartheta = q^* - \bar{q}$ and $k = \vartheta q^*$. By construction, $\vartheta \neq 0$ and $q^* \vartheta = k$. To see that $\bar{q}\vartheta < k$, it suffices to note that

$$\bar{q}\vartheta = (q^* - \vartheta)\vartheta = k - \|\vartheta\|^2 < k.$$

We need to prove that $q\vartheta \geqslant k$ for all $q \in Q$. We argue by contradiction: suppose that for some $q \in Q$, $q\vartheta < k$. Note that for every $\alpha \in [0, 1]$, $q_\alpha = \alpha q + (1 - \alpha)q^* \in Q$, since Q is convex. Now,

$$\begin{aligned}
\|\bar{q} - q^*\|^2 - \|\bar{q} - q_\alpha\|^2 &= \|\bar{q} - q^*\|^2 - \|\bar{q} - q^* - \alpha(q - q^*)\|^2 \\
&= 2\alpha(\bar{q} - q^*) \cdot (\bar{q} - q^*) - \alpha^2\|q - q^*\|^2 \\
&= \alpha[2(k - q^*\vartheta) - \alpha\|q - q^*\|^2].
\end{aligned}$$

By assumption $k - q^*\vartheta > 0$. For $alpha > 0$ small enough, then,

$$\|\bar{q} - q^*\|^2 - \|\bar{q} - q_\alpha\|^2 > 0,$$

so $\|\bar{q} - q^*\| > \|\bar{q} - q_\alpha\|0$, which is impossible since $q_\alpha \in Q$ and q^* solves $(*)$. *Q.E.D.*

8.3 Characterizing Maximizers

EVEN THOUGH MAXIMIZATION IS not a differential problem, when one has differentiability there are results that make it easy to find maximizers. For this section, we take set D to be open.

8.3.1 Problems in $\mathbb{R}$

For simplicity, we first consider the case $K = 1$.

LEMMA 8.3.1. *Suppose that $f : D \to \mathbb{R}$ is differentiable. Let $\bar{x} \in X$. If $f'(\bar{x}) > 0$, then there is some $\delta > 0$ such that for every $x \in B_\delta(\bar{x}) \cap D$ we have $f(x) > f(\bar{x})$ if $x > \bar{x}$, and that $f(x) < f(\bar{x})$ if $x < \bar{x}$.*

Proof. By assumption, we have $f'(\bar{x}) \in \mathbb{R}_{++}$. Then, by definition, there is some $\delta > 0$ such that for any $x \in B'_\delta(\bar{x}) \cap D$,

$$|\frac{f(x) - f(\bar{x})}{x - \bar{x}} - f'(\bar{x})| < f'(\bar{x}),$$

and, by Exercise 2.2.2, since $f'(\bar{x}) > 0$, $(f(x) - f(\bar{x}))(x - \bar{x}) > 0$. □

The analogous result for the case of a negative derivative is presented, without proof, as the following corollary.

COROLLARY 8.3.1. *Suppose that* $f : D \to \mathbb{R}$ *is differentiable. Let* $\bar{x} \in D$. *If* $f'(\bar{x}) < 0$, *then there is some* $\delta > 0$ *such that for every* $x \in B_\delta(\bar{x}) \cap D$ *we have* $f(x) < f(\bar{x})$ *if* $x > \bar{x}$, *and that* $f(x) > f(\bar{x})$ *if* $x < \bar{x}$.

THEOREM 8.3.1. *Suppose that* $f : D \to \mathbb{R}$ *is differentiable. If* $\bar{x} \in \text{int}(D)$ *is a local maximizer of* f *then* $f'(\bar{x}) = 0$.

Proof. Suppose not: $f'(\bar{x}) \neq 0$. If $f'(\bar{x}) > 0$, then, by Lemma 8.3.1, there is $\delta > 0$ such that for all $x \in B_\delta(\bar{x}) \cap D$ satisfying $x > \bar{x}$ we have that $f(x) > f(\bar{x})$. Since $\bar{x}$ is a local maximizer of f, then there is $\varepsilon > 0$ such that for all $x \in B_\varepsilon(\bar{x}) \cap D$ it is true that $f(x) \leqslant f(\bar{x})$. Since $\bar{x} \in \text{int}(D)$, there is $\gamma > 0$ such that $B_\gamma(\bar{x}) \subseteq D$. Let $\beta = \min\{\varepsilon, \delta, \gamma\} > 0$. Clearly, $(\bar{x}, \bar{x} + \beta) \subset B'_\beta(\bar{x}) \neq \varnothing$ and $B'_\beta(\bar{x}) \subseteq D$. Moreover, $B'_\beta(\bar{x}) \subseteq B_\delta(\bar{x}) \cap D$ and $B'_\beta(\bar{x}) \subseteq B_\varepsilon(\bar{x}) \cap D$. This implies that for some x one has $f(x) > f(\bar{x})$ and $f(x) \leqslant f(\bar{x})$, an obvious contradiction. A similar contradiction appears if $f'(\bar{x}) < 0$, by Corollary 8.3.1. □

THEOREM 8.3.2. *Let* $f : D \to \mathbb{R}$ *be of class* C^2. *If* $\bar{x} \in \text{int}(D)$ *is a local maximizer of* f *then* $f''(\bar{x}) \leqslant 0$.

Proof. Since $\bar{x} \in \text{int}(D)$, there is a $\varepsilon > 0$ for which $B_\varepsilon(\bar{x}) \subseteq D$. For every $h \in B_\varepsilon(0)$, since f is twice differentiable, by Taylor's theorem (Theorem 5.4.2), there is some x^*_h in the interval joining $\bar{x}$ and $\bar{x} + h$, such that

$$f(\bar{x} + h) = f(\bar{x}) + f'(\bar{x})h + \frac{1}{2} f''(x^*_h)h^2.$$

Since $\bar{x}$ is a local maximizer, there is a $\delta > 0$ such that $x \in B_\delta(\bar{x}) \cap D$ implies $f(x) \leqslant f(\bar{x})$. Let $\beta = \min\{\varepsilon, \delta\} > 0$. By construction, for any $h \in B'_\beta(0)$ one has that

$$f'(\bar{x})h + \frac{1}{2} f''(x^*_h)h^2 = f(\bar{x} + h) - f(\bar{x}) \leqslant 0.$$

By Theorem 8.3.1, since f is differentiable and $\bar{x}$ is a local maximizer, $f'(\bar{x}) = 0$, from where $h \in B'_\beta(0)$ implies that $f''(x^*_h)h^2 \leqslant 0$, and hence that $f''(x^*_h) \leqslant 0$. Now, letting $h \to 0$, we get, by Theorem 2.6.11, that $\lim_{h \to 0} f''(x^*_h) \leqslant 0$, and hence that $f''(\bar{x}) \leqslant 0$, since f'' is continuous and each x_h lies in the interval joining $\bar{x}$ and $\bar{x} + h$. □

EXERCISE 8.3.1. *Prove theorems analogous to the previous two, for the case of local minimizers.*

Notice that the last theorems only give us necessary conditions:[2] this is not a tool that tells us which points are local maximizers, but it tells us what points are not. A complete characterization requires both necessary and sufficient conditions. We now find sufficient conditions.

[2] And there are further necessary conditions.

THEOREM 8.3.3. *Suppose that* $f : D \to \mathbb{R}$ *is twice differentiable. Let* $\bar{x} \in \text{int}(D)$. *If* $f'(\bar{x}) = 0$ *and* $f''(\bar{x}) < 0$, *then* $\bar{x}$ *is a local maximizer.*

Proof. Since $f : D \to \mathbb{R}$ is twice differentiable and $f''(\bar{x}) < 0$, we have, by Corollary 8.3.1, that for some $\delta > 0$ it is true that whenever $x \in B_\delta(\bar{x}) \cap D$ we have that (i)$f'(x) < f'(\bar{x}) = 0$, when $x > \bar{x}$; and (ii)$f'(x) > f'(\bar{x}) = 0$, when $x < \bar{x}$. Since $x \in \int(D)$, there is an $\varepsilon > 0$ such that $B_\varepsilon(\bar{x}) \subseteq D$. Let $\beta = \min\{\delta, \varepsilon\} > 0$. By the Mean Value Theorem (Theorem 5.4.1), we have that for all $x \in B_\beta(\bar{x})$,

$$f(x) = f(\bar{x}) + f'(x^*)(x - \bar{x})$$

for some x^* in the interval between $\bar{x}$ and x (why?). Thus, if $x > \bar{x}$, we have $x^* \geqslant \bar{x}$, and, therefore, $f'(x^*) \leqslant 0$, so that $f(x) \geqslant f(\bar{x})$. On the other hand, if $x < \bar{x}$, then $f'(x^*) \geqslant 0$, so that $f(x) \leqslant f(\bar{x})$. □

EXERCISE 8.3.2. *Prove an analogous theorem, for the case of a local minimizer.*

Notice that the sufficient conditions are stronger than the set of necessary conditions: there is a little gap that the differential method does not cover.

8.3.2 Higher-dimensional problems

We now allow for functions defined on higher-dimesional domains (namely $K \geqslant 2$), and use the results of the one-dimensional case, using the definition on the appendix of Chapter 7.

LEMMA 8.3.2. *If $x^* \in D$ is a local maximizer of $f : D \to \mathbb{R}$, then for all $\Delta \in \mathbb{R}^K \setminus \{0\}$, 0 is a local maximizer of $\varphi_{x^*,\Delta}$.*

Proof. Let $\Delta \in \mathbb{R}^K \setminus \{0\}$. By construction there is $\varepsilon' > 0$ such that $f(x) \leqslant f(x^*)$ for all $x \in B_{\varepsilon'}(x^*) \cap D$. Define $\varepsilon = \varepsilon'/\|\Delta\|$ and suppose that $\delta \in B_\varepsilon(0) \cap D_{x^*,\Delta}$. By construction, $\|x^* - (x^* + \delta\Delta)\| = |\delta|\|\Delta\| < \varepsilon\|\Delta\| = \varepsilon'$, while $x^* + \delta\Delta \in D$, which implies that $x^* + \delta\Delta \in B_{\varepsilon'}(x^*) \cap D$ and, hence, $\varphi_{x^*,\Delta}(\delta) = f(x^* + \delta\Delta) \leqslant f(x^*) = \varphi_{x^*,\Delta}(0)$. □

The previous lemma implies the following result.

THEOREM 8.3.4. *If $f : D \to \mathbb{R}$ is differentiable and $x^* \in D$ is a local maximizer of f, then $Df(x^*) = 0$.*

Proof. By Lemma 8.3.2, for every $k = 1, \ldots, K$, 0 is a local maximizer of φ_{x^*,e_K}, where e_k is the k-th canonical vector in $\mathbb{R}^K$. Since φ_{x^*,e_k} is differentiable, by Lemma 7.6.1, it follows from Theorem 8.3.1 than $\varphi'_{x^*,e_k}(0) = 0$, whereas, again by Lemma 7.6.1, $\varphi'_{x^*,e_k}(0) = e_k \cdot Df(x^*) = \frac{\partial f}{\partial x_k}(x^*)$. □

THEOREM 8.3.5. *If $f : D \to \mathbb{R}$ is of class C^2 and $x^* \in D$ is a local maximizer of f, then $D^2 f(x^*)$ is negative semidefinite.*

Proof. Let $\Delta \in \mathbb{R}^K \setminus \{0\}$. By Lemma 8.3.2 and Theorem 8.3.2, $\varphi''_{x^*,\Delta}(0) \leqslant 0$, whereas, by Lemma 7.6.1, $\varphi''_{x^*,\Delta}(0) = \Delta^\top D^2 f(x^*)\Delta$. □

As before, these conditions do not tell us which points are maximizers, but only which ones are not. Before we can argue sufficiency, we need to introduce the following lemma.

LEMMA 8.3.3. *If $f : D \to \mathbb{R}$ is of class C^2 and $D^2 f(x^*)$ is negative definite, then there exists $\varepsilon > 0$ such that for every $x \in B_\varepsilon(x^*)$, $D^2 f(x)$ is negative definite.*

Proof. Suppose not. Then, for each $n \in \mathbb{N}$, there are an $x_n \in B_{1/n}(x^*)$ and a $\Delta_n \in \mathbb{R}^K \setminus \{0\}$ such that $\Delta_n^\top D^2 f(x_n)\Delta \geqslant 0$. Since $\Delta_n \neq 0$, we can assume, without losing generality, $\|\Delta_n\| = 1$. Then, it follows that for some subsequence $(x_{n(m)}, \Delta_{n(m)})_{m=1}^\infty$ we have that, for all $m \in \mathbb{N}$, $\Delta_{n(m)}^\top D^2(x_{n(m)})\Delta_{n(m)} \geqslant 0$, and $(x_{n(m)}, \Delta_{n(m)}) \to (x^*, \Delta)$ for some Δ such that $\|\Delta\| = 1$. Since $f \in C^2$, $\Delta^\top D^2 f(x^*)\Delta \geqslant 0$, contradicting the negative definiteness of $D^2 f(x^*)$. □

THEOREM 8.3.6. *Suppose that* $f : D \to \mathbb{R}$ *is of class* C^2 *and let* $\bar{x} \in D$*. If* $Df(\bar{x}) = 0$ *and* $D^2 f(\bar{x})$ *is negative definite, then* $\bar{x}$ *is a local maximizer.*

Proof. The argument is left as an exercise. (Hint: use the previous lemma.) □

8.4 Maxima and Concavity

FOR THE PURPOSES OF this section, we take $D \subseteq \mathbb{R}^K$, $K \in \mathbb{N}$, $D \neq \varnothing$ and drop the openness assumption.

Note that the results that we obtained in the previous sections hold only locally. We now study the extent to which local extrema are, in effect, global extrema.

THEOREM 8.4.1. *Suppose that* D *is a convex set and* $f : D \to \mathbb{R}$ *is a concave function. Then, if* $\bar{x} \in D$ *is a local maximizer of* f*, it is also a global maximizer.*

Proof. We argue by contradiction: suppose that $\bar{x} \in D$ is a local maximizer of f, but it is not a global maximizer. Then, there is $\varepsilon > 0$ such that for every $x \in B_\varepsilon(\bar{x}) \cap D$, $f(x)(\bar{x})$; and there is $x^* \in D$ such that $f(x^*) > f(\bar{x})$. Clearly, then, $x^* \notin B_\varepsilon(\bar{x})$, which implies that $\|x^* - \bar{x}\| \geqslant \varepsilon$. Now, since D is convex and f is concave, we have that for $\vartheta \in [0, 1]$,

$$f(\vartheta x^* + (1-\vartheta)\bar{x}) \geqslant \vartheta f(x^*) + (1-\vartheta) f(\bar{x}),$$

but, since $f(x^*) > f(\bar{x})$, we further have that if $\vartheta \in (0, 1]$, then $\vartheta f(x^*) + (1-\vartheta) f(\bar{x}) > f(\bar{x})$, so that $f(\vartheta x^* + (1-\vartheta)\bar{x}) > f(\bar{x})$.

Now, consider $\vartheta^* \in (0, \varepsilon/\|x^* - \bar{x}\|)$. Clearly, $\vartheta^* \in (0, 1)$, so $f(\vartheta^* x^* + (1-\vartheta^*)\bar{x}) > f(\bar{x})$. However, by construction,

$$\|(\vartheta^* x^* + (1-\vartheta^*)\bar{x}) - \bar{x}\| = \vartheta^* \|x^* - \bar{x}\| < (\frac{\varepsilon}{\|x^* - \bar{x}\|})\|x^* - \bar{x}\| = \varepsilon,$$

which implies that $(\vartheta^* x^* + (1-\vartheta^*)\bar{x}) \in B_\varepsilon(\bar{x})$, and, moreover, by convexity of D, we have that $(\vartheta^* x^* + (1-\vartheta^*)\bar{x}) \in B_\varepsilon(\bar{x}) \cap D$. This contradicts the fact that $f(x) \leqslant f(\bar{x})$ for all $x \in B_\varepsilon(\bar{x}) \cap D$. □

EXERCISE 8.4.1. *Prove an analogous theorem, for the case of a local minimizer.*

THEOREM 8.4.2. *Suppose that* D *is convex,* $f : D \to \mathbb{R}$ *is of class* C^2 *and for each* $x \in D$*,* $D^2 f(x)$ *is negative definite. Then, there exists at most one point* $\bar{x} \in D$ *such that* $Df(\bar{x}) = 0$*. If such point exists, it is a global maximizer.*

Proof. We first prove the last part of the theorem. Suppose that there is $\bar{x} \in D$ such that $Df(\bar{x}) = 0$. By assumption, $Df''(\bar{x})$ is negative definite, and therefore, by Theorem 8.3.6, $\bar{x}$ is a local maximizer. Since $Df(x)$ is negative definite everywhere, by Theorem 7.5.2, we have that f is concave and, therefore, by Theorem 8.4.1, $\bar{x}$ is a global maximizer.

We must now show that there cannot exist more than one such point. We argue by contradiction: suppose that there are distinct $\bar{x}_1, \bar{x}_2 \in D$ such that $f'(\bar{x}_1) = f'(\bar{x}_2) = 0$. By our previous argument, both $\bar{x}_1$ and $\bar{x}_2$ are global maximizers, so that $f(\bar{x}_1) = f(\bar{x}_2)$. Now, since $D^2 f(x)$ is negative definite everywhere, by Theorem 7.5.2, we have that f is strictly concave, and

$$f\left(\frac{1}{2}\bar{x}_1 + \frac{1}{2}\bar{x}_2\right) > \frac{1}{2} f(\bar{x}_1) + \frac{1}{2} f(\bar{x}_2) = f'(\bar{x}_1) = f'(\bar{x}_2),$$

contradicting the fact that both $\bar{x}_1$ and $\bar{x}_2$ are global maximizers, since D is convex. □

EXERCISE 8.4.2. *Prove an analogous theorem, for the case when* $D^2 f(x)$ *is positive definite everywhere.*

9
Constrained Maximization

SUPPOSE THAT $f : \mathbb{R} \to \mathbb{R}$ IS DIFFERENTIABLE, and suppose that for $a, b \in \mathbb{R}$, $a < b$, we want to find $x^* \in [a, b]$ such that $f(x) \leqslant f(x^*)$ at all $x \in [a, b]$. That is, we want to solve the problem

$$\max f(x) : x \geqslant a \text{ and } x \leqslant b.$$

If $x^* \in (a, b)$ solves the problem, then x^* is a local maximizer of f (why?), and it follows from Theorem 8.3.1 that $f'(x^*) = 0$. If, alternatively, $x^* = b$ solves the problem, then by Corollary 8.3.1, it must be that $f'(x^*) \geqslant 0$. Finally, if $x^* = a$ solves the problem, it follows from Lemma 8.3.1 that $f'(x^*) \leqslant 0$.

It is then straightforward that if x^* solves the problem, then there exist $\lambda_a^*, \lambda_b^* \in \mathbb{R}_+$ such that $f(x^*) - \lambda_b^* + \lambda_a^* = 0$, $\lambda_a^*(x^* - a) = 0$ and $\lambda_b^*(b - x^*) = 0$.[1] It is customary to define a function

$$\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}; \mathcal{L}(x, \lambda_a, \lambda_b) = f(x) + \lambda_b(b - x) + \lambda_a(a - x),$$

which is called *the Lagrangean*, and with which the first condition can be re-written as

$$\frac{\partial \mathcal{L}}{\partial x}(x^*, \lambda_a^*, \lambda_b^*) = 0.$$

In this section we show how these Lagrangean methods work, and emphasize when they fail.

[1] The second and third conditions simply express that (i) if $x^* \in (a, b)$, then $\lambda_a^* = 0$ and $\lambda_b^* = 0$; (ii) if $x^* = b$, then $\lambda_a^* = 0$ and $\lambda_b^* \geqslant 0$; and (iii) if $x^* = a$, then $\lambda_a^* \geqslant 0$ and $\lambda_b^* = 0$.

9.1 Equality constraints

FOR THIS SECTION, WE MAINTAIN the assumptions that $D \subseteq \mathbb{R}^K$, K finite, is open, and that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}^J$, with $J \leqslant K$.

Suppose that we want to solve the problem

$$\max_{x \in D} f(x) : g(x) = 0, \tag{9.1}$$

which means, in our previous notation, that we want to find $\max_{\{x \in D | g(x) = 0\}} f$. The method that is usually applied in economics consists of the following steps: (1) defining the Lagrangean function $\mathcal{L} : D \times \mathbb{R}^J \to \mathbb{R}$, by $\mathcal{L}(x, \lambda) = f(x) + \lambda \cdot g(x)$; and (2) finding $(x^*, \lambda^*) \in D \times \mathbb{R}^J$ such that $D\mathcal{L}(x^*, \lambda^*) = 0$. That is, a recipe is applied as though there is a “result” that states the following:

Let f *and* g *be differentiable.* $x^* \in D$ *solves Problem* (9.1) *if, and only if, there exists* $\lambda^* \in \mathbb{R}^J$ *such that* $Df(x^*) + \lambda^{*\top} Dg(x^*) = 0$.

Unfortunately, though, *such a statement is not true*, for reasons that we now study.

For simplicity of presentation, suppose that $D = \mathbb{R}^2$ and $J = 1$, and denote the typical element of $\mathbb{R}^2$ by (x, y). So, given $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$, we want to find

$$\max_{(x,y)\in\mathbb{R}^2} f(x, y) : g(x, y) = 0.$$

Let us suppose that we do not know the Lagrangean method, but are quite familiar with unconstrained optimization. A "crude" method suggests the following:

(1) Suppose that we can solve from the equation $g(x, y) = 0$, to express y as a function of x: we find a function $y : \mathbb{R} \to \mathbb{R}$ such that $g(x, y) = 0$ if, and only if $y = y(x)$.

(2) With the function y at hand, we study the unconstrained problem $\max_{x\in\mathbb{R}} F(x)$, where $F : \mathbb{R} \to \mathbb{R}$ is defined by $F(x) = f(x, y(x))$.

(3) Since we want to use calculus, if f and g are differentiable, we need to figure out function y'. Now, if $g(x, y(x)) = 0$, then, differentiating both sides, we get that $\partial_x g(x, y(x)) + \partial_y g(x, y(x))y'(x) = 0$, from where

$$y'(x) = -\frac{\partial_x g(x, y(x))}{\partial_y g(x, y(x))}.$$

(4) Now, with F differentiable, we know that x^* solves $\max_{x\in\mathbb{R}} F(x)$ locally, only if $F'(x^*) = 0$.

In our case, the last step is simply that

$$\partial_x f(x^*, y(x^*)) + \partial_y f(x^*, y(x^*))y'(x^*) = 0,$$

or, equivalently,

$$\partial_x f(x^*, y(x^*)) - \partial_y f(x^*, y(x^*)) \frac{\partial_x g(x^*, y(x^*))}{\partial_y g(x^*, y(x^*))} = 0.$$

So, if we define $y^* = y(x^*)$ and

$$\lambda^* = -\frac{\partial_y f(x^*, y(x^*))}{\partial_y g(x^*, y(x^*))} \in \mathbb{R},$$

we get that

$$\partial_x f(x^*, y(x^*)) + \lambda^* \partial_x g(x^*, y(x^*)) = 0,$$

whereas

$$\partial_y f(x^*, y(x^*)) + \lambda^* \partial_y g(x^*, y(x^*)) = 0.$$

Then, our method has apparently shown that:

Let f *and* g *be differentiable.* $x^* \in D$ locally *solves the Problem* (9.1),[2] *only if there exists* $\lambda^* \in \mathbb{R}^J$ *such that* $Df(x^*) + \lambda^{*\top} Dg(x^*) = 0$.

The latter means that: (i) as in the unrestricted case, the differential approach, at least in principle, only finds local extrema; and (ii) the Lagrangean condition is only

[2] That is, there is $\varepsilon > 0$ such that

$$f(x) \leqslant f(x^*)$$

for all

$$x \in B_\epsilon(x^*) \cap \{x \in D \mid g(x) = 0\}.$$

necessary and not sufficient by itself. So, we need to be careful and study further conditions for sufficiency. Also, we need to determine under what conditions can we find the function y and, moreover, be sure that it is differentiable.

For sufficiency, we can again appeal to our crude method and use the sufficiency results we inherit from unconstrained optimization. Since we now need F to be differentiable twice, so as to make it possible that $F''(x^*) < 0$, we must assume that so are f and g, and moreover, we need to know $y''(x)$. Since we already know $y'(x)$, by differentiation,

$$\begin{aligned} y''(x) &= -\frac{\partial}{\partial x}\left(\frac{\partial_x g(x, y(x))}{\partial_y g(x, y(x))}\right) \\ &= -\frac{1}{\partial_y g(x, y(x))}\begin{pmatrix} 1 & y'(x) \end{pmatrix} D^2 g(x, y(x)) \begin{pmatrix} 1 \\ y'(x) \end{pmatrix} \end{aligned}$$

Now, the condition that $F''(x^*) < 0$ is equivalent, by substitution,[3] to the requirement that

$$\begin{pmatrix} 1 & y'(x^*) \end{pmatrix} D^2_{(x,y)}\mathcal{L}(x^*, y^*, \lambda^*) \begin{pmatrix} 1 \\ y'(x^*) \end{pmatrix} < 0.$$

Obviously, this condition is satisfied if $D^2_{(x,y)}\mathcal{L}(x^*, y^*, \lambda^*)$ is negative definite, but this would be overkill: notice that

$$\begin{pmatrix} 1 & y'(x^*) \end{pmatrix} \cdot Dg(x^*, y^*) = 0,$$

so it suffices that we guarantee that for every $\Delta \in \mathbb{R}^2 \setminus \{0\}$ such that $\Delta \cdot Dg(x^*, y^*) = 0$ we have that $\Delta^\top D^2_{(x,y)}\mathcal{L}(x^*, y^*, \lambda^*)\Delta < 0$.

So, in summary, we seem to have argued to following result:

Suppose that $f, g \in C^1$*. Then:*
(1) $x^* \in D$ *locally solves Problem* (9.1), only if *there exists* $\lambda^* \in \mathbb{R}^J$ *such that* $D\mathcal{L}(x^*, \lambda^*) = 0$.
(2) If $f, g \in C^2$ *and there exists* $\lambda^* \in \mathbb{R}^J$ *such that (i)* $D\mathcal{L}(x^*, \lambda^*) = 0$*, and (ii) that for every* $\Delta \in \mathbb{R}^2 \setminus \{0\}$ *such that* $\Delta \cdot Dg(x^*, y^*) = 0$*, we have that* $\Delta^\top D^2_{(x,y)}\mathcal{L}(x^*, y^*, \lambda^*)\Delta < 0$*; then,* $x^* \in D$ *locally solves Problem* (9.1).

But we still need to argue that we can indeed solve y as a function of x . Notice that it has been crucial throughout our analysis that $\partial_y g(x^*, y^*) \neq 0$. Of course, even if the latter hadn't been true, but $\partial_x g(x^*, y^*) \neq 0$, our method would still have worked, mutatis mutandis. So, what we actually require is that $Dg(x^*, y^*)$ have rank 1, its maximum possible. The obvious question is: is this a general result, or does it only work in our simplified case?

To see that it is indeed a general result, we introduce without proof the following important result:

THEOREM 9.1.1 (The Implicit Function Theorem). *Let* $D \subseteq \mathbb{R}^{K+J}$ *and let* $g : D \to \mathbb{R}^J \in C^1$*. If* $(x^*, y^*) \in D$ *is such that* $\operatorname{rank}(D_y g(x^*, y^*)) = J$*, then there exist* $\varepsilon, \delta > 0$ *and* $\gamma : B_\varepsilon(x^*) \to B_\delta(y^*) \in C^1$ *such that:*

1. *for all* $x \in B_\varepsilon(x^*)$*,* $(x, \gamma(x)) \in D$*;*

2. *for all* $x \in B_\varepsilon(x^*)$*,* $g(x, y) = g(x^*, y^*)$ *for* $y \in B_\delta(y^*)$ *if, and only if* $y = \gamma(x)$*;*

3. *for all* $x \in B_\varepsilon(x^*)$*,* $D\gamma(x) = -D_y g(x, \gamma(x))^{-1} D_x g(x, \gamma(x))$.

[3] Note that $F''(x)$ equals

$$\begin{aligned} &\partial^2_{xx} f(x, y(x)) \\ &+ \partial^2_{xy} f(x, y(x))y'(x) \\ &+ \partial^2_{yx}(x, y(x))y'(x) \\ &+ \partial^2_{yy} f(x, y(x))y'(x)^2 \\ &+ \partial_y f(x, y(x))y''(x)), \end{aligned}$$

or, by substitution,

$$\begin{aligned} &(1, y'(x))D^2 f(x, y(x))\begin{pmatrix} 1 \\ y'(x) \end{pmatrix} \\ &- \frac{\partial_y f(x, y(x))}{\partial_y g(x, y(x))}\times \\ &(1, y'(x))D^2 g(x, y(x))\begin{pmatrix} 1 \\ y'(x) \end{pmatrix}. \end{aligned}$$

Substitution at x^* yields the expression that follows, by definition of y^* and λ^*.

This important theorem allows us to express y as a function of x and gives us the derivative of this function: exactly what we wanted! Of course, we need to satisfy the hypotheses of the theorem if we are to invoke it. In particular, the condition on the rank is known as "constraint qualification" and is crucial for the Lagrangean method to work (albeit it is oftentimes forgotten!). So, finally, the following result is true:

THEOREM 9.1.2 (Lagrange). *Let* $f : D \to \mathbb{R}$ *and* $g : D \to \mathbb{R}^J$ *be of class* C^1*, with* $J \leqslant K$*. Let* $x^* \in D$ *be such that* $\operatorname{rank}(Dg(x^*)) = J$*. Then,*

1. *If* x^* *locally solves Problem* (9.1)*, then there exists* $\lambda^* \in \mathbb{R}^J$ *such that* $D\mathcal{L}(x^*, \lambda^*) = 0$*.*

2. *If there exists* $\lambda^* \in \mathbb{R}^J$ *such that (i)* $D\mathcal{L}(x^*, \lambda^*) = 0$ *and (ii) for every* $\Delta \in \mathbb{R}^K \setminus \{0\}$ *such that* $Dg(x^*)\Delta = 0$*, it is true that* $\Delta^\top D^2_{x,x}\mathcal{L}(x^*, \lambda^*)\Delta < 0$*; then,* x^* *locally solves Problem* (9.1)*.*

9.2 Inequality constraints

9.2.1 Linear programming

Let A be an $m \times n$ matrix, and let $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.[4] Consider the following problem:

$$v_p = \max_{x \in \mathbb{R}^n} c \cdot x : Ax \leqslant b.$$

[4] We will follow the convention that all vectors are taken as columns.

This is one of several equivalent representations of linear programs: problems where a linear function is to be optimized over a polyhedron. There are several interesting results and well understood algorithms that solve this kind of problem. Here, we focus on two specific results.

Define the following "dual" problem,

$$v_d = \min_{y \in \mathbb{R}^m_+} b \cdot y : y^\top A = c^\top.$$

From now on, refer to the original problem as the "primal." Notice for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m_+$ such that $Ax \leqslant b$ and $y^\top A = c^\top$, it is true that $c \cdot x = y^\top Ax \leqslant y^\top b$, so it follows that, if they exist, $v_p \leqslant v_d$.

That is, for any $y \geqslant 0$ such that $y^\top A = c$, the number $b \cdot y$ is an upper bound to the solution of the primal problem; the dual problem finds the lowest such upper bound. Crucially, if both problems are feasible, then they both have solution and their solutions are the same!

THEOREM 9.2.1 (The Duality Theorem). *Suppose that there exists* $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m_+$ *such that* $A\bar{x} \leqslant b$ *and* $\bar{y}^\top A = c^\top$*. Then,* $v_p = v_d \in \mathbb{R}$*.*

Proof. It suffices to show that there exists $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m_+$ such that $Ax \leqslant b$, $y^\top A = c^\top$ and $c \cdot x \geqslant b \cdot y$. By the Theorem of the Alternative (or Farkas's lemma), it suffices to show that for any $(\alpha, \beta, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}_+ \times \mathbb{R}^n$, if $\alpha^\top A - \beta c^\top = 0$ and $\beta b^\top + \mu^\top A^\top = 0$, then $\alpha^\top b + \mu^\top c \geqslant 0$. Now, to see that this true, consider the following two cases:
(1) If $\beta > 0$, then

$$\alpha^\top b = \frac{\beta}{\beta} b^\top \alpha = -\frac{1}{\beta}\mu^\top A^\top \alpha = -\frac{1}{\beta}\alpha A^\top \mu = -\frac{1}{\beta}\beta c^\top \mu.$$

(2) If $\beta = 0$, then
$$\alpha^\top b \geqslant \alpha^\top A\bar{x} = 0 = \mu^\top A^\top \bar{y} = \mu^\top c.$$
□

THEOREM 9.2.2 (Complementary Slackness). *Suppose that* $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m_+$ *satisfies* $A\bar{x} \leqslant b$ *and* $\bar{y}^\top A = c^\top$. *The following statements are equivalent:*

1. $\bar{x}$ *solves the primal problem and* $\bar{y}$ *solves the dual problem;*

2. $\bar{y}^\top (b - A\bar{x}) = 0$.

Proof. To see that 1 implies 2, notice that, by the Duality Theorem, $c \cdot \bar{x} = b \cdot \bar{y}$, while $\bar{y}^\top A = c^\top$.

To see that 2 implies 1, it suffices to show that $c \cdot \bar{x} = b \cdot \bar{y}$. But this is immediate, since $\bar{y}^\top A = c^\top$, if $\bar{y}^\top (b - A\bar{x}) = 0$. □

9.2.2 Non-linear programming

As before, let $f : D \to \mathbb{R} \in C^1$ and $f : D \to \mathbb{R}^J \in C^1$. Now suppose that we have to solve the problem
$$\max_{x \in D} f(x) : g(x) \geqslant 0. \tag{9.2}$$
Again, the "usual" method says that one should try to find $(x^*, \lambda^*) \in D \times \mathbb{R}^J_+$ such that $D_x\mathcal{L}(x^*, \lambda^*) = 0$, $g(x^*) \geqslant 0$ and $\lambda^* \cdot g(x^*) = 0$. It is as though there is a theorem that states:

> *If* $x^* \in D$ *locally solves Problem* (9.2), *then there exists* $\lambda^* \in \mathbb{R}^J_+$ *such that* $D_x\mathcal{L}(x^*, \lambda^*) = 0$, $g(x^*) \geqslant 0$ *and* $\lambda^* \cdot g(x^*) = 0$.

Now, even though in this statement we are recognizing the local character and (only) the necessity of the result, we still have to worry about constraint qualification. To see that this is the case, consider the following example:

EXAMPLE 9.2.1. *Consider the problem*
$$\max_{(x,y) \in \mathbb{R}^2} -((x-3)^2 + y^2) : 0 \leqslant y \leqslant -(x-1)^3.$$
The Lagrangean of this problem can be written as
$$\mathcal{L}(x, y, \lambda_1, \lambda_2) = -(x-3)^2 - y^2 + \lambda_1(-(x-1)^3 - y) + \lambda_2 y.$$
Notice that, although $(1, 0)$ *solves the problem, there is no solution* $(x^*, \lambda_1^*, \lambda_2^*)$ *to the following system:*
(i) $-2(x^* - 3) + 3\lambda_1^*(x^* - 1)^2 = 0$ *and* $-2y^* - \lambda_1^* + \lambda_2^* = 0$;
(ii) $\lambda_1^* \geqslant 0$ *and* $\lambda_2^* \geqslant 0$;
(iii) $-(x^* - 1)^3 - y^* \geqslant 0$ *and* $y^* \geqslant 0$; *and*
(iv) $\lambda_1^*(-(x^* - 1)^3 - y^*) = 0$ *and* $\lambda_2^* y^* = 0$.

If the first order conditions were necessary even without the constraint qualification (i.e. if the statement were true) the system of equations in the previous example would necessarily have to have a solution. The point of the example is just that the theorem requires the constraint qualification condition: the following theorem is true.

THEOREM 9.2.3 (Kühn - Tucker). *Let* $f : D \to \mathbb{R} \in C^1$ *and* $g : D \to \mathbb{R}^J \in C^1$. *Let* $x^* \in D$ *be such that* $g(x^*) \geqslant 0$. *Define the set* $\mathcal{J} = \{j \in \{1, \dots, J\} \mid g_j(x^*) = 0\}$, *let* $I = \#\mathcal{J}$, *and suppose that* $\operatorname{rank}(D\tilde{g}(x^*)) = I$ *for* $\tilde{g} : D \to \mathbb{R}^I$ *defined by* $\tilde{g}(x) = (g_j(x))_{j \in \mathcal{J}}$. *Then,*

1. *If* x^* *is a local solution to Problem* (9.2), *then there exists* $\lambda^* \in \mathbb{R}^J_+$ *such that* $D_x \mathcal{L}(x^*, \lambda^*) = 0$, $g(x^*) \geqslant 0$ *and* $\lambda^* \cdot g(x^*) = 0$.

2. *Suppose that* $f, g \in C^2$ *and there exists* $\lambda^* \in \mathbb{R}^J_+$ *such that:*
   (i) $D_x \mathcal{L}(x^*, \lambda^*) = 0$,
   (ii) $g(x^*) \geqslant 0$,
   (iii) $\lambda^* \cdot g(x^*) = 0$, *and*
   (iv) $\Delta^\top D^2_{x,x} \mathcal{L}(x^*, \lambda^*) \Delta < 0$ *for all* $\Delta \in \mathbb{R}^I \setminus \{0\}$ *such that* $\Delta \cdot D\tilde{g}(x^*) = 0$.
   Then, x^* *is a local solution to Problem* (9.2).

As before, it must be noticed that there is a gap between necessity and sufficiency, and that the theorem only gives local solutions. For the former problem, there is no solution. For the latter, one can study concavity of the objective function and convexity of the feasible set. Importantly, notice that with inequality constraint the sign of λ does matter: this is because of the geometry of the theorem: a local maximizer is attained when the feasible directions, as determined by the gradients of the binding constraints is exactly opposite to the desired direction, as determined by the gradient of the objective function. Obviously, locally only the binding constraints matter, which explains why the constraint qualification looks more complicated here than with equality constraints. Finally, it is crucial to notice that the process does not amount to maximizing $\mathcal{L}$: in general, $\mathcal{L}$ does not have a maximum; what one finds is a saddle point of $\mathcal{L}$.

The proof of the following result is left as an exercise

THEOREM 9.2.4. *Suppose that* $f : \mathbb{R}^K \to \mathbb{R}$ *and* $g : \mathbb{R}^K \to \mathbb{R}^J$ *are both of class* C^1.

1. *Suppose that the set* $F = \{x \in \mathbb{R}^n \mid g(x) \geqslant 0\}$ *is compact, and that for every* $x \in F$, *if we denote* $\mathcal{J}(x) = \{j \in \{1, \dots, J\} \mid g_j(x) = 0\}$ *and* $I(x) = \#\mathcal{J}(x)$, *we have that*
$$\operatorname{rank}([Dg_j(x)]_{j \in \mathcal{J}(x)}) = I(x).$$
   If there exists $x^* \in F$ *such that*
   (i) there is some $\lambda^* \in \mathbb{R}^m_+$ *for which* $D_x \mathcal{L}(x^*, \lambda^*) = 0$ *and* $\lambda^* \cdot g^*(x^*) = 0$*; and*
   (ii) for every $x \in F \setminus \{x^*\}$ *and all* $\lambda \in \mathbb{R}^m_+$, *the equality* $D_x \mathcal{L}(x, \lambda) = 0$ *implies that* $\lambda \cdot g(x) > 0$*;*
   then x^* *uniquely solves Problem* (9.2).

2. *Suppose that there exists no pair* (x, λ) *for which* $D_x \mathcal{L}(x, \lambda) = 0$, $\lambda \geqslant 0$, $g(x) \geqslant 0$, *and* $\lambda \cdot g(x) = 0$. *Then,* x^* *locally solves Problem* (9.2) *only if*
$$\operatorname{rank}((Dg_i(x))_{i \in \mathcal{J}}) < I,$$
   where $\mathcal{J} = \{i \in \{1, \dots, J\} \mid g_i(x^*) = 0\}$ *and* $I = \#\mathcal{J}$.

9.3 Parametric programming

WE NOW STUDY HOW THE solution of a problem depends on the parameters that define the problem.

9.3.1 Continuity

Let $\Omega \subseteq \mathbb{R}^M$ be nonempty, and let $D : \Omega \twoheadrightarrow \mathbb{R}^K$ be a correspondence from Ω into $\mathbb{R}^K$: for every $\omega \in \Omega$, this correspondence assigns a subset $D(\omega) \subseteq \mathbb{R}^K$.

Suppose that $D : \Omega \twoheadrightarrow \mathbb{R}^K$ is nonempty- and compact-valued, in the sense that for all $\omega \in \Omega$, the set $D(\omega)$ is non-empty and compact.

DEFINITION 9.3.1. *Correspondence* D *is* upper hemicontinuous at $\omega \in \Omega$ *if for every pair of sequences* $(\omega_n)_{n=1}^{\infty}$ *in* $\mathbb{R}^M$, *and* $(x_n)_{n=1}^{\infty}$ *in* $\mathbb{R}^K$, *such that*
(i) for all $n \in \mathbb{N}$, $\omega_n \in \Omega$,
(ii) $\lim_{n\to\infty} \omega_n = \omega$ *and*
(iii) for all $n \in \mathbb{N}$, $x_n \in D(\omega_n)$,
there exist a subsequence $(x_{n_m})_{m=1}^{\infty}$ *of* $(x_n)_{n=1}^{\infty}$ *and a point* $x \in D(\omega)$ *such that*

$$\lim_{k\to\infty} x_{n_m} = x.$$

The correspondence is upper hemicontinuous *if it is upper hemicontinuous at every* $\omega \in \Omega$.

An upper hemicontinuous correspondence has the property that its graph is "closed" at points where the correspondence explodes.[5]

[5] More formally, what suffices for upper hemicontinuity is that the graph of the correspondence be closed, and that for all bounded $B \subseteq \Omega$, the set

$$\cup_{\omega \in B} D(\omega)$$

be bounded too.

DEFINITION 9.3.2. *Correspondence* D *is* lower hemicontinuous at $\omega \in \Omega$ *if for every sequence* $(\omega_n)_{n=1}^{\infty}$ *in* $\mathbb{R}^M$ *such that*
(i) for all $n \in \mathbb{N}$, $\omega_n \in \Omega$ *and*
(ii) $\lim_{n\to\infty} \omega_n = \omega$,
and for every point $x \in D(\omega)$, *there exists a sequence* $(x_n)_{n=1}^{\infty}$ *in* $\mathbb{R}^K$ *such that*
(iii) for all $n \in \mathbb{N}$, $x_n \in D(\omega_n)$, *and*
(iv) $\lim_{n\to\infty} x_n = x$.
The correspondence is lower hemicontinuous *if it is lower hemicontinuous at every* $\omega \in \Omega$.

A lower hemicontinuous correspondence has the property that its graph is "open" at points where the correspondence explodes.

DEFINITION 9.3.3. *Correspondence* D *is* continuous at $\omega \in \Omega$ *if it is upper and lower hemicontinuous at* ω. *It is* continuous *if it is continuous at every* $\omega \in \Omega$.

A continuous correspondence does not explode.

THEOREM 9.3.1. *If correspondence* $F : \Omega \twoheadrightarrow \mathbb{R}^K$ *is singleton-valued*[6] *and upper or lower hemicontinuous, then the function* $f : \Omega \to \mathbb{R}^K$, *defined implicitly by* $\{f(\omega)\} = F(\omega)$, *is continuous.*

[6] That is, if for all $\omega \in \Omega$, one has that $D(\omega)$ is a singleton set.

Proof. The argument is left as an exercise. □

In fact, the relationship between the concepts introduced in the previous theorem is stronger: in the case of single valued correspondence, both types of hemicontinuity are equivalent to continuity of the associated function (and equivalent to each other, then).

The importance of the concept of continuity of correspondences is given by the following result

THEOREM 9.3.2 (Theorem of the Maximum). *Let function* $f : \mathbb{R}^K \times \Omega \to \mathbb{R}$ *be continuous and let correspondence* $D : \Omega \twoheadrightarrow \mathbb{R}^K$ *be nonempty-, compact-valued and continuous. The correspondence* $X : \Omega \twoheadrightarrow \mathbb{R}^K$ *defined by*

$$X(\omega) = \text{argmax}_{x \in D(\omega)} f(x, \omega)$$

is upper hemicontinuous[7] *and the* value function $\nu : \Omega \to \mathbb{R}$, *defined by*

$$\nu(\omega) = \max_{x \in D(\omega)} f(x, \omega),$$

is continuous.

[7] And nonempty- and compact-valued.

Proof. Since D is nonempty- and compact-valued, and f is continuous (in x), Weierstrass's Theorem guarantees that X is non-empty valued and ν is well defined. These same assumptions imply that X is compact-valued.

To see that X is upper hemicontinuous, fix $\omega \in \Omega$ and take sequences $(\omega_n)_{n=1}^{\infty}$ and $(x_n)_{n=1}^{\infty}$ such that the three conditions in Definition 9.3.1 are satisfied. Since D is upper hemicontinuous, there exists a subsequence $(x_{n_m})_{m=1}^{\infty}$ such that $x_{n_m} \to x$, for some $x \in D(\omega)$. Now, let $x' \in D(\omega)$. Since D is *lower* hemicontinuous at ω, there exists a sequence $(x'_m)_{m=1}^{\infty}$ such that $x'_m \in D(\omega_{n_m})$, for all $m \in \mathbb{N}$ and $x'_m \to x'$. By construction (property ii), we have that $f(x_{n_m}, \omega_{n_m}) \geqslant f(x'_m, \omega_{n_m})$. Since $\omega_{n_m} \to \omega$, $x_{n_m} \to x$ and $x'_m \to x'$, and since f is continuous,

$$f(x, \omega) = \lim_{m \to \infty} f(x_{n_m}, \omega_{n_m}) \geqslant \lim_{m \to \infty} f(x'_m, \omega_{n_m}) = f(x', \omega).$$

Since the latter is true for all $x' \in D(\omega)$, we have that $x \in X(\omega)$.

Now, to see that ν is continuous, fix $\omega \in \Omega$ and suppose that ν is *not* continuous at ω. Then, for some $\varepsilon > 0$ we can construct a sequence $(\omega_n)_{n=1}^{\infty}$ in Ω that converges to ω, but such that $|\nu(\omega_n) - \nu(\omega)| \geqslant \varepsilon$ for all $n \in \mathbb{N}$. Fix $x_n \in X(\omega_n)$ and $x \in X(\omega)$. Since X is upper hemicontinuous, we can find a subsequence $(x_{n_m})_{m=1}^{\infty}$ that converges to x. It is immediate that $(x_{n_m}, \omega_{n_m}) \to (x, \omega)$, but, by construction, $|f(x_{n_m}, \omega_{n_m}) - f(x, \omega)| \geqslant \varepsilon$ for all $m \in \mathbb{N}$. This implies that $f(x_{n_m}, \omega_{n_m}) \nrightarrow f(x, \omega)$, which is impossible since f is continuous. $\square$

9.4 Application: Continuity of the Marshallian demand

WE NOW WANT TO STUDY WHETHER the Marshallian demand of an individual is continuous. By the Theorem of the Maximum, if the individual's utility function is continuous, all we need to show is that the budget correspondence is continuous as well.

THEOREM 9.4.1. *The budget correspondence* $B \twoheadrightarrow \mathbb{R}^L_{++} \times \mathbb{R}_+ \twoheadrightarrow \mathbb{R}^L_+$ *is continuous.*

Proof. Fix $(p, m) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$ and a sequence $(p_n, m_n)_{n=1}^{\infty}$, defined in $\mathbb{R}^L_{++} \times \mathbb{R}_+$ and such that $(p_n, m_n) \to (p, m)$. Let $(x_n)_{n=1}^{\infty}$ be a sequence such that $x_n \in B(p_n, m_n)$ for all $n \in \mathbb{N}$.

For each $l \in \{1, \dots, L\}$, since $p_{l,n} \to p_l \in \mathbb{R}_{++}$, there is some $p^*_l \in \mathbb{R}_{++}$ such that, for all $n \in \mathbb{N}$, $p_{l,n} \geqslant p^*_l$. Denote $p^* = (p^*_1, \dots, p^*_L)$. Since $m_n \to m$, there exists $m^* \in \mathbb{R}_+$ for which $m_n \leqslant m^*$, for all $n \in \mathbb{N}$. By construction, $x_n \in B(p_n, m_n) \subseteq B(p^*, m^*)$.

By Theorem 2.6.5, there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ that converges to some $x \in \mathbb{R}_+^L$. By continuity of the inner product, $p_{n_k} \cdot x_{n_k} \to p \cdot x$, while $p_{n_k} \cdot x_{n_k} \leqslant m_{n_k} \to m$ implies that $p \cdot x \leqslant m$, so that $x \in B(p, m)$. This shows that B is upper hemicontinuous at (p, m). Since (p, m) was arbitrary, it follows that B is upper hemicontinuous.

Now, let $(p_n, m_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}_{++}^L \times \mathbb{R}_+$ such that $(p_n, m_n) \to (p, m)$, and let $x \in B(p, m)$. If $m = 0$, it is trivial that B is lower hemicontinuous at (p, m), so let us assume that $m > 0$. Define, for all $l \in \{1, \ldots, L\}$,

$$s_l = \frac{p_l x_l}{m}$$

and, for all $n \in \mathbb{N}$,

$$x_{l,n} = \frac{s_l m_n}{p_{l,n}}$$

and $x_n = (x_{1,n}, \ldots, x_{L,n})$. Clearly $x_n \in B(p_n, m_n)$, and, each $l \in \{1, \ldots, L\}$,

$$x_{l,n} = \frac{p_l x_l}{m} \frac{m_n}{p_{l,n}} \to x_l,$$

so $x_n \to x$. This shows that B is upper hemicontinuous at (p, m), and, hence that it is upper hemicontinuous, since (p, m) was arbitrary. □

Corollary 9.4.1. *If $u : \mathbb{R}_+^L \to \mathbb{R}$ is a continuous utility function, then its Marshallian demand correspondence is upper hemicontinuous, and its indirect utility function is continuous. If, in addition, u is strictly quasiconcave, then its Marshallian demand is a continuous function.*

Corollary 9.4.2. *If $\succsim$ is a complete, strictly monotone, convex and continuous pre-order, then its Marshallian demand correspondence is upper hemicontinuous. If, in addition, $\succsim$ is strictly convex, then its Marshallian demand is a continuous function.*

9.5 Differentiability

Suppose now that both sets $D \subseteq \mathbb{R}^K$ and $\Omega \subseteq \mathbb{R}^M$, are open and finite-dimensional. Suppose that $f : D \times \Omega \to \mathbb{R}$ and $g : D \times \Omega \to \mathbb{R}^J$, and consider the following (simplified) parametric problem: given $\omega \in \Omega$, let

$$v(\omega) = \max_{x \in D} f(x, \omega) : g(x, \omega) = 0.$$

Suppose that the differentiability and second-order conditions are given, so that a point x^* solves this maximization problem if, and only if, there exists a $\lambda^* \in \mathbb{R}^J$ such that $D\mathcal{L}(x^*, \lambda^*, \omega) = 0$.

Suppose furthermore that we can define functions $x : \Omega \to D$ and $\lambda : \Omega \to \mathbb{R}^J$, given by the solution of the problem and the associated multiplier, for every ω. Then, it follows directly from the Implicit Function Theorem that if, for a given $\bar{\omega} \in \Omega$,

$$\operatorname{rank} \begin{pmatrix} 0_{J \times J} & D_x g(x^*, \bar{\omega}) \\ D_x g(x^*, \bar{\omega})^\top & D_{x,x}^2 \mathcal{L}(x^*, \lambda^*, \bar{\omega}) \end{pmatrix} = J + K,$$

then there exists some $\epsilon > 0$ such that on $B_\epsilon(\bar{\omega})$ the functions x and λ are differentiable and

$$\begin{pmatrix} D\lambda(\bar{\omega}) \\ Dx(\bar{\omega}) \end{pmatrix} = -\begin{pmatrix} 0_{J\times J} & D_x g(x^*,\bar{\omega}) \\ D_x g(x^*,\bar{\omega})^\top & D^2_{x,x}\mathcal{L}(x^*,\lambda^*,\bar{\omega}) \end{pmatrix}^{-1} \begin{pmatrix} D_\omega g(x(\bar{\omega}),\bar{\omega}) \\ D^2_{\omega,x}\mathcal{L}(x(\bar{\omega}),\lambda(\omega),\bar{\omega}) \end{pmatrix}.$$

It is then immediate that ν is differentiable at $\bar{\omega}$ and

$$D\nu(\bar{\omega}) = D_x f(x(\bar{\omega}),\bar{\omega})Dx(\bar{\omega}) + D_\omega(\bar{\omega}).$$

A simpler method, however, is given by the following theorem

THEOREM 9.5.1 (The Envelope Theorem). *If, under the assumptions of this subsection, ν is continuously differentiable at $\bar{\omega}$, then $D\nu(\bar{\omega}) = D_\omega\mathcal{L}(x(\bar{\omega}),\lambda(\bar{\omega}),\bar{\omega})$.*

Proof. One just needs to use the Chain Rule: by assumption,

$$D_x f(x(\omega),\omega) + D_x g(x(\omega),\omega)^\top\lambda(\omega) = 0,$$

whereas $g(x(\omega),\omega) = 0$, so

$$D_x g(x(\omega),\omega)Dx(\omega) + D_\omega g(x(\omega),\omega) = 0;$$

meanwhile,

$$\begin{aligned} D\nu(\omega) &= Dx(\omega)^\top D_x f(x(\omega),\omega) + D_\omega f(x(\omega),\omega) \\ &= -D_x(\omega)^\top D_x g(x(\omega),\omega)^\top\lambda(\omega) + D_\omega f(x(\omega),\omega) \end{aligned}$$

and

$$\begin{aligned} D_\omega\mathcal{L}(x(\omega),\lambda(\omega),\omega) &= D_\omega f(x(\omega),\omega) + D_\omega g(x(\omega),\omega)^\top\lambda(\omega) \\ &= D_\omega f(x(\omega),\omega) - D_x(\omega)^\top D_x g(x(\omega),\omega)^\top\lambda(\omega), \end{aligned}$$

which gives the result. □

EXERCISE 9.5.1. *Let $f:\mathbb{R}^K\to\mathbb{R}$, $g:\mathbb{R}^K\to\mathbb{R}^J\in C^2$, with $J\leqslant K\in\mathbb{N}$. Suppose that for all $\omega\in\mathbb{R}^m$, the problem*

$$\max f(x) : g(x) = \omega$$

has a solution, which is characterized by the first order conditions of the Lagrangean defined by $\mathcal{L}(x,\lambda,\omega) = f(x) + \lambda\cdot(\omega - g(x))$. Suppose furthermore that these conditions define differentiable functions $x:\mathbb{R}^J\to\mathbb{R}^K$ and $\lambda:\mathbb{R}^J\to\mathbb{R}^J$. Prove that $D\nu(\omega) = \lambda(\omega)$, for all ω, where $\nu:\mathbb{R}^J\to\mathbb{R}$ is the value function of the problem.

10

Fixed Point Theory

TWO KEY RESULTS IN mathematical economics are presented next.

10.1 Kakutani's fixed-point theorem

THEOREM 10.1.1 (Kakutani). *Let $\Delta \subseteq \mathbb{R}^L$ and let $\Gamma : \Delta \twoheadrightarrow \Delta$ be a non-empty-valued correspondence. If Δ is compact and convex, and Γ is convex-valued and upper hemicontinuous, then there exists $\delta \in \Delta$ such that $\delta \in \Gamma(\delta)$.*

When Γ is single-valued (i.e. a function), the result is referred to as Brower's fixed-point theorem and is easy enough to visualize in the case $L = 1$.

A complete proof of Kakutani's theorem is far from simple and we will not attempt it here. To see the general strategy of such proof, here we will see its simplest version, in $\mathbb{R}$. That is, we will argue the following result:

> Let $\Gamma : [0,1] \to [0,1]$ be nonempty-, convex- and compact-valued, and upper hemicontinuous. There exists $\delta \in [0,1]$ such that $\delta \in \Gamma(\delta)$.

Proof of Theorem 10.1.1, in $\mathbb{R}$: The following algorithm constructs a sequence in $[0,1]^4$.

0 Let $n = 0$, $\underline{\delta}_0 = 0$, $\overline{\delta}_0 = 1$, $\underline{\gamma}_0 \in \Gamma(0)$ and $\overline{\gamma}_0 \in \Gamma(1)$.

1 Let $n = n + 1$. If

$$\exists \delta \in \Gamma\left(\frac{\underline{\delta}_{n-1} + \overline{\delta}_{n-1}}{2}\right) : \delta \geqslant \frac{\underline{\delta}_{n-1} + \overline{\delta}_{n-1}}{2},$$

go to 2; else, go to 3.

2 Define

$$\underline{\delta}_n = \frac{\underline{\delta}_{n-1} + \overline{\delta}_{n-1}}{2},\ \overline{\delta}_n = \overline{\delta}_{n-1},\ \underline{\gamma}_n = \delta \text{ and } \overline{\gamma}_n = \overline{\gamma}_{n-1}$$

and go to 1.

3 Find

$$\delta \in \Gamma\left(\frac{\underline{\delta}_{n-1} + \overline{\delta}_{n-1}}{2}\right) : \delta > \frac{\underline{\delta}_{n-1} + \overline{\delta}_{n-1}}{2},$$

and define

$$\underline{\delta}_n = \underline{\delta}_{n-1},\ \overline{\delta}_n = \frac{\underline{\delta}_{n-1} + \overline{\delta}_{n-1}}{2},\ \underline{\gamma}_n = \underline{\gamma}_{n-1} \text{ and } \overline{\gamma}_n = \delta.$$

Go to 1.

This algorithm constructs a sequence $(\underline{\delta}_n, \underline{\gamma}_n, \overline{\gamma}_n, \overline{\delta}_n)_{n=1}^{\infty}$, in $[0,1]^4$, that satisfies the following properties, for all $n \in \mathbb{N}$:

(i) $\underline{\delta}_n \leqslant \underline{\gamma}_n$ and $\overline{\gamma}_n \leqslant \overline{\delta}_n$;

(ii) $0 \leqslant \underline{\gamma}_n < \overline{\gamma}_n \leqslant 1$; and

(iii) $\overline{\gamma}_n - \underline{\gamma}_n \leqslant 2^{-n}$.

Since this sequence is bounded, by Theorem 2.6.5 it possesses a convergent subsequence. Since Γ is upper hemicontinuous, we can pick this subsequence,

$$(\underline{\delta}_{n_m}, \underline{\gamma}_{n_m}, \overline{\gamma}_{n_m}, \overline{\delta}_{n_m})_{m=1}^{\infty},$$

such that

$$(\underline{\delta}_{n_m}, \underline{\gamma}_{n_m}, \overline{\gamma}_{n_m}, \overline{\delta}_{n_m}) \to (\underline{\delta}^*, \underline{\gamma}^*, \overline{\gamma}^*, \overline{\delta}^*)$$

with $\underline{\gamma}^* \in \Gamma(\underline{\delta}^*)$ and $\overline{\gamma}^* \in \Gamma(\overline{\delta}^*)$. By condition (i), $\underline{\delta}^* \leqslant \underline{\gamma}^*$ and $\overline{\gamma}^* \leqslant \overline{\delta}^*$. By conditions (ii) and (iii), $|\overline{\delta}_{n_m} - \underline{\delta}_{n_m}| \to 0$, so $\underline{\delta}^* = \overline{\delta}^*$. Let $\delta = \underline{\delta}^*$, and note that $\overline{\gamma}^* \leqslant \delta \leqslant \underline{\gamma}^*$. Since $\underline{\gamma}^* \in \Gamma(\underline{\delta}^*) = \Gamma(\delta)$ and $\overline{\gamma}^* \in \Gamma(\overline{\delta}^*) = \Gamma(\delta)$ and Γ is convex-valued, we have that $\delta \in \Gamma(\delta)$.[1] *Q.E.D.*

[1] Recall that $\Gamma(\delta^*) \subseteq \mathbb{R}$.

10.1.1 *Application: Existence of Nash Equilibrium*

SUPPOSE THAT THERE IS A FINITE SET of players (society), each of which we denote by $i \in \{1, \dots, I\}$. Suppose that player i can choose an action from a set of alternatives Σ^i, which is assumed to be nonempty. Denote by s^i the action taken by i; the full profile of actions chosen by the society, $s = (s^1, \dots, s^I)$, determines the outcome of the game, and in fact, we identify the profile of actions with the outcome of the game. Players (may) care about the outcome of the game: i derives utility (receives payoff) $u^i(s^1, \dots, s^I)$ from the outcome of the game. These elements describe a simultaneous-move game: formally, a *simultaneous-move game* is

$$(\{1, \dots, I\}, (\Sigma^i, u^i : \times_{j=1}^{I} \Sigma^j \to \mathbb{R})_{i=1}^{I}).$$

In the present setting, where players play only once and all actions have to be chosen simultaneously, we will also refer to actions as *strategies*, and will refer to each set Σ^i as the set of strategies of player i. For simplicity of notation, denote by $\Sigma^{\neg i}$ the set of strategies of all players other than i, and use $s^{\neg i}$ to denote its generic element.[2] Also, when necessary, use $u^i(s^i, s^{\neg i})$ to denote $u^i((s^1, \dots, s^{i-1}, s^i, s^{i+1}, \dots, s^I)$. The thought process in individual i's mind will be the following: if the other players are playing $s^{\neg i}$, the best i can choose for herself is s^i that makes $u^i(s^i, s^{\neg i})$ as high as possible.[3] Formally, for player i, s^i is a *best response* to $s^{\neg i}$ if it solves the problem $\max_{\hat{s} \in \Sigma^i} u^i(\hat{s}, s^{\neg i})$.

A profile of strategies is a *Nash equilibrium* if each player is satisfied with her choice, given what her opponents are choosing.[4] That is, profile of strategies $(s^1, \dots, s^I) \in \times_{i=1}^{I} \Sigma^i$ is a *Nash equilibrium in pure strategies* if, for each i, s^i is a best response to $s^{\neg i}$.

[2] That is, $\Sigma^{\neg i} = \times_{j \neq i} \Sigma^j$ and $s^{\neg i} = (s^1, \dots, s^{i-1}, s^{i+1}, \dots, s^I)$.

[3] This behavior is known as *best response*: given everything that is not (directly) under her control, an individual's choice should be optimal for herself, according to her own preferences: she should not regret the choice that she makes, *at the time he is making it*.

[4] One may argue that best-response behavior is a very weak principle: why would players not question the motives and choices of their opponents? However, notice that if one finds a profile of choices where every player is best-responding to the rest of the actions, the criticism is less powerful! This leads to the basic concept of equilibrium in games.

Intuitively, a Nash equilibrium is an outcome of a game in which no individual player would find that it is for her own benefit to deviate unilaterally from what is being played. We do not know whether multilateral deviations would benefit some players, nor do say how the game is played when there are multiple outcomes that have the equilibrium property. But if players are going to choose their actions individually, outcomes that are not Nash equilibrium do not result appealing as predictions of the theory: at an outcome like that, at least one player would regret her choice.[5] As in other concepts of economic equilibrium, Nash equilibrium arises as a solution that appeals in the sense that it seems "sustainable," although we do not specify a "dynamics" for which equilibrium corresponds to a "resting point." It is also important to notice that Nash equilibrium, or best-response behavior for that matter, does not give an algorithm to determine how a player chooses her play (or how we should choose ours when we play games): if a player knew what her opponents are choosing, then we would have such an algorithm; but remember that this is a simultaneous-move situation, so we cannot interpret equilibrium as an algorithm for choice! What the definition of equilibrium does is simply to distinguish situations where, under the best-response principle, individuals do not regret the choices they make at the time when they are making them, form situations in which at least one of them does.

[5] Of course, non-equilibrium outcomes can be played. But we do not want to construct a theory in which predict that an individually rational decision maker will choose suboptimal choices. Or, at least, that is not the route that classical game theory has followed.

Suppose that players consider the possibility of choosing their actions randomly. Then, best-response behavior can be applied by considering players that are more sophisticated in their thought processes: given that her opponents are choosing their actions randomly (with certain probabilities), how should a player randomize between her possible actions so as to maximize her expected payoff?[6]

[6] We will accept that expected payoffs correctly represent the individual's preferences, although this too can be questionable.

Let Δ^i be the set of probability distributions over Σ^i. For simplicity of notation, denote by $\Delta^{\neg i} = \times_{j \neq i} \Delta^j$, the set of probability distributions of all players other than i, and use $p^{\neg i}$ to denote its generic element. If the other players are using $p^{\neg i}$ to choose their plays, the best i can choose for herself is p^i that makes $\mathbb{E}_{p^i}[\mathbb{E}_{p^{\neg i}}[u^i(s)]]$ as high as possible.[7] Thus, a profile of probability distributions $(p^1, \ldots, p^I) \in \times_{i=1}^I \Delta^i$ is a *Nash equilibrium in mixed strategies* if for each i, p^i solves the problem

$$\max_{\hat{p} \in \Delta^i} \mathbb{E}_{\hat{p}}[\mathbb{E}_{p^{\neg i}}[u^i(s)]].$$

[7] When all strategy sets are finite, $\mathbb{E}_{p^i}[\mathbb{E}_{p^{\neg i}}[u^i(s)]]$ equals, simply,

$$\sum_s \prod_{j=1}^I p^j(s^j) u^i(s).$$

A game will be said to be *finite* if it has finitely many players and each of them has only finitely many strategies that she can play. Obviously, PRS is finite, and it turns out that finite games often have no Nash equilibrium in pure strategies. As in PRS however, the use of mixed strategies always gives us an equilibrium in this type of game.

THEOREM 10.1.2 (Nash). *Any finite game has a Nash equilibrium in mixed strategies.*

Proof. Let $\Delta = \times_{i=1}^I \Delta^i$, and define the correspondence $B^i : \Delta^{\neg i} \twoheadrightarrow \Delta^i$, by letting

$$B^i(p^{\neg i}) = \operatorname{argmax}_{\hat{p} \in \Delta^i} \mathbb{E}_{\hat{p}}[\mathbb{E}_{p^{\neg i}}[u^i(s)]].$$

This correspondence is known as the *best-response correspondence of player* i. Since Δ^i is compact and the expectation operator is continuous in the probabilities, the correspondence is non-empty- and compact-valued. Since the expectation operator is linear in the probabilities and set Δ^i is convex, it is also convex-valued. By the Theorem of the Maximum, the correspondence is also upper hemi-continuous.

Now, define correspondence $B : \Delta \twoheadrightarrow \Delta$ by $B(p) = \times_i B^i(p^{-i})$. This correspondence inherits all the properties of the best-response correspondences and, hence, by Kakutani's Fixed-Point Theorem, there exists some $p \in \Delta$ such that $B(p) \ni p$. By construction, p is a Nash equilibrium in mixed strategies. $\square$

Luckily there also exist general existence results for Nash equilibrium in pure strategies. The most classical one is presented next, without a proof (but it should not be hard for you to give one, if you understood the argument just given for Nash's theorem).

THEOREM 10.1.3 (Glicksberg). *Suppose that for every player i, the set of strategies is a compact and convex subset of some finite-dimensional Euclidean space. If for every player one has that the payoff function u^i is concave in s^i and continuous, then the game has a Nash equilibrium in pure strategies.*

10.1.2 Application: Existence of Competitive Equilibrium

ASSUME A SOCIETY POPULATED by a finite number of individuals, which we denote by $i = 1, \dots, I$. In this society, we will consider the case in which only exchange of commodities takes place.

Consumer i is modeled by what she likes and what she has. For simplicity of expression, we will assume here that our consumers have preferences that are representable by utility functions $u^i : \mathbb{R}^L_+ \to \mathbb{R}$.[8] In general equilibrium, we want to endogenize the individuals' nominal income, so we will assume that they are endowed with a bundle, $w^i \in \mathbb{R}^L_+$, of commodities.[9]

An *exchange economy* is defined by a society, and by the full description of its members,

$$\{\{1, \dots, I\}, (u^i, w^i)_{i=1}^I\}.$$

Let $p = (p_1, \dots, p_L) \in \mathbb{R}^L$ denote commodity prices, and use x^i to denote individual i's consumption plan.[10]

In an exchange economy with competitive markets, consumers take all prices as independent of their demands, and the only constraint that individual i recognizes is that she cannot spend more than her nominal wealth, which is the nominal value of her endowment.

DEFINITION. *In an exchange economy $\{\{1, \dots, I\}, (u^i, w^i)_{i=1}^I\}$, a* competitive equilibrium *is a pair consisting of a vector of prices and a profile of consumption plans, $(p, (x^i)_{i=1}^I)$, such that: (i) for each consumer i, bundle x^i solves the problem $\max_x u^i(x) : p \cdot x \leqslant p \cdot w^i$; and (ii) all markets clear: $\sum_{i=1}^I x^i = \sum_{i=1}^I w^i$.*

The definition of equilibrium takes preferences and endowments as given fundamentals, and determines values for all endogenous variables of the problem; in the case of an exchange economy, the endogenous variables are all the prices and the consumption decisions of all individuals. Equilibrium is then the requirement that all these variables be feasible and that no agent regret the decision she is making at the time she is making it.[11]

The following property is well known, and simplifies the treatment of competitive equilibrium.

[8] The standard properties of preferences may be invoked. Here, we will interchangeably say that the individual has convex preferences or that her utility function is quasiconcave.

[9] Notice that, by the latter assumption, we are introducing one institution in our society: private property.

[10] We want to study situations where agents trade voluntarily and where they think that their actions do not impact the aggregate conditions at which trade takes place. We, then, are adding a second institution, competitive markets, which are exchange facilities where an anonymous price is announced for each commodity, denoted p_l, and where all traders can trade at that given price.

[11] Importantly, notice that in the interpretation of the definition of competitive equilibrium, there are endogenous variables that are not decided by any one particular agent: while prices are endogenous to the whole economy, each decision-maker thinks that she cannot affect them. Notice also that the definition of equilibrium does not say what occurs in the economy when it is not in equilibrium. Finally, notice the assumptions implicit in the definition: (i) it is assumed, as an institution, the existence of complete competitive markets; (ii) it is assumed, as a rule of behavior, that all agents are price takers; (iii) each individual's behavior affects her well-being only; and (iv) no unit of a commodity can be consumed by more than one consumer. Many results crucially depend on these assumptions.

Proposition (Walras's law). *Fix an exchange economy* $\{(u^i, w^i)_{i=1}^I\}$ *in which all consumers have locally nonsatiated preferences, and at least one of them has strongly monotone preferences. Suppose that* $(p, (x^i)_{i=1}^I)$ *satisfies that:*

1. *for each individual* i, x^i *solves* $\max_x u^i(x) : p \cdot x \leqslant p \cdot w^i$*;*
2. *for all* $l \in \{1, ..., L-1\}$, $\sum_{i=1}^I x_l^i = \sum_{i=1}^I w_l^i$.

Then, all prices are positive, $p \gg 0$*, and the following are all competitive equilibria:*

$$(p, (x^i)_{i=1}^I), \left(\frac{1}{p_1}p, (x^i)_{i=1}^I\right), \left(\frac{1}{||p||}p, (x^i)_{i=1}^I\right) \text{ and } \left(\frac{1}{\sum_l p_l}p, (x^i)_{i=1}^I\right).$$

Proof. Since one individual's preferences are strictly monotone, it follows from condition 1 that allprices must be strictly positive. Since all consumers are locally nonsatiated, condition 1 also implies, by the version of Walras's covered in Consumer's theory, that $\sum_{i=1}^I p \cdot (x^i - w^i) = 0$. But then, by condition 2, $p_L \sum_{i=1}^I (x_L^i - w_L^i) = 0$, which implies that $\sum_{i=1}^I (x_L^i - w_L^i) = 0$, since $p_L > 0$. This means that $(p, (x^i)_{i=1}^I)$ is a competitive equilibrium. That the other pairs are equilibria too follows from homogeneity of degree zero of Marshallian demand. □

The result says that when looking for general equilibria of an economy with strongly monotone consumers, it suffices to guarantee that all of the markets but one clear. This says that the $L \times L$ system of market-clearing equations is underdetermined (as a function of prices), and is in fact an $L \times (L-1)$ system. So, one can drop one variable by letting, for instance, $p_1 = 1$, and solving a $(L-1) \times (L-1)$ system.

Theorem. *Suppose that* $\sum_{i=1}^I w^i \gg 0$ *and that each* u^i *is continuous, strongly quasiconcave and strictly monotone. Then, there exists a competitive equilibrium.*

Denote by Δ the $(L-1)$-dimensional unit simplex, and let $\Delta^o = \Delta \cap \mathbb{R}_{++}^L$ and $\Delta^\partial = \Delta \setminus \Delta^o$. (These sets are known as the relative interior and the boundary of the simplex.) The aggregate excess demand function over strictly positive prices, $Z : \Delta^o \to \mathbb{R}^L$, is defined by

$$Z(p) = \sum_i [x^i(p) - w^i],$$

where

$$x^i(p) = \text{argmax}_x u^i(x) : p \cdot x \leqslant p \cdot w^i.$$

You can take for granted that, under the assumptions of the theorem, Z is continuous and bounded below; satisfies that, for all $p \in \Delta^o$, $p \cdot Z(p) = 0$; and is such that

$$\max_{l=1,...,L} \{Z_l(p_n)\} \to \infty$$

for all sequence $(p_n)_{n=1}^\infty$ in Δ^o such that $p_n \to p \in \Delta^\partial$.

Define correspondence $\Gamma : \Delta \twoheadrightarrow \Delta$ as follows:

$$\Gamma(p) = \begin{cases} \text{argmax}_{\gamma \in \Delta} Z(p) \cdot \gamma, & \text{if } p \in \Delta^o; \\ \{\gamma \in \Delta \mid p \cdot \gamma = 0\}, & \text{if } p \in \Delta^\partial. \end{cases}$$

The exercises that follow will provide a proof of the Theorem:

Exercise 10.1.1. *Argue that* Γ *is nonempty-, compact- and convex-valued.*

EXERCISE 10.1.2. *Argue that if* $p \in \Delta^o$ *and* $Z(p) \neq (0,\dots,0)$ *then* $\Gamma(p) \subseteq \Delta^\partial$.

EXERCISE 10.1.3. *Argue that if* $p \in \Delta^\partial$, *then* $p \notin \Gamma(p)$.

EXERCISE 10.1.4. *Argue that* Γ *is upper hemicontinuous at all* $p \in \Delta^o$.

EXERCISE 10.1.5. *Fix* $p \in \Delta^\partial$, $(p_n)_{n=1}^\infty$ *in* Δ *such that* $p_n \to p$, *and* $(\gamma_n)_{n=1}^\infty$ *in* Δ *such that* $\gamma_n \in \Gamma(p_n)$ *for each* n.

1. *Argue that there exist a subsequence of* $(\gamma_n)_{n=1}^\infty$, $(\gamma_{n_m})_{m=1}^\infty$, *and a* $\gamma \in \Delta$ *such that* $\gamma_{n_m} \to \gamma$.
2. *Fix the subsequence constructed in 1, and suppose that* $(p_{n_m})_{m=1}^\infty$ *has no subsequences in* Δ^o. *Argue that* $p \cdot \gamma = 0$.
3. *Fix the subsequence constructed in 1, and suppose that* $(p_{n_m})_{m=1}^\infty$ *has a subsequence in* Δ^o, $(p_{n_{m_k}})_{k=1}^\infty$. *Argue that there exists* $k^* \in \mathbb{N}$ *such that, for all* $k \geqslant k^*$ *and all* $l \in \{1,\dots,L\}$ *such that* $p_l > 0$,
$$Z_l(p_{n_{m_k}}) < \max_{l' \in \{1,\dots,L\}} \{Z_{l'}(p_{n_{m_k}})\}.$$
Conclude that, hence, $p \cdot \gamma = 0$.
4. *Argue that* Γ *is upper hemicontinuous at all* $p \in \Delta^\partial$.

EXERCISE 10.1.6. *Argue that there exists some* $p \in \Delta$ *such that* $p \in \Gamma(p)$, *and conclude that, hence,* $Z(p) = 0$. *Argue that this proves the theorem.*

10.2 Banach's Fixed-Point Theorem

DEFINITION 10.2.1. *A function* $f : \mathbb{R}^K \to \mathbb{R}^K$ *is said to be a* contraction *if there exists a number* $\alpha < 1$ *such that, for all* $x, x' \in \mathbb{R}^K$,
$$\|f(x) - f(x')\| \leqslant \alpha \|x - x'\|.$$
When that is the case, we define
$$\inf\{\alpha \in \mathbb{R} \mid \forall x, x' \in \mathbb{R}^K \|f(x) - f(x')\| \leqslant \alpha\|x - x'\|\}$$
as the Lipschitz modulus *of* f.

THEOREM 10.2.1 (Banach). *If* $f : \mathbb{R}^K \to \mathbb{R}^K$ *is a contraction, there exists an* $\bar{x} \in \mathbb{R}^K$ *for which* $f(\bar{x}) = \bar{x}$. *Moreover, such* $\bar{x}$ *is unique and, for every* $x \in \mathbb{R}^K$, *the sequence defined by*
$$x_1 = x \text{ and } \forall n \geqslant 2, x_n = f(x_{n-1}) \tag{10.1}$$
converges to $\bar{x}$.

Proof. Let α be the modulus of contraction f. For the sake of clarity, we divide the argument in three steps.

Step 1: existence of some $\bar{x}$. Fix *any* $x \in \mathbb{R}^K$, and let the sequence $(x_n)_{n=1}^\infty$ be defined recursively by Eq. (10.1). By construction,
$$\|x_3 - x_2\| = \|f(x_2) - f(x_1)\| \leqslant \alpha\|x_2 - x_1\|,$$

and, thus,

$$\|x_4 - x_3\| = \|f(x_3) - f(x_2)\| \leqslant \alpha\|x_3 - x_2\| \leqslant \alpha^2\|x_2 - x_1\|,$$

and so on, so that, in general,

$$\|x_{n+1} - x_n\| \leqslant \alpha^{n-1}\|x_2 - x_1\|.$$

Since $\alpha \leqslant 1$, it follows that $(x_n)_{n=1}^{\infty}$ is Cauchy and, hence, it converges to some $\bar{x} \in \mathbb{R}^K$.

Step 2: $\bar{x}$ is indeed a fixed point. By construction, for any $\varepsilon > 0$, we can find $n^* \in \mathbb{N}$ such that for all $n \geqslant n^*$, $\|x_n - \bar{x}\| < \varepsilon/2$. Notice, then, that for all $\varepsilon > 0$,

$$\begin{aligned}
\|f(\bar{x}) - \bar{x}\| &\leqslant \|f(\bar{x}) - x_{n^*+1}\| + \|x_{n^*+1} - \bar{x}\| \\
&= \|f(\bar{x}) - f(x_{n^*})\| + \|x_{n^*+1} - \bar{x}\| \\
&\leqslant \alpha\|\bar{x} - x_{n^*}\| + \|x_{n^*+1} - \bar{x}\| \\
&< \frac{1+\alpha}{2}\varepsilon \\
&< \varepsilon,
\end{aligned}$$

where first inequality follows from triangle inequality, the second one by the definition of α, and the last one from the fact that $\alpha < 1$, since f is a contraction.

Now, this is possible only if $\|f(\bar{x}) - \bar{x}\| = 0$, which means that $f(\bar{x}) = \bar{x}$.

Step 3: uniqueness of fixed points. Now, let $x' \in \mathbb{R}^K$ be such that $f(x') = x'$. Then,

$$0 \leqslant \|\bar{x} - x'\| = \|f(\bar{x}) - f(x')\| \leqslant \alpha\|\bar{x} - x'\|.$$

Since $\alpha < 1$, this is possible only if $\|\bar{x} - x'\| = 0$, which means that $\bar{x} = x'$. □

It is important to notice that, since the proof took the point x arbitrarily, we indeed showed that for every $x \in \mathbb{R}^K$, the sequence defined by Eq. (10.1) converges to $\bar{x}$.

11
Riemann Integration

IN THE FIRST PART of this chapter, we assume that $a, b \in \mathbb{R}$, $a < b$. As before, we assume that $X \subseteq \mathbb{R}$.

11.1 The Riemann integral

There are different, but equivalent, ways to define the Riemann integral. We now introduce the simplest (although not necessarily the best) one.

A function $s : [a, b] \to \mathbb{R}$ is said to be a *step function* if there exist a monotonically increasing finite sequence $(x_{1,\ldots,}x_{n^*})$ such that $x_1 > a$, $x_{n^*} = b$, and a finite sequence $(s_{1,\ldots,}s_{n^*})$ satisfying that for all $n \in \{1, 2, \ldots, n^*\}$ it is true that for all $x \in (x_{n-1}, x_n)$, $s(x) = s_n$, where we define $x_0 = a$.

EXAMPLE 11.1.1. *Consider* $s : [-2, 2] \to \mathbb{R}$ *defined by*

$$s(x) = \begin{cases} -1 & \text{if } -2 \leqslant x < 0 \\ 0 & \text{if } \quad x = 0 \\ 1 & \text{if } \quad 0 < x \leqslant 2 \end{cases}$$

It is easy the see that s *is a step function: use* $\{x_1, x_2\} = \{0, 2\}$ *and* $\{s_1, s_2\} = \{-1, 1\}$, *and let* $x_0 = -2$*; then, we have that for all* $x \in (x_0, x_1) = (-2, 0), s(x) = -1 = s_1$ *and for all* $x \in (x_1, x_2) = (0, 2), s(x) = 1 = s_2$.

It is important to notice, both in the definition and in the example, that the values of $s(a)$ and $s(b)$ do not matter. Similarly, the value of s at any point of discontinuity is irrelevant (e.g. $s(0)$ in the example,) but there can be only finitely many such points. It should also be clear that any step function on $[a, b]$ is bounded, because it takes at most $(2n^* + 1)$ (a finite number) different values.

DEFINITION 11.1.1. *Given a step function* $s : [a, b] \to \mathbb{R}$, *we define the integral of* s *from* a *to* b *by*

$$\int_a^b s(x)\, dx = \sum_{n=1}^{n^*} s_n(x_n - x_{n-1}),$$

where $x_0 = a$, $(x_{1,\ldots,}x_{n^*})$ *is a monotonically increasing finite sequence such that* $x_1 > x_0$, $x_{n^*} = b$, *and* $(s_{1,\ldots,}s_{n^*})$ *is a finite sequence satisfying that for all* $n \in \{1, 2, \ldots, n^*\}$ *it is true that for all* $x \in (x_{n-1}, x_n)$, $s(x) = s_n$.

Notice that the integral of a step function on $[a, b]$ is always a real number. Also, it should be clear that the integral is unique, so that no matter what particular sequences one uses to find it, the summation is always the same.

EXAMPLE 11.1.2. *For* s *defined as in Example 11.1.1, we have*

$$\int_{-2}^{2} s(x)\, dx = -1(0-(-2)) + 1(2-0) = 0$$

Again, notice that the integral is independent of $s(a)$ and $s(b)$, and of the value of s at any point of discontinuity (e.g. 0 in our example).

DEFINITION 11.1.2. *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function. If there exists a unique* $I \in \mathbb{R}$ *such that*

$$\int_{a}^{b} s(x)\, dx \leqslant I \leqslant \int_{a}^{b} t(x)\, dx$$

for every pair of step functions $s : [a, b] \to \mathbb{R}$ *and* $t : [a, b] \to \mathbb{R}$ *such that* $s(x) \leqslant f(x) \leqslant t(x)$ *for all* $x \in [a, b]$*, then* f *is said to be integrable (on* $[a, b]$*), and* I *is said to be the integral of* f *from* a *to* b*, which we denote by*

$$\int_{a}^{b} f(x)\, dx = I.$$

It is important to notice that I is required by the definition to be finite and unique.

Let $f : [a, b] \to \mathbb{R}$ be integrable. Then, we define

$$\int_{b}^{a} f(x)\, dx = -\int_{a}^{b} f(x)\, dx$$

and $\int_{a}^{a} f(x)\, dx = 0$. Also, the following definition is (has to be) given for formal completeness: let $c, d \in [a, b]$, $c \leqslant d$; we define the *integral of* f *from* c *to* d as

$$\int_{c}^{d} f(x)\, dx = \int_{c}^{d} \tilde{f}(x)\, dx$$

where $\tilde{f} : [c, d] \to \mathbb{R}$ is defined by $\tilde{f}(x) = f(x)$, for all $x \in [c, d]$.

11.2 Properties of the Riemann integral

The following results list (some) important properties of the Riemann integral. We state them without proof.

THEOREM 11.2.1. *Suppose that the functions* $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *are integrable and* $\alpha, \beta \in \mathbb{R}$*. Then,* $(\alpha f + \beta g) : [a, b] \to \mathbb{R}$ *is integrable and*

$$\int_{a}^{b} (\alpha f + \beta g)(x)\, dx = \alpha \int_{a}^{b} f(x)\, dx + \beta \int_{a}^{b} g(x)\, dx$$

THEOREM 11.2.2. *Suppose that* $f : [a, b] \to \mathbb{R}$ *is integrable and* $c \in [a, b]$*. Then,*

$$\int_{a}^{b} f(x)\, dx = \int_{a}^{c} f(x)\, dx + \int_{c}^{b} f(x)\, dx$$

THEOREM 11.2.3. *Suppose that the functions* $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *are integrable and, for all* $x \in [a, b]$*,* $f(x) \leqslant g(x)$*. Then,* $\int_{a}^{b} f(x)\, dx \leqslant \int_{a}^{b} g(x)\, dx$.

Additionally, if $f : [a, b] \to \mathbb{R}$ is integrable, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$, then

$$\int_a^b f(x)\ dx = \int_{a+\alpha}^{b+\alpha} f(x-\alpha)\ dx$$

and

$$\int_a^b f(x)\ dx = \frac{1}{\beta}\int_{\beta a}^{\beta b} f(\frac{x}{\beta})\ dx$$

THEOREM 11.2.4. *If $f : [a, b] \to \mathbb{R}$ is either monotonic or continuous, then it is integrable.*

It must be pointed out that the fact that the domain of f is assumed bounded in the previous theorem is crucial.

11.3 The Fundamental Theorems of Calculus

Following are the two most important results relating integral and differential calculus. (Their proofs use the Mean Value Theorem that we learned in Chapter 5.4.)

THEOREM 11.3.1 (First Fundamental Theorem of Calculus). *If $f : [a, b] \to \mathbb{R}$ is differentiable, and $f' : [a, b] \to \mathbb{R}$ is integrable, then $\int_a^b f'(x)\ dx = f(b) - f(a)$.*

THEOREM 11.3.2 (Second Fundamental Theorem of Calculus). *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable. Define $F : [a, b] \to \mathbb{R}$, for all $x \in [a, b]$, by $F(x) = \int_a^x f(t)\ dt$. If f is continuous at $\bar{x} \in X$, then $F'(\bar{x}) = f(\bar{x})$.*

11.4 Antiderivatives (indefinite integrals)

The results in last section show the tight relation that exists between differential and integral calculus. We now show how we can take advantage of such relation in order to find the integral

DEFINITION 11.4.1. *A function $F : X \to \mathbb{R}$ is said to be an antiderivative of $f : X \to \mathbb{R}$ if for all $x \in X$, $F'(x) = f(x)$.*

Suppose that $F : X \to \mathbb{R}$ is an antiderivative of $f : X \to \mathbb{R}$. Then, we will also write $F = \int f(x)\ dx$. Notice that antiderivatives are not unique.

The following results (which establish the most useful properties of antiderivatives,) come straightforwardly from Theorems 5.1.2, 5.1.3 and 5.2.1: $\int(\alpha f)(x)\ dx = \alpha \int f(x)\ dx$ for any $\alpha \in \mathbb{R}$; $\int(f+g)(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$;

$$\int x^n\ dx = \frac{x^{n+1}}{n} + K, \text{ if } n \neq -1;$$

$$\int \frac{1}{x}\ dx = \ln(x) + K;$$

$\int e^x\ dx = e^x + K$; $\int e^{f(x)} f'(x)\ dx = e^{f(x)} + K$;

$$\int (f(x))^n f'(x)\ dx = \frac{1}{n+1}(f(x))^{n+1} + K,$$

if $n \neq -1$; and,

$$\int \frac{1}{f(x)} f'(x)\ dx = \ln(f(x)) + K.$$

Here, $K \in \mathbb{R}$ is arbitrary. Notice that the first two results assume that antiderivatives for f and g exist (on the domain in which the functions are defined).

The importance of these rules is that, together with the Fundamentals Theorems of Calculus, they make the computation of Riemann integrals a process in which one just has to reverse the one of differentiation. In particular, if we can find the antiderivative of an integrable function (with bounded domain), we can use the First Fundamental Theorem of Calculus in the computation of the Riemann integral.

EXAMPLE 11.4.1. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is defined by* $f(x) = x^3 - 3x$, *for all* $x \in \mathbb{R}$. *Now, we can use the first, second and third rules that we just introduced, to show that*

$$\int f(x)\, dx = \frac{1}{4}x^4 - \frac{3}{2}x^2 + K$$

where $K \in \mathbb{R}$. *Now, let* $F : [a, b] \to \mathbb{R}$ *be defined as for all* $x \in [a, b]$,

$$F(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2$$

Clearly, $F \in C^\infty$ *so that* F *is differentiable and* F' *is integrable (by Theorem 11.2.4). By the First Fundamental Theorem of Calculus, it follows that* $\int_a^b F'(x)\, dx = F(b) - F(a)$, *which means that*

$$\int_a^b (x^3 - 3x)\, dx = (\frac{1}{4}b^4 - \frac{3}{2}b^2) - (\frac{1}{4}a^4 - \frac{3}{2}a^2) = (\frac{1}{4}x^4 - \frac{3}{2}x^2)_a^b$$

where the last inequality is just introducing some new notation. Alternative notation would be:

$$(\frac{1}{4}x^4 - \frac{3}{2}x^2)\Big|_a^b$$

For example,

$$\int_1^2 (x^3 - 3x)\, dx = (\frac{1}{4}x^4 - \frac{3}{2}x^2)_1^2 = -\frac{3}{4}.$$

EXAMPLE 11.4.2. *Suppose that we are interested in* $\int e^{-\frac{x^2}{2}} x\, dx$. *By our first rule,*

$$\int e^{-\frac{x^2}{2}} x\, dx = -\int e^{-\frac{x^2}{2}} (-x)\, dx.$$

Now, let $f(x) = -\frac{x^2}{2}$, *so that* $f'(x) = -x$ *and*

$$\int e^{-\frac{x^2}{2}} x\, dx = -\int e^{f(x)} f'(x)\, dx = -e^{f(x)} + K = -e^{-\frac{x^2}{2}} + K,$$

using the sixth rule. As before, by the First Fundamental Theorem, for all $a \in \mathbb{R}$

$$\int_{-a}^{a} e^{-\frac{x^2}{2}} x\, dx = (-e^{-\frac{x^2}{2}} + K)_{-a}^{a} = 0.$$

11.5 Integration by parts

Suppose that we have two functions $u : X \to \mathbb{R}$ and $v : X \to \mathbb{R}$, both of which are differentiable. We know from Chapter 5 that

$$(u.v)'(x) = u(x)v'(x) + v(x)u'(x)$$

so that

$$u(x)v'(x) = (u.v)'(x) - v(x)u'(x)$$

Therefore, by the rules that we introduced in the last section

$$\begin{aligned}\int u(x)v'(x)\ dx &= \int [(u.v)'(x) - v(x)u'(x)]\ dx \\ &= \int (u.v)'(x)\ dx - \int v(x)u'(x)\ dx \\ &= (u.v)(x) - \int v(x)u'(x)\ dx \\ &= u(x).v(x) - \int v(x)u'(x)\ dx\end{aligned}$$

EXAMPLE 11.5.1. *Suppose we want to find* $\int e^x x\ dx$. *Then, let* $v'(x) = e^x$ *and* $u(x) = x$. *Clearly,* $v(x) = e^x$ *and* $u'(x) = 1$. *Then, we have*

$$\begin{aligned}\int e^x x\ dx &= \int u(x)v'(x)\ dx \\ &= u(x).v(x) - \int v(x)u'(x)\ dx \\ &= e^x x - \int e^x\ dx \\ &= e^x x - e^x\end{aligned}$$

Therefore, by the First Fundamental Theorem of Calculus, $\int_0^1 e^x x\ dx (e^x x - e^x)_0^1 =$ 1.

11.6 Improper integrals

So far, we have restricted our definition of Riemann Integral of the function f to the case in which for $a, b \in \mathbb{R}$, the function f is defined on $[a, b]$ and is bounded. It is convenient, however, to generalize the definition of the integral.

Suppose initially that $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$, $b > a$, and consider the function $f : [a, b) arrow \mathbb{R}$. Then, if for all $d \in [a, b)$, the function is integrable when its domain is restricted to $[a, d]$, then, we define

$$\int_a^b f(x)\ dx = \lim_{d \to b} \int_a^d f(x)\ dx$$

provided that the limit exists.

Similarly, suppose now that $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R}$, $a < b$, and consider the function $f : (a, b] arrow \mathbb{R}$. Then, if for all $d \in (a, b]$, the function is integrable when its domain is restricted to $[d, b]$, then, we define

$$\int_a^b f(x)\ dx = \lim_{d \to a} \int_d^b f(x)\ dx$$

provided that the limit exists.

Finally, if we have $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$, $b > a$, and consider the function $f : (a, b) arrow \mathbb{R}$. Then, if for all $c, d \in (a, b)$, $c < d$, the function is integrable when its domain is restricted to $[c, d]$, then, we define

$$\int_a^b f(x)\ dx = \int_a^\gamma f(x)\ dx + \int_\gamma^b f(x)\ dx$$

for (any) $\gamma \in (a, b)$, provided that both integrals and its sum exist (one should be particularly worried about the result $\infty + (-\infty)$, which does not exist).

EXAMPLE 11.6.1. *Recall Example 11.5.1. It follows that*

$$\int_0^\infty e^x x\ dx = \lim_{d\to\infty} (e^x x - e^x)_0^d = \lim_{d\to\infty} (e^d(d-1)) + 1 = \infty.$$

EXERCISE 11.6.1. *(From Apostol, Calculus:) For* $a \in \mathbb{R}$, $b \in \mathbb{R}_{++}$, *compute the following integrals:* $\int_0^a (1 + x + x^2)\ dx$, $\int_0^{2a} (1 + x + x^2)\ dx$, $\int_{-1}^{2a} (1 + x + x^2)\ dx$, $\int_{-2}^{a} x^2(1 + x^2)\ dx$, $\int_a^{a^2} (1 + x^2)^2\ dx$, $\int_1^b (1 + x^{1/2})\ dx$, *and* $\int_b^{b^2} (x^{1/4} + x^{1/2})\ dx$.

EXERCISE 11.6.2. *(From Apostol, Calculus:) Show that:*

$$\int \sqrt{1 - x^2}\ dx = x\sqrt{1 - x^2} + \int \frac{x^2}{\sqrt{1 - x^2}}\ dx$$

EXERCISE 11.6.3. *Recall Example 11.4.2. Show that*

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} x\ dx = 0$$

EXERCISE 11.6.4. *Recall Example 11.4.1. Show that* $\int_{-\infty}^{\infty} (x^3 - 3x)\ dx$ *does not exist.*

11.7 *Integration in higher-dimensional spaces*

Recall that one of the goals of this notes was to introduce concepts in such a way that generalizing to multiple dimensions was relatively simple. Surprisingly, one of the easiest concepts to generalize is the one of Riemann integral.

From now on, we maintain the assumption that $a, b, c, d \in \mathbb{R}$, $a < b$ and $c < d$. A function $s : [a, b] \times [c, d] \to \mathbb{R}$ is a *step function* if there exist finite monotonically increasing sequences $(x_n)_{n=1}^{n^*}$ and $(y_m)_{m=1}^{m^*}$ and a finite double array $((s_{n,m})_{n=1}^{n^*})_{m=1}^{m^*}$ such that $x_1 > a$, $x_{n^*} = b$, $y_1 > c$, $y_{m^*} = d$ and that for all $n \in \{1, \dots, n^*\}$, for all $m \in \{1, \dots, m^*\}$ and for all $(x, y) \in (x_{n-1}, x_n) \times (y_{m-1}, y_m)$,[1] $s(x, y) = s_{n,m}$, where $x_0 = a$ and $y_0 = c$.

Notice that this is a straightforward generalization of definition in the one-dimensional case, with the sole exception that step functions need no longer be bounded (why?).

[1] This is where notation exhibits a conflict: the left-hand side of the expression is an ordered pair, whereas each one of the terms in the Cartesian product on the right-hand side is an open interval. Had we used $]a, b[$ for open intervals, the conflict would not have arisen, but this would be nonstandard.

EXAMPLE 11.7.1. *Consider* $s : [-2, 2] \times [0, 1] \to \mathbb{R}$, *defined by:*

$$s(x, y) = \begin{cases} -1, & \text{if } -2 \leqslant x < 0; \\ y, & \text{if } x = 0; \\ 1, & \text{if } 0 < x < 1 \text{ and } 0 < y \leqslant 0.5; \\ -1, & \text{if } 0 < x < 1 \text{ and } 0.5 < y \leqslant 1; \\ -2, & \text{if } 1 \leqslant x \leqslant 2 \text{ and } 0 < y < 0.5; \\ 2, & \text{if } 1 \leqslant x \leqslant 2 \text{ and } 0.5 \leqslant y \leqslant 1; \end{cases}$$

To see that s *is step, define* $\{x_n\}_{n=1}^3 = \{0, 1, 2\}$, $\{y_m\}_{m=1}^2 = \{0.5, 1\}$ *and define*

$\{\{s_{n,m}\}_{n=1}^{3}\}_{m=1}^{2}$ *by* $s_{1,1} = -1$, $s_{1,2} = -1$, $s_{2,1} = 1$, $s_{2,2} = -1$, $s_{3,1} = -2$, *and* $s_{3,2} = 2$.

Notice that the definition of $s(x, y)$ when $x = 0$ does not matter. If we had defined for all $y \in [0, 1]$, $s(0, y) = 0$ the same definitions would imply that s is step. Moreover, these definitions would still work if we defined $s(0, 0) = 0$ and for all $y \in (0, 1]$, $s(0, y) = \ln(y)$, but in this case s would not be bounded!

From now, we will denote $Q = [a, b] \times [c, d]$. Notice that Q is a closed cube. Again, a straightforward generalization is the following:

DEFINITION 11.7.1. *Given a step function* $s : Q \to \mathbb{R}$*, we define the integral of* s *on* Q *by:*

$$\int_Q s(x, y)\ dx\ dy = \sum_{m=1}^{m^*} \sum_{n=1}^{n^*} s_{n,m}(x_n - x_{n-1})(y_m - y_{m-1})$$

where $x_0 = a$, $y_0 = c$, $(x_n)_{n=1}^{n^*}$ *and* $(y_m)_{m=1}^{m^*}$ *are two finite monotonically increasing sequences and* $((s_{n,m})_{n=1}^{n^*})_{m=1}^{m^*}$ *is a finite double array such that* $x_1 > a$, $x_{n^*} = b$, $y_1 > c$, $y_{m^*} = d$ *and that for all* $n \in \{1, \ldots, n^*\}$*, for all* $m \in \{1, \ldots, m^*\}$ *and for all* $(x, y) \in (x_{n-1}, x_n) \times (y_{m-1}, y_m)$, $s(x, y) = s_{n,m}$.

EXAMPLE 11.7.2. *For* s *defined as in Example 11.7.1,* $Q = [-2, 2] \times [0, 1]$ *and*

$$\begin{aligned}\int_Q s(x, y)\ dx\ dy &= (-1)(0 - (-2)) + 1(1 - 0)(0.5 - 0) \\ &\quad + (-2)(2 - 1)(0.5 - 0) + (-1)(0 - (-2))(1 - 0.5) \\ &\quad + (-1)(1 - 0)(1 - 0.5) + 2(2 - 1)(1 - 0.5) \\ &= -1 + 0.5 - 1 - 1 - 0.5 + 1 \\ &= -2\end{aligned}$$

Given definition 11.7.1, the following should appear natural as a generalization of definition 11.1.2.

DEFINITION 11.7.2. *Let* $f : Q \to \mathbb{R}$ *be a bounded function. If there exists a unique* $I \in \mathbb{R}$ *such that*

$$\int_Q s(x, y)\ dx\ dy \leqslant I \leqslant \int_Q t(x, y)\ dx\ dy$$

for every pair of step functions $s : Q \to \mathbb{R}$ *and* $t : Q \to \mathbb{R}$ *such that* $s(x, y) \leqslant f(x, y) \leqslant t(x, y)$ *for all* $(x, y) \in Q$*, then* f *is said to be integrable, and* I *is said to be the integral of* f*, which we denote by*

$$\int_Q f(x, y)\ dx\ dy = I.$$

Notice that the definition requires I to exist and be unique. As before, the definition results clumsy to compute the integral. However, the following theorem simplifies such computation.

THEOREM 11.7.1. *Let* $f : Q \to \mathbb{R}$ *be bounded and integrable. Suppose that for all* $y \in [c, d]$, $f(\cdot, y) : [a, b] \to \mathbb{R}$ *is integrable. Let* $A : [c, d] \to \mathbb{R}$ *be defined by for all* $y \in [c, d]$,

$$A(y) = \int_a^b f(x, y)\ dx$$

If A is integrable, then

$$\int_Q f(x,y)\,dx\,dy = \int_c^d A(y)\,dy$$

This result is normally expressed by saying that

$$\int_Q f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy$$

or simply that

$$\int_Q f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Notice, however, that the latter is not a definition. It is the implication of a theorem and applies only when its assumptions hold.

EXERCISE 11.7.1. *The following proof of this theorem is relatively simple; try and fill in the gaps.*

Proof. Let $s : Q \to \mathbb{R}$ and $t : Q \to \mathbb{R}$ be step functions such that $s(x,y) \leqslant f(x,y) \leqslant t(x,y)$ for all $(x,y) \in Q$. Fix $y \in [c,d]$. Clearly, $s(\cdot,y)$ and $t(\cdot,y)$ are also step functions and for all $x \in [a,b]$, $s(x,y) \leqslant f(x,y) \leqslant t(x,y)$. Then, by definition (which one?)

$$\int_a^b s(x,y)\,dx \leqslant A(y) \leqslant \int_a^b t(x,y)\,dx$$

Now, the left-most and right-most terms on the previous expression are themselves step functions (of y on $[c,d]$; check this!), and by assumption the term in the middle is integrable, so that, by Theorem 11.2.3,

$$\int_c^d \left(\int_a^b s(x,y)\,dx\right)\,dy \leqslant \int_c^d A(y)\,dy \leqslant \int_c^d \left(\int_a^b t(x,y)\,dx\right)\,dy$$

and then, by properties of sums (justify this!),

$$\int_Q s(x,y)\,dx\,dy \leqslant \int_c^d A(y)\,dy \leqslant \int_Q t(x,y)\,dx\,dy$$

and, finally, since s and t were arbitrary,

$$\int_Q f(x,y)\,dx\,dy = \int_c^d A(y)\,dy.$$

□

Moreover, most usually one can use the following:

THEOREM 11.7.2. *If $f : Q \to \mathbb{R}$ is continuous, then it is integrable and*

$$\int_Q f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx.$$

EXAMPLE 11.7.3. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by $f(x,y) = x^3y - 3xy^2$. f is* *continuous and*

$$\begin{aligned}\iint_{[0,1]\times[0,1]} f(x,y)\ dx\ dy &= \int_0^1 \left[\int_0^1 (x^3y - 3xy^2)\ dx\right]\ dy \\ &= \int_0^1 \left(\tfrac{1}{4}x^4y - \tfrac{3}{2}x^2y^2\right)_0^1\ dy \\ &= \int_0^1 \left(\tfrac{1}{4}y - \tfrac{3}{2}y^2\right)\ dy \\ &= \left(\tfrac{1}{8}y^2 - \tfrac{1}{2}y^3\right)_0^1 \\ &= -\frac{3}{8}\end{aligned}$$

EXERCISE 11.7.2. *Show that for all* $a \in \mathbb{R}_+$ *and for all* $c, d \in \mathbb{R} : c > d$*,*

$$\iint_{[-a,a]\times[c,d]} e^{-\frac{x^2+y^2}{2}} xy\ dx\ dy = 0$$

(This result will be extremely useful in statistics. Do you know why?)

So far we have restricted attention to functions defined on rectangles (or cubes) only. This is clearly limited, but easy to overcome: let $S \subseteq Q$; if $f : D \to \mathbb{R}$ and $S \subseteq D \subseteq Q$, define

$$\int_S f(x,y)\ dx\ dy = \int_Q \tilde{f}(x,y)\ dx\ dy$$

where $\tilde{f} : Q \to \mathbb{R}$ is defined by

$$\tilde{f}(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in S; \\ 0, & \text{otherwise.}\end{cases}$$

EXERCISE 11.7.3. *Let* $Q = [0,2] \times [0,2]$ *and suppose that* $f : Q \to \mathbb{R}$ *is defined by for all* $(x,y) \in Q$*,* $f(x,y) = xy$*. Let*

$$S = \{(x,y) \in \mathbb{R} \mid x^2 \leqslant y < x\}$$

Show that $S \subseteq Q$*. Show that* $S \neq \varnothing$*. Show that*

$$\int_S f(x,y)\ dx\ dy = \frac{1}{24}$$

Can you prove the last result using two different arguments (orders of integration)?

12
Probability

12.1 Measure Theory

Suppose that we have fixed a universe S. Denote by $\mathcal{P}(S)$ the set of all subsets of S (that is, $E \in \mathcal{P}(S)$ if, and only if, $E \subseteq S$. Obviously, $\varnothing \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$. One can also say that $\{\varnothing, S\} \subseteq \mathcal{P}(S)$ and $\varnothing \subseteq \mathcal{P}(S)$, but it would be a mistake to say that $S \subseteq \mathcal{P}(S)$.

12.1.1 Algebras and σ-algebras:

Our problem now is to define, in a consistent manner, the size of (some of) the subsets of S. The consistency of our definition will require some "structure" on the family of subsets whose size we define: (i) we should be able to tell the size of the set with no elements in it; (ii) if we are able to measure a set, we should also be able to measure the rest of the universe; and (iii) if we are able to measure a series of sets, then we should also be able to measure their union.

For this:

Definition 12.1.1. *A family of subsets of S, $\Sigma \subseteq \mathcal{P}(S)$, is* an algebra *if:*

1. *it contains the empty set: $\varnothing \in \Sigma$;*
2. *it is closed under complement: if $A \in \Sigma$, then $S \setminus A \in \Sigma$; and*
3. *it is closed under finite union: if $\{A_n\}_{n=1}^N \subseteq \Sigma$ is a finite set, then $\cup_{n=1}^N A_n \in \Sigma$.*

The following theorem is almost immediate from the definition.

Theorem 12.1.1. *If Σ is an algebra, then:*

1. *it contains S: $S \in \Sigma$; and*
2. *it is closed under finite intersection: if $\{A_n\}_{n=1}^N \subseteq \Sigma$ is a finite set, then $\cap_{n=1}^N A_n \in \Sigma$.*

Proof. Left as an exercise. For the second part, recall De Morgan's laws. □

Theorem 12.1.2. *$\mathcal{P}(S)$ is an algebra.*

Proof. Left as an exercise. □

Theorem 12.1.3. *Let $\mathfrak{A} \neq \varnothing$ be a collection of Algebras. $\Sigma = \cap_{\Sigma' \in \mathfrak{A}} \Sigma'$ is an algebra.*

Proof. By construction, $\varnothing \in \Sigma'$, for all $\Sigma' \in \mathfrak{A}$, which implies that $\varnothing \in \Sigma$. Suppose now that $A \in \Sigma$. By definition, $A \in \Sigma'$, for all $\Sigma' \in \mathfrak{A}$, which implies that $S \setminus A \in \Sigma'$, for all of them, and hence that $S \setminus A \in \Sigma$. Finally, let $\{A_n\}_{n=1}^N$ be a finite subset of Σ. By construction, for all n and all $\Sigma' \in \mathfrak{A}$, we have that $A_n \in \Sigma'$, which implies that, also, $A_n \in \Sigma'$; this means that $\cup_{n=1}^N A_n \in \Sigma'$, for all $\Sigma' \in \mathfrak{A}$, and hence that $\cup_{n=1}^N A_n \in \Sigma$. □

Exercise 12.1.1. *Is it true that if $\mathfrak{A}$ is a collection of algebras and $\mathfrak{A} \neq \varnothing$, then $\Sigma = \cup_{\Sigma' \in \mathfrak{A}} \Sigma'$ is an algebra?*

Theorem 12.1.4. *For every $\mathcal{A} \subseteq \mathcal{P}(S)$, there is an algebra $\Sigma \subseteq \mathcal{P}(S)$ such that:*

1. *$\mathcal{A} \subseteq \Sigma$; and*
2. *if $\Sigma' \subseteq \mathcal{P}(S)$ is an algebra and $\mathcal{A} \subseteq \Sigma'$, then $\Sigma \subseteq \Sigma'$.*

Proof. Consider the set

$$\mathfrak{A} = \{\Sigma' \subseteq \mathcal{P}(S) \mid \Sigma' \text{ is an algebra and } \mathcal{A} \subseteq \Sigma'\}.$$

Since $\mathcal{P}(S) \in \mathfrak{A}$, it follows that $\mathfrak{A} \neq \varnothing$. Let $\Sigma = \cap_{\Sigma' \in \mathfrak{A}} \Sigma'$. That Σ satisfies the properties of the statement is immediate, while it follows from Theorem 12.1.3 that Σ is an algebra. □

The algebra Σ of the previous theorem is known as the algebra generated by $\mathcal{A}$. Notice that it is *the algebra* and not *an algebra*, because, so defined, Σ is unique. Notice also that the conditions of the definition of algebra have the intuition we wanted. For some purposes, however, we need to strengthen the third property:

Definition 12.1.2. *A family of subsets of S, $\Sigma \subseteq \mathcal{P}(S)$, is* a σ-algebra *if:*

1. *it contains the empty set: $\varnothing \in \Sigma$;*
2. *it is closed under complement: if $A \in \Sigma$, then $S \setminus A \in \Sigma$; and*
3. *it is closed under countable union: if $\{A_n\}_{n=1}^\infty \subseteq \Sigma$, then $\cup_{n=1}^N A_n \in \Sigma$.*

The following theorems are left as exercises.

Theorem 12.1.5. *$\mathcal{P}(S)$ is a σ-algebra.*

Theorem 12.1.6. *If Σ is a σ-algebra, then it is an algebra. When S is finite, if Σ is an algebra, then it is a σ-algebra.*

Theorem 12.1.7. *If Σ is a σ-algebra, then:*

1. *it contains S: $S \in \Sigma$; and*
2. *it is closed under countable intersection: if $\{A_n\}_{n=1}^\infty \subseteq \Sigma$, then $\cap_{n=1}^N A_n \in \Sigma$.*

Theorem 12.1.8. *Let $\mathfrak{S} \neq \varnothing$ be a collection of σ-algebras in S. Then, $\Sigma = \cap_{\Sigma' \in \mathfrak{S}} \Sigma'$ is a σ-algebra.*

Exercise 12.1.2. *Is it true that if $\mathfrak{S}$ is a collection of σ-algebras of S and $\mathfrak{S} \neq \varnothing$, then $\Sigma = \cup_{\Sigma' \in \mathfrak{S}} \Sigma'$ is a σ-algebra?*

Theorem 12.1.9. *For every $\mathcal{A} \subseteq \mathcal{P}(S)$, there is a σ-algebra $\Sigma \subseteq \mathcal{P}(S)$ such that:*

1. *$\mathcal{A} \subseteq \Sigma$; and*

2. *if $\Sigma' \subseteq \mathcal{P}(S)$ is a σalgebra and $\mathcal{A} \subseteq \Sigma'$, then $\Sigma \subseteq \Sigma'$.*

The σ-algebra Σ of the previous theorem is the σ -algebra generated by $\mathcal{A}$. Notice that it is *the*, and not *a*, as in the case of algebras. The σ-algebra generated by $\mathcal{A}$ is denoted by $\sigma(\mathcal{A})$.

There is an argument, often used when dealing with σ-algebras, known as *the good-set principle*:

> Let $\Sigma \subseteq \mathcal{P}(S)$ be a σ-algebra for S. Think of Σ as the family of all the subsets of S satisfying some property, or the "good" subsets of S. If $\mathcal{A}$ is an arbitrary family of good subsets of S, then all the sets in $\sigma(\mathcal{A})$ are good.

As a result, this is is trivial since, by hypothesis, the class of all good subsets is a σ-algebra, so, by definition, $\sigma(\mathcal{A}) \subseteq \Sigma$. But it is useful as it gives a correct intuition: if the good subsets form a σ-algebra, then all the sets in the σ-algebra generated by a family of good subsets are good as well.

EXERCISE 12.1.3. *Let $\mathcal{A}$ be a class of subsets of S and $A \subseteq S$. For any $\mathcal{E} \subseteq \mathcal{P}(S)$, denote by $\mathcal{E} \cap A$ the class $\{B \cap A : B \in \mathcal{E}\}$. Show that $\sigma_A(\mathcal{A} \cap A) \subseteq \sigma_S(\mathcal{A}) \cap A$, where $\sigma_A(\mathcal{A} \cap A)$ denotes the σ-algebra generated by $\mathcal{A} \cap A$ relative to the universe A and $\sigma_S(\mathcal{A})$ is the σ-algebra generated for $\mathcal{A}$ relative to universe S.*

The good-set principle allows us to show that the relationship between $\sigma_A(\mathcal{A} \cap A)$ and $\sigma_S(\mathcal{A}) \cap A$ is stronger that the previous exercise suggests: define

$$\Sigma = \{E \in \sigma_S(\mathcal{A}) \mid E \cap A \in \sigma_A(\mathcal{A} \cap A)\};$$

notice that Σ is a σ-algebra (for S) and satisfies that $\mathcal{A} \subseteq \Sigma$; the latter implies that $\sigma_S(\mathcal{A}) \subseteq \Sigma$, so $E \in \sigma_S(\mathcal{A})$ suffices to imply that $E \cap A \in \sigma_A(\mathcal{A} \cap A)$, and, then, $\sigma_S(\mathcal{A}) \cap A \subseteq \sigma_A(\mathcal{A} \cap A)$. Here, Σ is the family of all subsets of S. This result and the exercise together imply that $\sigma_A(\mathcal{A} \cap A) = \sigma_S(\mathcal{A}) \cap A$.

12.1.2 Measure

If Σ is a σ-algebra for S, then (S, Σ) is said to be *a measurable space*. The idea here is that Σ is the collection of subsets of S that we can "measure." Now, what do we understand by "measuring"? Intuitively, what we want to do is to associate each set to a number. Of course, this assignment cannot be arbitrary: (i) sizes cannot be negative, and we must consider the possibility of finding an "infinitely large" set; (ii) a set that contains nothing must have zero measure; and (iii) if we take a collection of mutually separated sets and we measure them, and we then measure their union, the sum of the first measures must equal the last measure.

Formally, denote $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$, which is usually called the *extended real line*, and let us take its positive orthant: $\mathbb{R}^*_+ = \mathbb{R}_+ \cup \{\infty\}$. Let Σ be an algebra for S and let $\mu : \Sigma \to \mathbb{R}^*$. Function μ is said to be *finitely additive* if for any finite collection of mutually disjoint sets in Σ, $\{A_n\}_{n=1}^N$, one has that $\mu(\cup_{n=1}^N A_n) = \sum_{n=1}^N \mu(A_n)$. It is said to be σ-*additive* if for any sequence $\{A_n\}_{n=1}^\infty$ of mutually disjoint sets in Σ, similarly, $\mu(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$.

Obviously, if μ is σ-additive, then it is also finitely additive. It is also immediate that if S is finite, Σ is an algebra for S and $\mu : \Sigma \to \mathbb{R}^*$is finitely additive, then (S, Σ)

is a measurable space and μ is σ-additive. The proof of the following result is also left as an exercise.

THEOREM 12.1.10. *Let Σ be an algebra for S and let $\mu : \Sigma \to \mathbb{R}^*$ be finitely additive. If there is an $A \in \Sigma$ such that $\mu(A) \in \mathbb{R}$, then $\mu(\varnothing) = 0$.*

EXERCISE 12.1.4. *Let S be an infinite, countable set. Define the following class of subsets of S,*

$$\Sigma = \{A \subseteq S \mid \textit{either } A \textit{ or } A^c \textit{ is finite}\},$$

and define $\mu : \Sigma \to \{0, 1\}$ by $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A^c is finite. Show that Σ is an algebra, and that μ is finitely additive but not σ-additive. Show also that there exists a sequence $(A_n)_{n=1}^\infty$ in Σ such that for every $n \in \mathbb{N}$ one has that $\mu(A_n) = 0$, but $\mu(\cup_{n=1}^\infty A_n) = 1$.[1]

[1] Hint: find $(A_n)_{n=1}^\infty$ such that each $A_n \subseteq A_{n+1}$ and $\cup_{n=1}^\infty A_n = S$

It is obvious that the structure imposed when we consider arbitrary sequences is more than when we consider finite sequences only. But, is the extra complication necessary? To see that it is, consider the following experiment: a coin is tossed until if comes head. Suppose that we want to measure the probability that the experiment stops at an even toss. We need to consider countable but infinite sequences! Notice that σ-additivity corresponds to the condition (iii) that we want to impose to our measures.

Let (S, Σ) be a measurable space and let $\mu : \Sigma \to \mathbb{R}^*$. Function μ is said to be *a measure* if it is σ-additive and satisfies that $\mu(\varnothing) = 0$ and $\mu(A) \in \mathbb{R}^*_+$ for all $A \in \Sigma$. A *measure space* is (S, Σ, μ), where (S, Σ) is a measurable space and $\mu : \Sigma \to \mathbb{R}^*_+$ is a measure.

Again, the proofs of the following theorems are left as exercises.

THEOREM 12.1.11. *Let (S, Σ) be a measurable space and let $\mu, \mu' : \Sigma \to \mathbb{R}^*_+$ be measures. Then, $\mu^* = \mu + \mu'$ is a measure as well.*

THEOREM 12.1.12. *Let (S, Σ) be a measurable space and fix $A^* \in \Sigma$. Define $\mu^* : \Sigma \to \mathbb{R}^*_+$ by $\mu^*(A) = \mu(A \cap A^*)$. Then, μ^* is a measure.*

THEOREM 12.1.13. *Let (S, Σ, μ) be a measure space. Then, for all $A, A' \in \Sigma$, if $A \subseteq A'$, then $\mu(A) \leqslant \mu(A')$ If, additionally, $\mu(A) \in \mathbb{R}$, then $\mu(A) = \mu(A') - \mu(A' \setminus A)$.*

EXERCISE 12.1.5. *Prove the following results:*

1. *If $X : S \to \mathbb{R}$ is $\{\varnothing, S\}$-measurable, then $X(s) = X(s')$ for all $s, s' \in S$.*
2. *Any function $X : S \to \mathbb{R}$ is $\mathcal{P}(S)$-measurable.*
3. *$\sigma(\{S\}) = \{\varnothing, S\}$.*
4. *If Σ is a σ-algebra, then $\sigma(\Sigma) = \Sigma$.*

12.1.3 Example: Lebesgue measure

Let L be a finite number, and denote by $\mathcal{I}$ the class of subsets I of $\mathbb{R}^L$ that can be written as $I = \prod_{l=1}^L [a_l, b_l]$, for $(a_l, b_l)_{l=1}^L$ such that $a_l < b_l$. Define the function $\nu : \mathcal{I} \to \mathbb{R}$, by $\nu(\prod_{l=1}^L [a_l, b_l]) = \prod_{l=1}^L (b_l - a_l)$. Define the *outer measure* function, $m : \mathcal{P}(\mathbb{R}^L) \to \mathbb{R}^*_+$, by

$$m(A) = \inf \left\{ \sum_{n=1}^{\infty} \nu(I_n) : I_n \in \mathcal{I} \text{ and } A \subseteq \cup_{n=1}^\infty I_n \right\}.$$

Set $A \subseteq \mathbb{R}^L$ is *Lebesgue-measurable* if for every $\varepsilon > 0$, there is an open set $O \subseteq \mathbb{R}^L$ such that $m(O \setminus A) < \varepsilon$. Denote by $\mathcal{L}_L \subseteq \mathcal{P}(\mathbb{R}^L)$ the class of all Lebesgue-measurable sets, and define the *Lebesgue measure* $\mu : \mathcal{L}_L \to \mathbb{R}^*_+$ by $\mu(A) = m(A)$.

THEOREM 12.1.14. *$(\mathbb{R}^L, \mathcal{L}_L, \mu)$ is a measure space.*

Proof. This proof is beyond these notes. It can be found in standard textbooks on the topic. □

12.2 Probability

When the space S we are dealing with is the space of all possible results of an experiment, the subsets we want to measure are called events and their measures are understood as probabilities (which can be understood from either a frequentist or a likelihood perspective). Now, if (S, Σ, p) is a measure space and $p(S) = 1$, we say that S is a *sample space*, that (S, Σ, p) is a *probability space* and that p is a *probability measure*. So defined, the properties we impose for p to be considered a probability measure are known as *Kolmogorov's axioms*; their arguments are left as exercises.

THEOREM 12.2.1. *Let (S, Σ, p) be a probability space. For every $E \in \Sigma$, $p(E) \leqslant 1$ and $p(E^c) = 1 - p(E)$, where $E^c = S \setminus E$.*

THEOREM 12.2.2. *Let (S, Σ, p) be a probability space. For every $E, E' \in \Sigma$, $p(E' \cap E^c) = p(E') - p(E' \cap E)$ and $p(E \cup E') = p(E) + p(E') - p(E \cap E')$.*

EXERCISE 12.2.1. *A partition of S is a sequence $(E_n)_{n=1}^N$, with N finite or equal to infinity, of pairwise disjoint sets in Σ such that $\cup_{n=1}^N E_n = S$. With this definition, prove the following generalization of Theorem 12.2.2:* let (S, Σ, p) be a probability space. Then, for every $E \in \Sigma$ and every partition $(E_n)_{n=1}^N$ of S, one has that $p(E) = \sum_{n=1}^N p(E \cap E_n)$.

THEOREM 12.2.3 (Bonferroni's simple inequality). *Let (S, Σ, p) be a probability space. For every $E, E' \in \Sigma$, $p(E \cap E') \geqslant p(E) + p(E') - 1$.*

THEOREM 12.2.4 (Boole's inequality). *Let (S, Σ, p) be a probability space. For every sequence $(E_n)_{n=1}^\infty$ of sets in Σ, $p(\cup_{n=1}^\infty E_n) \leqslant \sum_{n=1}^\infty p(E_n)$.*

THEOREM 12.2.5. *Let (S, Σ, p) be a probability space, and let $(E_n)_{n=1}^\infty$ be a sequence in Σ such that $E_n \subseteq E_{n+1}$ for each n. Then, $p(\cup_{n=1}^\infty E_n) = \lim_{n\to\infty} p(E_n)$.*

Proof. Since p is σ-additive,

$$\begin{aligned}
p(\cup_{n=1}^\infty E_n) &= p(E_1) + \sum_{n=1}^\infty p(E_{n+1} \setminus E_n) \\
&= p(E_1) + \sum_{n=1}^\infty [p(E_{n+1}) - p(E_n)] \\
&= p(E_1) + \lim_{N\to\infty} \sum_{n=1}^N [p(E_{n+1}) - p(E_n)] \\
&= p(E_1) + \lim_{N\to\infty} [p(E_{N+1}) - p(E_1)] \\
&= \lim_{N\to\infty} p(E_N).
\end{aligned}$$

where the second step comes from Theorem 12.2.2 and the limit exists because it is a monotone and bounded sequence. □

COROLLARY 12.2.1. *Let* (S, Σ, p) *be a probability space, and let* $(E_n)_{n=1}^{\infty}$ *be a sequence in* Σ *such that* $E_n \supseteq E_{n+1}$ *for each* n. *Then,* $p(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} p(E_n)$.

12.3 Conditional probability

Henceforth, we maintain a probability space, (S, Σ, p), fixed. Let $E^* \in \Sigma$ be such that $p(E^*) > 0$. The *probability measure given (or conditional on)* E^* is defined by $p(\cdot \mid E^*) : \Sigma \to [0, 1]$, with $p(E \mid E^*) = p(E \cap E^*)/p(E^*)$.

THEOREM 12.3.1. *Let* $E, E' \in \Sigma$ *and suppose that* $p(E) \in (0, 1)$. *Then,*

$$p(E') = p(E) \cdot p(E' \mid E) + (1 - p(E)) \cdot p(E' \mid E^c).$$

Proof. By definition,

$$\begin{aligned} p(E)p(E' \mid E) + (1 - p(E))p(E' \mid E^c) &= p(E)\frac{p(E' \cap E)}{p(E)} + (1 - p(E))\frac{p(E' \cap E^c)}{p(E^c)} \\ &= p(E' \cap E) + p(E' \cap E^c) \\ &= p(E'), \end{aligned}$$

because of σ-additivity. □

The previous theorem, in fact, admits the following generalization, whose proof is left as an exercise.

THEOREM 12.3.2. *Let* $(E_n)_{n=1}^{N}$ *be a partition of* S *such that every* $p(E_n) > 0$. *Then, for any* $E' \in \Sigma$, *one has that* $p(E') = \sum_{n=1}^{N} p(E_n)p(E' \mid E_n)$.

THEOREM 12.3.3. *Let* $(E_n)_{n=1}^{N}$ *be a finite sequence of sets in* Σ, *such that* $p(\cap_{n=1}^{N-1} E_n) > 0$. *Then,*

$$p(\cap_{n=1}^{N} E_n) = p(E_1)p(E_2 \mid E_1)p(E_3 \mid E_1 \cap E_2) \ldots p(E_N \mid \cap_{n=1}^{N-1} E_n).$$

Proof. The proof is left as an exercise. (Hint: recall mathematical induction!) □

12.4 Independence

A family of events $\mathcal{E} \subseteq \Sigma$ is *pairwise independent* if $p(E \cap E') = p(E)p(E')$ for any two distinct $E, E' \in \mathcal{E}$. It is *independent* if

$$p(\cap_{n=1}^{N} E_n) = \prod_{n=1}^{N} p(E_n)$$

for any finite subfamily $\{E_n\}_{n=1}^{N}$ of distinct sets in $\mathcal{E}$.

THEOREM 12.4.1. *If* $\mathcal{E}$ *is independent, then it is pairwise independent.*

EXAMPLE 12.4.1. *Notice that pairwise independence does not suffice for independence. Let* $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *and* $\Sigma = \mathcal{P}(S)$, *and suppose that* : $p(\{s\}) = 1/9$

for each $s \in S$. *Let* $E_1 = \{1, 2, 7\}$, $E_2 = \{3, 4, 7\}$ *and* $E_3 = \{5, 6, 7\}$, *so*

$$p(E_1) = p(E_2) = p(E_3) = \frac{1}{3}.$$

Now, if $i, j \in \{1, 2, 3\}$, $i \neq j$, *then* $p(E_i \cap E_j) = 1/9 = p(E_i)p(E_j)$, *but*

$$p(E_1 \cap E_2 \cap E_3) = \frac{1}{9} \neq \frac{1}{27} = p(E_1)p(E_2)p(E_3),$$

so this family is pairwise independent, but not independent.

Notice that $\cap_{E \in \mathcal{E}} E = \varnothing$ is neither necessary nor sufficient for independence.

Theorem 12.4.2. *If* $\{E, E'\}$ *is independent, then so is* $\{E^c, E'\}$.

Proof. Just note that

$$\begin{aligned} p(E^c \cap E') &= p(E') - p(E \cap E') \\ &= p(E') - p(E)p(E') \\ &= (1 - p(E))p(E') \\ &= p(E^c)p(E'), \end{aligned}$$

by additivity and independence of $\{E, E'\}$. □

Corollary 12.4.1. *If* $\{E, E'\}$ *is independent, then so is* $\{E^c, (E')^c\}$.

Theorem 12.4.3. *Let* $\mathcal{E}$ *be independent. Then,* $\mathcal{E}^* = \{E \in \Sigma : E^c \in \mathcal{E}\}$ *is independent.*

The proofs of the last two results are left as exercises.

12.5 Random variables

Fix a measurable space (S, Σ). Function $X : S \to \mathbb{R}$ is *measurable with respect to* Σ *(or* Σ*-measurable)* if for every $x \in \mathbb{R}$, one has that

$$\{s \in S \mid X(s) \leqslant x\} \in \Sigma.$$

Theorem 12.5.1. *If* X *is* Σ*-measurable, then for all* $x \in \mathbb{R}$, *the following sets lie in* Σ: $\{s \in S \mid X(s) \geqslant x\}$, $\{s \in S \mid X(s) < x\}$, $\{s \in S \mid X(s) > x\}$ *and* $\{s \in S \mid X(s) = x\}$.

A *random variable (in* $\mathbb{R}$*)* is a Σ-measurable function $X : S \to \mathbb{R}$. Let us now endow the measurable space with a probability measure p and fix a random variable $X : S \to \mathbb{R}$. The *distribution function of* X is $F_X : \mathbb{R} \to [0, 1]$, defined by $F_X(x) = p(\{s \in S \mid X(s) \leqslant x\})$. (Note that F_X is well defined because X is Σ-measurable.)

Theorem 12.5.2. *Let* F_X *be the distribution function of* X. *Then,*

1. $\lim_{x \to -\infty} F_X(x) = 0$ *and* $\lim_{x \to \infty} F_X(x) = 1$;

2. $F_X(x) \geqslant F_X(x')$ *whenever* $x \geqslant x'$;

3. F_X *is right continuous: for all* $x \in \mathbb{R}$, $\lim_{h \downarrow 0} F_X(x + h) = F_X(x)$.

Proof. To see that $\lim_{x\to-\infty} F_X(x) = 0$, consider

$$(E_n)_{n=1}^{\infty} = (\{s \in S \mid X(s) \leqslant -n\})_{n=1}^{\infty},$$

which is a sequence in Σ. Notice that each $E_n \supseteq E_{n+1}$, so

$$\begin{aligned}
\lim_{x\to-\infty} F_X(x) &= \lim_{n\to\infty} p(E_n) \\
&= p(\cap_{n=1}^{\infty} E_n) \\
&= p(\cap_{n=1}^{\infty}\{s \in S \mid X(s) \leqslant -n\}) \\
&= p(\{s \in S \mid \forall n \in \mathbb{N}, X(s) \leqslant -n\}) \\
&= p(\varnothing) \\
&= 0,
\end{aligned}$$

where the second equality comes from Corollary 12.2.1.

Proving that $\lim_{x\to\infty} F_X(x) = 1$ and that $x \geqslant x'$ implies that $F_X(x) \geqslant F_X(x')$ is left as an exercise.

Now, fix $x \in \mathbb{R}$ and consider $(E_n)_{n=1}^{\infty} = (\{s \in S \mid X(s) \leqslant x + 1/n\})_{n=1}^{\infty}$. Notice that each $E_n \supseteq E_{n+1}$, so

$$\begin{aligned}
\lim_{h\downarrow 0} F_X(x) &= \lim_{n\to\infty} p(E_n) \\
&= p(\cap_{n=1}^{\infty} E_n) \\
&= p(\cap_{n=1}^{\infty}\{s \in S \mid X(s) \leqslant x + 1/n\}) \\
&= p(\{s \in S \mid \forall n \in \mathbb{N}, X(s) \leqslant x + 1/n\}) \\
&= p(\{s \in S \mid X(s) \leqslant x\}) \\
&= F_X(x),
\end{aligned}$$

where the second equality comes from Corollary 12.2.1. □

Notice that it is not necessarily true that $\lim_{h\uparrow 0} F_X(x+h) = F_X$, so we cannot guarantee that F is continuous. It is a good exercise to find a case in which $\lim_{h\uparrow 0} F_X(x+h) \neq F_X$. It is also important to see which step in the obvious proof of left-continuity would fail:

$$\{s \in S \mid \exists n \in \mathbb{N} : X(s) \leqslant x - 1/n\} = \{s \in S \mid X(s) < x\},$$

which may be a proper subset of $\{s \in S \mid X(s) \leqslant x\}$.

THEOREM 12.5.3. *Let F_X be the distribution function of X. Then,*

1. $p(\{s \in S \mid X(s) > x\}) = 1 - F_X(x)$;
2. $p(\{s \in S \mid x < X(s) \leqslant x'\}) = F_X(x') - F_X(x)$, *whenever* $x \leqslant x'$;
3. $p(\{s \in S \mid X(s) = x\}) = F_X(x) - \lim_{h\uparrow 0} F_X(x+h)$.

Proof. Part 1 is left as an exercise. To see the second part, notice that

$$\begin{aligned}
p(\{s \in S \mid x < X(s) \leqslant x'\}) &= p(\{s \in S \mid X(s) \leqslant x'\} \setminus \{s \in S \mid X(s) \leqslant x\}) \\
&= p(\{s \in S \mid X(s) \leqslant x'\}) - p(\{s \in S \mid X(s) \leqslant x\}) \\
&= F_X(x') - F_X(x),
\end{aligned}$$

where the second equality follows since $x \leqslant x'$.

For the third part, consider $(E_n)_{n=1}^{\infty} = (\{s \in S \mid X(s) \leqslant x - 1/n\})_{n=1}^{\infty}$. Notice that each $E_n \subseteq E_{n+1}$, so

$$\begin{aligned}
\lim_{h \uparrow 0} F_X(x+h) &= \lim_{n \to \infty} p(E_n) \\
&= p(\cup_{n=1}^{\infty} E_n) \\
&= p(\cup_{n=1}^{\infty} \{s \in S \mid X(s) \leqslant x - 1/n\}) \\
&= p(\{s \in S \mid \exists n \in \mathbb{N} : X(s) \leqslant x - 1/n\}) \\
&= p(\{s \in S \mid X(s) < x\}) \\
&= p(\{s \in S \mid X(s) \leqslant x\} \setminus \{s \in S \mid X(s) = x\}) \\
&= p(\{s \in S \mid X(s) \leqslant x\}) - p(\{s \in S \mid X(s) = x\}) \\
&= F_X(x) - p(\{s \in S \mid X(s) = x\}),
\end{aligned}$$

where the second equality comes from Theorem 12.2.5 and the seventh since $\{s \in S \mid X(s) = x\} \subseteq \{s \in S \mid X(s) \leqslant x\}$. □

The distribution function of a random variable characterizes (totally defines) its associated probability measure.

THEOREM 12.5.4. *Let F_X be the distribution function of X. Let $g : \mathbb{R} \to \mathbb{R}$ be strictly increasing and define the random variable $Y = g \circ X : S \to \mathbb{R}$. Denote by F_Y the distribution function of Y. Then, for all $y \in \mathbb{R}$, $F_Y(y) = F_X(g^{-1}(y))$.*

Proof. Let $y \in \mathbb{R}$, and note that

$$\begin{aligned}
F_Y(y) &= p(\{s \in S \mid Y(s) \leqslant y\}) \\
&= p(\{s \in S \mid g(X(s)) \leqslant y\}) \\
&= p(\{s \in S \mid X(s) \leqslant g^{-1}(y)\}) \\
&= F_X(g^{-1}(y)),
\end{aligned}$$

where existence of g^{-1} and the third equality follow from the fact that g is strictly increasing. □

Random variable X is said to be *continuous* if F_X is continuous. X is said to be *absolutely continuous* if there exists an integrable function $f_X : \mathbb{R} \to \mathbb{R}_+$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

for all $x \in \mathbb{R}$; in this case, f_X is said to be a *density function* of X.

EXAMPLE 12.5.1 (Standard Uniform Distribution). *Suppose that the distribution function of X is*

$$F_X(x) = \begin{cases} 0, & \textit{if } x < 0; \\ x, & \textit{if } 0 \leqslant x \leqslant 1; \\ 1, & \textit{if } x > 1. \end{cases}$$

Notice that X *is absolutely continuous, and one density function is*

$$f_X(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } 0 \leqslant x \leqslant 1; \\ 0, & \text{if } x > 1. \end{cases}$$

THEOREM 12.5.5. *Let* F_X *be the distribution function of* X. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be strictly decreasing and define the random variable* $Y = g \circ X : S \to \mathbb{R}$. *Denote by* F_Y *the distribution function of* Y. *Then, for all* $y \in \mathbb{R}$,

$$F_Y(y) = 1 - \lim_{h \uparrow 0} F_X(g^{-1}(y) + h).$$

Moreover, if X *is continuous,* $F_Y(y) = 1 - F_X(g^{-1}(y))$.

COROLLARY 12.5.1. *Let* f_X *be a density function of an absolutely continuous random variable* X. *Let* $g : \mathbb{R} \to \mathbb{R} \in \mathbb{C}^1$ *be strictly increasing and define the random variable* $Y = g \circ X : S \to \mathbb{R}$. *Define* $f_Y : \mathbb{R} \to \mathbb{R}$ *by*

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}.$$

Function f_Y *is a density function for* Y.

The proofs of the previous two results are left as exercises (Hint: in the latter case, remember the Chain Rule and the Inverse Function Theorem).

COROLLARY 12.5.2. *Let* f_X *be a density function of an absolutely continuous random variable* X. *Let* $g : \mathbb{R} \to \mathbb{R} \in \mathbb{C}^1$ *be strictly decreasing and define the random variable* $Y = g \circ X : S \to \mathbb{R}$. *Define* $f_Y : \mathbb{R} \to \mathbb{R}$ *by*

$$f_Y(y) = -\frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}.$$

Function f_Y *is a density function for* Y.

The last two theorems and corollaries have been stated under assumptions stronger than needed: it suffices that g be increasing in the closure of the set on which F is increasing. The latter set is known as the *support of* X.

EXERCISE 12.5.1. *Let* F_X *be the distribution function of a continuous random variable* X . *Let* $g : \mathbb{R} \to \mathbb{R}; g(x) = x^2$ *and define the random variable* $Y = g \circ X$. *Denote by* F_Y *the distribution function of* Y. *Show that* $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$, *and find a density function for* Y, *under the assumption that* X *is absolutely continuous.*

EXERCISE 12.5.2. *Prove the following result:* Let F_X be the distribution function of X. Suppose that F_X is strictly increasing and define the random variable $Y = F_X \circ X$. Y follows the Standard Uniform distribution (see Example 12.5.1).

12.6 Moments

Henceforth, we assume that X is an absolutely continuous random variable, with density f_X. We assume that for every set $D \subseteq \mathbb{R}$ such that $\int_D f_X(x)\,dx$ exists, it is true that

$$p(\{s \in S \mid X(s) \in D\}) = \int_D f_X(x)\,dx.$$

Moreover, for simplicity, the notation $p(X \in D)$ will replace $p(\{s \in S \mid X(s) \in D\})$ from now on.

Let $g : \mathbb{R} \to \mathbb{R}$ and define the random variable $g \circ X : S \to \mathbb{R}$. The *expectation of* $g \circ X$ is defined as

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx,$$

if the integral exists. Whenever there is $x^* \in \mathbb{R}$ such that $\int_{-\infty}^{x^*} xf(x)\,dx = -\infty$ and $\int_{x^*}^{\infty} xf(x)\,dx = \infty$, we say that $E(X)$ does not exist.[2] Notice, in particular, that even if for every $x^* \in \mathbb{R}_+$ one has that

$$\int_{-x^*}^{0} |x| f_X(x)\,dx = \int_{0}^{x^*} |x| f_X(x)\,dx \in \mathbb{R},$$

the integral $E(X)$ may fail to exist. Notice also that if

$$\int_{-\infty}^{\infty} |x| f_X(x)\,dx \in \mathbb{R}$$

then $E(X)$ exists.

[2] The reason is simple:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)\,dx \\ &= \int_{-\infty}^{x^*} xf(x)\,dx \\ &\quad + \int_{x^*}^{\infty} xf(x)\,dx \\ &= -\infty + \infty \end{aligned}$$

will not be defined.

THEOREM 12.6.1. *Let* $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ *and* $a, b, c \in \mathbb{R}$. *Then,*

1. $E(ag_1(X) + bg_2(X) + c) = aE(g_1(X)) + bE(g_2(X)) + c$*;*

2. *if* $g_1(x) \geqslant 0$ *for all* $x \in \mathbb{R}$*, then* $E(g_1(X)) \geqslant 0$*;*

3. *if* $g_1(x) \geqslant g_2(x)$ *for all* $x \in \mathbb{R}$*, then* $E(g_1(X)) \geqslant E(g_2(X))$*;*

4. *if* $a \leqslant g_1(x) \leqslant b$ *for all* $x \in \mathbb{R}$*, then* $a \leqslant E(g_1(x)) \leqslant b$.

THEOREM 12.6.2 (Chebychev's Inequality). *Let* $g : \mathbb{R} \to \mathbb{R}_+$ *be such that* $E(g(X)) \in \mathbb{R}$. *Then, for all* $r > 0$*,*

$$p(g(X) \geqslant r) \leqslant \frac{E(g(X))}{r}.$$

Proof. By definition,

$$\begin{aligned} E(g(X)) &= \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \\ &\geqslant \int_{\{x \in \mathbb{R} | g(x) \geqslant r\}} g(x) f_X(x)\,dx \\ &\geqslant \int_{\{x \in \mathbb{R} | g(x) \geqslant r\}} r f_X(x)\,dx \\ &= r \int_{\{x \in \mathbb{R} | g(x) \geqslant r\}} f_X(x)\,dx \\ &= rp(g(X) \geqslant r), \end{aligned}$$

where the first inequality follows since, for all $x \in \mathbb{R}$, $g(x) \geqslant 0$. □

Strictly speaking, in the previous proof we needed to argue that $\int_{\{x\in\mathbb{R}|g(x)\geqslant r\}} f_X(x)\, dx$ exists. For this, it would suffice, for example, that g be continuous.

For a finite integer k, the k-*th (non-central) moment of* X is $E(X^k)$, whenever it exists. If $E(X)$ exists in $\mathbb{R}$, the k-*th central moment of* X is $E((X-E(X))^k)$, whenever this integral exists. The first moment of X is its *expectation* or *mean*, and its second central moment is its *variance* and is denoted $V(X)$.

COROLLARY 12.6.1. *Suppose that* $E(X)$ *and* $V(X) > 0$ *exist. Then, for all* $t > 0$ *one has that*

$$p(|X-E(X)| \geqslant t\sqrt{V(X)}) \leqslant \frac{1}{t^2}.$$

Proof. Define $g : \mathbb{R} \to \mathbb{R}_+$ by

$$g(x) = \frac{(x-E(X))^2}{V(X)}.$$

By Chebychev's inequality,

$$\begin{aligned} p(|X-E(X)| \geqslant t\sqrt{V(X)}) &= p(\frac{(X-E(X))^2}{V(X)} \geqslant t^2) \\ &\leqslant \frac{1}{t^2}E(\frac{(X-E(X))^2}{V(X)}) \\ &= \frac{1}{t^2}. \end{aligned}$$

□

EXERCISE 12.6.1. *Prove the following corollary:* if $E(X)$ and $V(X) > 0$ exist, then

$$p(|X-E(X)| < 2\sqrt{V(X)}) \geqslant \frac{3}{4}.$$

Notice the surprising implication of the previous exercise: the probability that the realization of a random variable be at least two standard deviations from its mean is at least 0.75, regardless of its distribution!

The *Moment Generating Function* of random variable X is $M_X : \mathbb{R} \to \mathbb{R}$, defined by $M_X(t) = E(e^{tX})$, whenever the integral exists in $\mathbb{R}$.

THEOREM 12.6.3. *For all* $k \in \mathbb{N}$, *if the derivative exists,* $M_X^{(k)}(0) = E(X^k)$.

Proof. If the derivative exists,

$$\begin{aligned} M'_X(t_0) &= \frac{\partial E(e^{tX})}{\partial t}(t_0) \\ &= \frac{\partial \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx}{\partial t}(t_0) \\ &= \int_{-\infty}^{\infty} \frac{\partial e^{tx} f_X(x)}{\partial t}\, dx|_{t_0} \\ &= \int_{-\infty}^{\infty} \frac{\partial e^{tx} f_X(x)}{\partial t}\, dx|_{t_0} \\ &= \int_{-\infty}^{\infty} x e^{t_0 x} f_X(x)\, dx. \end{aligned}$$

Now, suppose that $k \in \mathbb{N} \setminus \{1\}$ and

$$M_X^{(k-1)}(t_0) = \int_{-\infty}^{\infty} x^{k-1} e^{t_0 x} f_X(x)\ dx.$$

Then,

$$\begin{aligned} M_X^{(k)}(t_0) &= \frac{\partial \int_{-\infty}^{\infty} x^{k-1} e^{tx} f_X(x)\ dx}{\partial t}(t_0) \\ &= \int_{-\infty}^{\infty} \frac{\partial x^{k-1} e^{tx} f_X(x)}{\partial t}\ dx|_{t_0} \\ &= \int_{-\infty}^{\infty} x^k e^{t_0 x} f_X(x)\ dx. \end{aligned}$$

By mathematical induction, it follows that for all $k \in \mathbb{N}$,

$$M_X^{(k)}(t_0) = \int_{-\infty}^{\infty} x^k e^{t_0 x} f_X(x)\ dx$$

and hence that

$$M_X^{(k)}(0) = \int_{-\infty}^{\infty} x^k f_X(x)\ dx = \mathrm{E}(X^k)$$

□

Notice that the previous theorem assumes that the derivative exists and replaces the derivative of an integral by the integral of the derivative, which amounts to replacing the limit of an integral by the integral of a limit. When X has bounded support, this is just fine. In other cases, it suffices to show that there exist a random variable with larger absolute value than the integrand and finite integral (in which case one can appeal to a result known as the Lebesgue's Dominated Convergence Theorem).

It is important to know that the moment generating function completely characterizes a random variable's distribution: if $M_X = M_{X'}$ then $F_X = F_{X'}$.

Exercise 12.6.2 (Standard Uniform Distribution). *Suppose that the distribution function of* X *is the one introduced in Example 12.5.1. Find* $\mathrm{E}(X)$ *and* M_X*. Show that* M_X *is not differentiable at* 0.

Exercise 12.6.3 (Standard Normal distribution). *Suppose that the density function of* X *is given by*

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

for all $x \in \mathbb{R}$*. Show that for all* $t \in \mathbb{R}$ *one has that* $M_X(t) = e^{\frac{t^2}{2}}$, $\mathrm{E}(X) = 0$ *and* $\mathrm{V}(X) = 1$.

Exercise 12.6.4 (Exponential distribution). *Suppose that the density function of* X *is given, for all* $x \in \mathbb{R}$*, by*

$$f_X(x) = \begin{cases} 0, & \text{if } x < 0; \\ \frac{1}{\beta} e^{-\frac{x}{\beta}}, & \text{if } x \geqslant 0, \end{cases}$$

where $\beta > 0$*. Find* $\mathrm{E}(X)$ *and* $\mathrm{E}(X^2)$*. Show that* $M_X : (-\infty, \frac{1}{\beta}) \to \mathbb{R}$ *is*

$$M_X(t) = \frac{1}{1 - \beta t}.$$

Use M_X to verify that $M_X'(0) = E(X)$ and $M_X''(0) = E(X^2)$.

12.7 Independence of random variables

Let $(X_n)_{n=1}^N$ be a finite sequence of random variables. The *joint distribution of* $(X_n)_{n=1}^N$ is $F_{(X_n)_{n=1}^N} : \mathbb{R}^N \to [0,1]$, defined by

$$F_{(X_n)_{n=1}^N}(x) = p(\{s \in S \mid (X_n(s))_{n=1}^N \leqslant x\}).$$

Sequence $(X_n)_{n=1}^N$ is said to be *absolutely continuous* if there exists a function $f_{(X_n)_{n=1}^N} : \mathbb{R}^N \to \mathbb{R}_+$ such that

$$F_{(X_n)_{n=1}^N}(x) = \int_{v \leqslant x} f_{(X_n)_{n=1}^N}(u)\, du$$

for all $x \in \mathbb{R}^N$; in this case, function $f_{(X_n)_{n=1}^N}$ is said to be a *joint density function* for $(X_n)_{n=1}^N$. If the sequence is absolutely continuous, and if each random variable X_n has function f_{X_n} as a density function, then $(X_n)_{n=1}^N$ is said to be *independent* if[3]

$$f_{(X_n)_{n=1}^N} = \prod_{n=1}^N f_{X_n}.$$

[3] This definition is not totally general in that independence does not really require absolute continuity. For the purposes of these notes, however, the definition suffices.

A sequence $(X_n)_{n=1}^\infty$ of random variables is said to be *independent* if every finite sequence $(X_{n_k})_{k=1}^K$ constructed with elements of $(X_n)_{n=1}^\infty$ is independent.

As before, if $(X_n)_{n=1}^N$ is a sequence of absolutely continuous random variables and $g : \mathbb{R}^N \to \mathbb{R}$, the *expectation of* $g((X_n)_{n=1}^N)$ is defined as

$$E(g((X_n)_{n=1}^N)) = \int_{-\infty}^{\infty} g(x) f_{(X_n)_{n=1}^N}(x)\, dx,$$

whenever the integral exists.

Theorem 12.7.1. *Let $(X_n)_{n=1}^N$ be a finite sequence of random variables. If $(X_n)_{n=1}^N$ is independent, then*

$$E(\prod_{n=1}^N X_n) = \prod_{n=1}^N E(X_n).$$

Proof. By definition

$$\begin{aligned}
E(\prod_{n=1}^{N} X_n) &= \int_{-\infty}^{\infty} (\prod_{n=1}^{N} x_n) f_{(X_n)_{n=1}^{N}}(x)\, dx \\
&= \int_{-\infty}^{\infty} (\prod_{n=1}^{N} x_n)(\prod_{n=1}^{N} f_{X_n}(x_n))\, dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{n=1}^{N} (x_n f_{X_n}(x_n))\, dx_N \dots\, dx_2\, dx_1 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \prod_{n=1}^{N-1} (x_n f_{X_n}(x_n))(\int_{-\infty}^{\infty} x_N f_{X_N}(x_N)\, dx_N) \dots\, dx_2\, dx_1 \\
&= E(X_N) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \prod_{n=1}^{N-1} (x_n f_{X_n}(x_n))\, dx_{N-1} \dots\, dx_2\, dx_1 \\
&\vdots \\
&= E(X_N)E(X_{N-1})\dots E(X_2) \int_{-\infty}^{\infty} x_1 f_{X_1}(x_1)\, dx_1 \\
&= E(X_N)E(X_{N-1})\dots E(X_2)E(X_1),
\end{aligned}$$

where the second inequality follows from independence. □

Strictly speaking, the third equality in the last expression also has to be justified. This would follow from a result known as Fubini's theorem.

EXERCISE 12.7.1. *Prove the following corollary: if* (X_1, X_2) *is independent, then*

$$\mathrm{Cov}(X_1, X_2) = E((X_1 - E(X_1))(X_2 - E(X_2))) = 0.$$

Perhaps because of the previous example, the idea of "no correlation" is oftentimes confused with independence. One must be careful about this: is two random variables are independent, then their correlation is zero; but the other causality is not true: if X is normal, then X and X^2 are uncorrelated, but they certainly are not independent.

12.8 Conditional Density and Expectation

For the purposes of this section, let (X_1, X_2) be a pair of random variables, assume that it is absolutely continuous and denote by $f_{(X_1,X_2)}$ its density function. The *marginal density of* X_1 is the function defined by $f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{(X_1,X_2)}(x_1, x_2)\, dx_2$, for each $x_1 \in \mathbb{R}$. If $f_{X_1}(x_1) > 0$, the *conditional density of* X_2 *given that* $X_1 = x_1$ is given by the function

$$f_{X_2|X_1}(x_2 \mid x_1) = \frac{f_{(X_1,X_2)}(x_1, x_2)}{f_{X_1}(x_1)},$$

for each $x_2 \in \mathbb{R}$.

These definitions are useful when one needs to "decompose" the bivariate problem: if one needs to know the probability that $X_1 \in [a, b]$, by definition one needs to compute

$$\int_a^b \int_{-\infty}^{\infty} f_{(X_1,X_2)}(x_1, x_2)\, dx_2\, dx_1 = \int_a^b f_{X_1}(x_1)\, dx_1;$$

and, if one knows that the realization of X_1 is x_1, and one needs to compute the probability that $X_2 \in [a, b]$ given that knowledge, one simply needs to know $\int_a^b f_{X_2|X_1}(x_2 \mid x_1)\, dx_2$, as the conditional density given that $X_1 = x_1$ re-normalizes the "prior" density $f_{(X_1,X_2)}$ to take into account the knowledge on X_1.

THEOREM 12.8.1. *If* (X_1, X_2) *is independent, then, for any* x_1 *such that* $f_{X_1}(x_1) > 0$, *one has that* $f_{X_2|X_1}(x_2 \mid x_1) = f_{X_2}(x_2)$.

Let $g : \mathbb{R} \to \mathbb{R}$. The *conditional expectation of* $g(X_2)$ *given that* $X_1 = x_1$ is

$$E(g(X_2) \mid X_1 = x_1) = \int_{-\infty}^{\infty} g(x_2) f_{X_2|X_1}(x_2 \mid x_1)\, dx_2.$$

Similarly, the *conditional expectation of* $g(X_2)$ *given* X_1, which we denote $E(g(X_2) \mid X_1)$, is the random variable that takes value $E(g(X_2) \mid X_1 = x_1)$ in any state in which $X_1 = x_1$; in particular, the probability that $E(g(X_2) \mid X_1 = x_1) \in [a, b]$ equals

$$\int_{\{x_1 : E(g(X_2)|X_1=x_1)\in[a,b]\}} f_{X_1}(x_1)\, dx_1.$$

THEOREM 12.8.2 (The Law of Iterated Expectations). *Let* (X_1, X_2) *be a pair of random variables.* $E(X_2) = E(E(X_2 \mid X_1))$.

Proof. By direct computation,

$$\begin{aligned} E(X_2) &= \int\int x_2 f_{(X_1,X_2)}(x_1, x_2)\, dx_2\, dx_1 \\ &= \int_{\{x_1|f_{X_1}(x_1)\}} (\int x_2 f_{X_2|X_1}(x_2 \mid x_1)\, dx_2) f_{X_1}(x_1)\, dx_1 \\ &= \int E(X_2 \mid X_1 = x_1) f_{X_1}(x_1)\, dx_1 \\ &= E(E(X_2 \mid X_1)). \end{aligned}$$

□

12.9 Convergence of random variables

There are several concepts of convergence for random variables. We consider three of them: a sequence $(X_n)_{n=1}^{\infty}$ of random variables

(a) *converges in probability to the random variable* X if for all $\varepsilon > 0$ one has that

$$\lim_{n\to\infty} p(|X_n - X| < \varepsilon) = 1;$$

(b) it *converges almost surely to the random variable* X if $p(\lim_{n\to\infty} X_n = X) = 1$; and

(c) it *converges in distribution to the random variable* X if at all $x \in \mathbb{R}$ at which F_X is continuous, one has that $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$, where each F_{X_n} is the distribution function of X_n and F_X is the one of X.

For notational conciseness, we denote the three types of convergence by $X_n \xrightarrow{p} X$, $X_n \xrightarrow{c.s.} X$ and $X_n \xrightarrow{d} X$, respectively. It is important to understand the relationship between these three concepts, which we now do, albeit in a somewhat informal manner.

We first introduce, without proof, two very intuitive results (formally, they follow from Lebesgue's dominated convergence theorem):

(Fact 1:) If $X_n \overset{c.s.}{\to} X$ and $g : \mathbb{R} \to \mathbb{R}$ is bounded, then $E(X_n) \to E(X)$.
(Fact 2:) If $X_n \overset{p}{\to} X$ and $g : \mathbb{R} \to \mathbb{R}$ is bounded, then $E(X_n) \to E(X)$.

With these two results, we can argue that (1) if $X_n \overset{c.s.}{\to} X$ then $X_n \overset{p}{\to} X$; and (2) if $X_n \overset{p}{\to} X$ then $X_n \overset{d}{\to} X$. For simplicity, let us consider only the case in which all X_n and X are continuous. For statement (1), fix $\varepsilon > 0$ and define $I_{\geqslant\varepsilon} : \mathbb{R} \to \{0, 1\}$ by saying

$$I_{\geqslant\varepsilon}(x) = \begin{cases} 1, & \text{if } x \geqslant \varepsilon; \\ 0, & \text{if } x < \varepsilon. \end{cases}$$

Since $X_n \overset{c.s.}{\to} X$, it follows that $I_{\geqslant\varepsilon}(|X_n - X|) \overset{c.s.}{\to} 0$, which guarantees, by the first fact, that $E(I_{\geqslant\varepsilon}(|X_n - X|)) \to 0$, or, equivalently, that $p(|X_n - X| \geqslant \varepsilon) \to 0$, which yields the result.

For the second statement, fix $x^* \in \mathbb{R}$, a point of continuity of F_X, and define, similarly to the previous argument, the function $I_{\leqslant x^*} : \mathbb{R} \to \{0, 1\}$ by

$$I_{\leqslant x^*}(x) = \begin{cases} 1, & \text{if } x \leqslant x^*; \\ 0, & \text{if } x > x^*. \end{cases}$$

By the second fact, we have that $E(I_{\leqslant x^*}(X_n)) \to E(I_{\leqslant x^*}(X))$, which is equivalent to saying that $p(X_n \leqslant x^*) \to p(X \leqslant x^*)$.

Now, neither one of the opposite causalities is true. When the limit variable is constant, convergence in distribution implies convergence in probability, as the following theorem states.

THEOREM 12.9.1. *Let $(X_n)_{n=1}^{\infty}$ be a sequence of continuous random variables and let $x^* \in \mathbb{R}$ be such that, $X_n \overset{d}{\to} x^*$. Then, $X_n \overset{p}{\to} x^*$.*[4]

[4] One must understant what the statement $X_n \overset{d}{\to} x^*$ means: what it says is that $X_n \overset{d}{\to} X$ where $X : S \to \mathbb{R}$ is the random variable constant in x^*.

Proof. Denote by F the distribution function of the random variable constant in x^*. Fix $\varepsilon > 0$. By definition,

$$\begin{aligned} p(|X_n - x^*| \geqslant \varepsilon) &= p(X_n \leqslant x^* - \varepsilon) + p(X_n \geqslant x^* + \varepsilon) \\ &= F_{X_n}(x^* - \varepsilon) + 1 - \lim_{h \uparrow 0} F_{X_n}(x^* + \varepsilon + h) \\ &= F_{X_n}(x^* - \varepsilon) + 1 - F_{X_n}(x^* + \varepsilon) \\ &\to F(x^* - \varepsilon) + 1 - F(x^* + \varepsilon) \\ &= 0 + 1 - 1 \\ &= 0, \end{aligned}$$

where the third equality follows because F_{X_n} is continuous and convergent because $x^* + \varepsilon$ is a point of continuity of F, since

$$F(x) = \begin{cases} 1, & \text{if } x \geqslant x^*; \\ 0, & \text{if } x < x^*. \end{cases}$$

□

However, if the limit variable is not a constant, convergence in distribution does not ensure convergence in probability and, in any case, convergence in probability does not ensure almost sure convergence, as shown in the following example.

EXAMPLE 12.9.1. *Suppose $S = [0, 1]$, endowed with the Uniform measure. For* *every interval* $[a,b]$*, define* $I_{[a,b]} : \mathbb{R} \to \{0,1\}$ *by*

$$I_{[a,b]}(s) = \begin{cases} 1, & \textit{if } s \in [a,b]; \\ 0, & \textit{if } s \notin [a,b]. \end{cases}$$

Consider the sequence $(X_n)_{n=1}^{\infty}$ *of random variables, defined as follows:*

$$\begin{aligned} X_1(s) &= s + I_{[0,1]}(s) \\ X_2(s) &= s + I_{[0,1/2]}(s) \\ X_3(s) &= s + I_{[1/2,1]}(s) \\ X_4(s) &= s + I_{[0,1/3]}(s) \\ X_5(s) &= s + I_{[1/3,2/3]}(s) \\ X_6(s) &= s + I_{[2/3,1]}(s) \\ X_7(s) &= s + I_{[0,1/4]}(s) \\ &\vdots \end{aligned}$$

and define X *by* $X(s) = s$*. We shall show that* $X_n \xrightarrow{p} X$*, but it is not true that* $X_n \xrightarrow{c.s.} X$*:*
(1) To see that $X_n \xrightarrow{p} X$*, notice that the interval in which* $X_n \neq X$ *gets smaller and smaller as* n *grows and, since* S *is endowed with the uniform measure, if* $\varepsilon < 1$*, then* $p(|X_n - X| > \varepsilon) \to 0$*.*
(2) Now, to see that it is not true that $X_n \xrightarrow{c.s.} X$*, simply notice that there is no* $s \in [0,1]$ *for which* $X_n(s) \to s = X(s)$*, since, in fact, for no* $s \in [0,1]$ *is* $X_n(s)$ *convergent, so,* $p(\lim_{n\to\infty} X_n = X) = 0$*.*

EXAMPLE 12.9.2. *Suppose a sequence* $(X_n : S \to \mathbb{R}_{++})_{n=1}^{\infty}$ *such that* $X_n \xrightarrow{p} x^* \in \mathbb{R}_{++}$*. Let* $\varepsilon \in (0, \sqrt{x^*}]$ *and* $\gamma \in (0, x^*]$ *be such that* $\varepsilon = \sqrt{x^*} - \sqrt{x^* - \gamma}$*. Notice that*

$$\begin{aligned} |x - x^*| < \gamma \quad &\Leftrightarrow \quad x^* - \gamma < x < x^* + \gamma \\ &\Leftrightarrow \quad \sqrt{x^* - \gamma} < \sqrt{x} < \sqrt{x^* + \gamma} \\ &\Leftrightarrow \quad \sqrt{x^* - \gamma} - \sqrt{x^*} < \sqrt{x} - \sqrt{x^*} < \sqrt{x^* + \gamma} - \sqrt{x^*} \\ &\Rightarrow \quad |\sqrt{x} - \sqrt{x^*}| < \sqrt{x^*} - \sqrt{x^* - \gamma} = \varepsilon, \end{aligned}$$

which implies that

$$p(|\sqrt{X_n} - \sqrt{x^*}| < \varepsilon) \geqslant p(|X_n - x^*| < \gamma) \to 1.$$

Now, if $\varepsilon > \sqrt{x^*}$*, then,* $p(|\sqrt{X_n} - \sqrt{x^*}| < \varepsilon) \geqslant p(|\sqrt{X_n} - \sqrt{x^*}| < \sqrt{x^*}) \to 1$*, which implies that* $\sqrt{X_n} \xrightarrow{p} \sqrt{x^*}$*.*

EXERCISE 12.9.1. *Consider a sequence* $(X_n : S \to [\underline{x}, \infty))_{n=1}^{\infty}$*, where* $\underline{x} \in \mathbb{R}_{++}$*, such that* $X_n \xrightarrow{p} x^* \in \mathbb{R}$*. Show that* $x^* \in [\underline{x}, \infty)$ *and* $x^*/X_n \xrightarrow{p} 1$*.*

12.10 The (weak) law of large numbers

A sequence $(X_n)_{n=1}^{\infty}$ of random variables is said to be *i.i.d.* if it is independent and for every n and n', $p(X_n \in D) = p(X_{n'} \in D)$ for all $D \subseteq \mathbb{R}$.

THEOREM 12.10.1 (The Weak Law of Large Numbers). *Let* $(X_n)_{n=1}^{\infty}$ *be an i.i.d. sequence of random variables, and suppose that, for all* n, $E(X_n) = \mu \in \mathbb{R}$ *and* $V(X_n) = \sigma^2 \in \mathbb{R}_{++}$. *The sequence of random variables given by* $\bar{X}_n = \sum_{k=1}^{n} X_k/n$ *converges in probability to* μ.

Proof. Given that $(X_n)_{n=1}^{\infty}$ is i.i.d. $E(\bar{X}_n) = \mu$ and $V(\bar{X}_n) = \sigma^2/n$.[5] Now, by Corollary 12.6.1, for $\varepsilon > 0$ one has that

$$p(|\bar{X}_n - \mu| \geqslant \varepsilon) \leqslant \frac{\sigma^2}{n\varepsilon^2},$$

so $\lim_{n\to\infty} p(|\bar{X}_n - \mu| \geqslant \varepsilon) \leqslant \lim_{n\to\infty} \sigma^2/(n\varepsilon^2) = 0$. □

[5] Showing this is left as an exercise.

EXERCISE 12.10.1. *Let* $(X_n)_{n=1}^{\infty}$ *be a sequence of random variables. Define the sequence* $(\bar{X}_n)_{n=1}^{\infty}$ *as in Theorem 12.10.1, and the sequence*

$$V_n = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X}_n)^2$$

for all $n \in \mathbb{N}$. *Show that* $\bar{X}_{n+1} = (X_{n+1} + n\bar{X}_n)/(n+1)$ *and*

$$nV_{n+1} = (n-1)V_n + (\frac{n}{n+1})(X_{n+1} - \bar{X}_n)^2.$$

Moreover, show that, under the assumptions of the Theorem, $E(\sqrt{n}(\bar{X}_n - \mu)/\sigma) = 0$ *and* $V(\sqrt{n}(\bar{X}_n - \mu)/\sigma) = 1$.

EXERCISE 12.10.2. *Prove the following result:* let X be an absolutely continuous random variable, and fix $\Omega \subseteq \mathbb{R}$ such that $p(X \in \Omega) \in (0,1)$. Consider the experiment where $n \in \mathbb{N}$ realizations of X are taken independently, and let G_n be the relative frequency with which a realization in Ω is obtained in the experiment. Then, $G_n \xrightarrow{p} p(X \in \Omega)$.

The "Strong" Law of Large Numbers gives an analogous result for almost sure convergence. In econometrics, the weak law usually suffices.

12.11 Central limit theorem

THEOREM 12.11.1 (The Central Limit Theorem). *Let* $(X_n)_{n=1}^{\infty}$ *be an i.i.d. sequence of random variables, and suppose that, for every* n, $E(X_n) = \mu \in \mathbb{R}$, $V(X_n) = \sigma^2 \in \mathbb{R}_{++}$ *and* $M_{X_n} = M_X$ *is defined in an open neighborhood of* 0. *Define the sequence* $(\bar{X}_n)_{n=1}^{\infty}$ *as in Theorem 12.10.1. Then,*

$$\sqrt{n}\frac{\bar{X}_n - \mu}{\sigma} \xrightarrow{d} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

Proof. Define the sequence $Y_n = (X_n - \mu)/(\sigma)$, and denote by M_Y the moment generating function common to all the Y_n variables (which we can do, because $(X_n)_{n=1}^{\infty}$ is i.i.d.). Note that $E(Y_n) = 0$ and $V(Y_n) = 1$, and that

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{k=1}^{n} Y_k &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{X_k-\mu}{\sigma} \\
&= \frac{\sqrt{n}}{\sigma}\sum_{k=1}^{n}(\frac{X_k}{n}-\frac{\mu}{n}) \\
&= \frac{\sqrt{n}}{\sigma}(\bar{X}_n-\mu),
\end{aligned}$$

so

$$\begin{aligned}
M_{\frac{\sqrt{n}}{\sigma}(\bar{X}_n-\mu)}(t) &= M_{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}Y_k}(t) \\
&= E(e^{t\frac{1}{\sqrt{n}}\sum_{k=1}^{n}Y_k}) \\
&= E(\prod_{k=1}^{n} e^{t\frac{1}{\sqrt{n}}Y_k}) \\
&= \prod_{k=1}^{n} E(e^{t\frac{1}{\sqrt{n}}Y_k}) \\
&= \prod_{k=1}^{n} M_Y(t\frac{1}{\sqrt{n}}) \\
&= M_Y(t\frac{1}{\sqrt{n}})^n.
\end{aligned}$$

Taking a Taylor expansion to the right-hand side of the previous expression, around 0, we get that

$$M_Y(t\frac{1}{\sqrt{n}}) = M_Y(0) + M_Y'(0)\frac{t}{\sqrt{n}} + \frac{1}{2}M_Y''(0)\frac{t^2}{n} + R^3(\frac{t}{\sqrt{n}}),$$

where, by Taylor's theorem,

$$\lim_{n\to\infty}(\frac{t}{\sqrt{n}})^{-2}R^3(\frac{t}{\sqrt{n}}) = 0.$$

By construction, $M_Y'(0) = E(Y_n) = 0$ and $M_Y''(0) = E(Y_n^2) = V(Y_n) = 1$, which implies that

$$M_Y(t\frac{1}{\sqrt{n}}) = 1 + \frac{1}{2}\frac{t^2}{n} + R^3(\frac{t}{\sqrt{n}}).$$

It then follows that

$$\begin{aligned}
\lim_{n\to\infty} M_Y(t\frac{1}{\sqrt{n}})^n &= \lim_{n\to\infty}(1+\frac{1}{2}\frac{t^2}{n}+R^3(\frac{t}{\sqrt{n}}))^n \\
&= \lim_{n\to\infty}(1+\frac{1}{n}(\frac{1}{2}t^2+nR^3(\frac{t}{\sqrt{n}})))^n \\
&= e^{\frac{t^2}{2}},
\end{aligned}$$

where the second equality follows from the fact that

$$\lim_{n\to\infty}(\frac{t}{\sqrt{n}})^{-2}R^3(\frac{t}{\sqrt{n}}) = 0$$

implies that

$$\lim_{n\to\infty} nR^3(\frac{t}{\sqrt{n}}) = 0.$$

But the latter suffices, since $e^{\frac{t^2}{2}}$ is the moment generating function of

$$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

$\square$

EXERCISE 12.11.1. *How can both the law of large numbers and the central limit theorem be true? That is, if the law says that* $\bar{X}_n$ *converges in probability to a constant* (μ)*, and convergence in probability implies convergence in distribution, then how can* $\bar{X}_n$ *also converge in distribution to the standard normal?*

Take a good look at what the Central Limit Theorem implies. In particular, it does not imply that every "large" sample of realizations of a random variable "tends" to be distributed normally!