On the sophistication of financial investors and the information revealed by prices

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Abstract

Does sufficient sophistication on the reasoning of financial traders lead to rational expectations equilibrium? We provide an answer by studying a simple exchange economy with complete markets and asymmetric information. Traders are classified as fundamentalists, who know the true probability distributions of random shocks, or speculators, who try to infer the true probabilities from asset prices. Starting with the naive beliefs that asset prices transmit no information, the speculators learn the mapping from asset prices to probabilities through level-k reasoning. We characterize the necessary conditions on convergence to rational expectations equilibrium for some specific utility functions (CRRA and CARA) and discuss the general case. Our results are that: (1) convergence to rational expectations requires that speculators have less market impact than fundamentalists; (2) convergence, when it takes place, occurs in an oscillating manner; and (3) asset prices can be more volatile than at rational expectations equilibrium when speculators display low sophistication.

Keywords: rational expectations, level-k reasoning, information revealed by prices, bounded reasoning, Radner equilibrium.
JEL classification: D53, D84, G02, G14

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1 Introduction

Rational expectations equilibrium (REE, henceforth) has been an utmost important idea in economic theory and is widely used in both micro and macro economics. The seminal work by Muth (1961) introduces rational expectations to study how agents predict price movements, while Radner (1979) models REE as self-fulling beliefs, where agents maximize their utility based on their beliefs and the market clearing outcome confirms these beliefs. In particular, Radner (1979) models REE as a mapping from the set of states of the world to the set of commodity prices, and proves the generic existence and invertibility of REE—prices are hence fully revealing, and all agents can figure out all the private information in the economy once they observe them.

Two limitations of Radner (1979), however, are the implicit assumption that the rational mapping from states to prices (and its inverse) are common knowledge between all agents, and the fact that the model is silent on how the agents would learn the mapping. In order to address these limitations, McAllister (1990) incorporates rational expectations equilibrium in a decision making framework and constructs the rational expectation mapping at the individual level. He considers the space of uncertainty for each agent to be the product of the set of exogenous states and the set of all possible asset positions of other traders. A REE is then defined as an admissible prior, a price vector, and asset positions such that traders are optimizing while market clears for all states.

Later, Dutta and Morris (1997) generalize McAllister (1990) by relaxing the assumptions of common knowledge of consensus beliefs and degeneracy of expectations. They introduce the concepts of *belief equilibrium*, where agents might have disagreements on their prior beliefs, and of *common belief equilibrium*, where agents hold the same beliefs. There, an REE is a restriction to common belief equilibrium where the agents only consider the exogenous states of nature and the mapping from states to prices is consensus between all agents. In this line of research, Ben-Porath and Heifetz (2011) is, to the best of our knowledge, the most general result on the epistemic foundation of rational expectations equilibrium literature. They show that common knowledge of rationality and market clearing (CKRMC) is sufficient to yield REE.

However, Dubey et al. (1987) criticize the REE approach in general equilibrium models with asymmetric information since it fails to explain how information gets encoded into asset prices. Unlike the literature above, this paper regards the question of how agents learn the rational expectation mapping. To be sure, there have been many discussions on convergence to rational expectations in both the macro and the micro literatures. Shiller (1978), for instance, studies convergence to the rational expectation forecast in Muth (1961), while DeCanio (1979) studies convergence to rational expectations in a linear forecast model with general autocorrelation structure. Closer to our paper, Bray (1982) studies a setting where agents are learning the relation between asset returns and prices using OLS estimation. Bray's results suggest the learning process could converge to rational expectations even if agents are using misspecified models. Blume and Easley (1984) study a dynamic market process in which agents learn a payoff relevant parameter by conditioning on past endogenously generated data. They define REE as the limit of the learning process once all the agents' beliefs converge to the true parameter almost surely.

Instead of studying the convergence to a single parameter, as this past literature did, our paper investigates more general conditions on which the mappings from states to prices converges to the REE.

Our paper incorporates level-k reasoning to a general equilibrium setting and defines an iterative reasoning process where agents' mappings evolve as they become more sophisticated. Level-k reasoning is an alternative to Nash equilibrium that describes how strategic sophistication determines players' strategies. Importantly, this approach is supported by ample experimental evidence—see Nagel (1995), Stahl and Wilson (1994) and Stahl and Wilson (1995).[1]

In addition to experimental studies, researchers have applied level-k thinking model to macroeconomics. For instance, Farhi and Werning (2019) provide an explanation to forward guidance puzzle with level-k thinking agents, where Angeletos and Lian (2017) argue that level-k thinking explains the slow adjustments of general equilibrium effects.

However, these previous studies ignore the issue of convergence and Farhi and Werning (2019) suggest that convergence to rational expectations occurs in a monotonic way. Our paper investigates convergence of level-k mapping in a general equilibrium setting and our results are that (1) convergence to REE is not guaranteed, and (2) convergence, if it occurs, takes place in a non-monotonic way.

2 A Simple Radner Economy

For simplicity, we consider a minimal, two-period exchange economy with uncertainty, where the state space for the future period is just $\Sigma = \{1, 2\}$. There is only one commodity in the economy, and consumption takes place only in the second period.

There are two types of agent in the economy. *Fundamentalists* know that the probability that state of the world $\sigma = 1$ will realize in the future is π. *Speculators* don't know this, and only understand that π comes from a probability distribution, Π, with support Δ. We will use super-index $a \in \{F, S\}$ to denote the agents' types.

Besides the information the two types have, they also differ in their preferences and future wealth. In state σ, agents of type a will be endowed with a wealth ω^a_σ. We assume that there is a continuum of agents of each type, with masses μ^F and μ^S.

In the present, the agents trade the elementary securities corresponding to the two states of nature. They have expected-utility preferences with type-dependent Bernoulli index $\upsilon^a : \mathbb{R} \to \mathbb{R}$. Denoting holdings by agents of type a of the security that pays in state σ as y^a_σ, the ex-ante utility of a fundamentalist is

$$\pi \cdot \upsilon^F(\omega^F_1 + y^F_1) + (1 - \pi) \cdot \upsilon^F(\omega^F_2 + y^F_2); \tag{1}$$

for a speculator, if she receives or discerns information $\mathcal{I} \subseteq \Delta$, it is

$$\mathbb{E}\left[\Pi \cdot \upsilon^S(\omega^S_1 + y^S_1) + (1 - \Pi) \cdot \upsilon^S(\omega^S_2 + y^S_2) \mid \mathcal{I}\right]. \tag{2}$$

What information the speculators use will depend, of course, on the sophistication of their reasoning.

We normalize the price of the security for $\sigma = 2$ to unity, and denote by q the price of the security for

[1] Crawford et al. (2013) provides a thorough review on level-k thinking models and several supporting experimental evidence. Other experimental evidence on level-k thinking includes Arad and Rubinstein (2012), Costa-Gomes and Crawford (2006), Crawford and Iriberri (2007a), Crawford and Iriberri (2007b) and Bosch-Domenech et al. (2002).

$\sigma = 1$. When an agent of type a chooses her portfolio, the only constraint she faces is that

$$q \cdot y_1^a + y_2^a = 0. \tag{3}$$

Note that one can use Eq. (3) to solve for the holdings of the second security and then rewrite Eqs. (1) and (2) in terms of the first security only. Using this simplification, we can write the optimal demands for the first security as $Y^F(q;\pi)$ for the fundamentalists, and as $Y^S(q;\mathcal{I})$ for the speculators.

Market clearing requires that the aggregate of the two types' demands vanish:

$$\mu^F \cdot Y^F(q;\pi) + \mu^S \cdot Y^S(q;\mathcal{I}) = 0. \tag{4}$$

3 Rational Expectations

A *rational expectations equilibrium* is a function $\bar{Q} : \Delta \to \mathbb{R}$ such that, for all π, Eq. (4) is satisfied for $q = \bar{Q}(\pi)$ and

$$\mathcal{I} = \bar{Q}^{-1}(q); \tag{5}$$

the equilibrium is *fully revealing* if it is injective, namely if

$$\bar{Q}^{-1}(\bar{Q}(\pi)) = \{\pi\} \tag{6}$$

for all π.

Eq. (5) requires that speculators discern all the information transmitted by prices *correctly at equilibrium.* Eq. (6) further requires that prices transmit *all* of the information available to the fundamentalists. It has been known, since Radner (1979), that mild conditions on preferences guarantee the existence of fully-revealing REE, generically on the agents' wealth levels.

4 Level-k Reasoning

The concept of REE assumes implicitly that speculators know the equilibrium itself and use it to infer information they lack. It is, in that sense, analogous to the concept of Nash equilibrium in game theory. An alternative approach is provided by the model of level-k reasoning. In this setting one starts by stipulating what the most naive behavior of an agent is, and proceeds recursively to define higher levels of sophistication as the reasoning of an agent who thinks that everybody else in the setting is one level below. We adapt this approach to the reasoning of speculators in their understanding of the informational content of prices.

4.1 Level 0

The least sophisticated a speculator can be is to fail to realize that the price depends on the information received by the fundamentalists. We call this level of naïveté *level-0 reasoning*, and define the corresponding demand of the speculator as $Y_0^S(q) = Y^S(q;\Delta)$. This is an agent who uses un-conditional expectations on her portfolio problem (Eq. (2) with $\mathcal{I} = \Delta$, the full support of Π) regardless of q. By linearity, that is

$$Y_0^S(q) = \arg\max_y \left\{\mathbb{E}[\Pi] \cdot \upsilon(\omega_1^S + y) + (1 - \mathbb{E}[\Pi]) \cdot \upsilon(\omega_2^S - q \cdot y)\right\}.$$

Assuming that it exists, we define the market-clearing pricing function $Q_0 : \Delta \to \mathbb{R}$ by the solution of Eq. (4) with level-0 reasoning by the speculators; explicitly

$$\mu^F \cdot Y^F(Q_0(\pi); \pi) + \mu^S \cdot Y_0^S(Q_0(\pi)) = 0,$$

for all π.

4.2 *Level* k

For any natural number k, suppose that a speculator understands the dependence of prices on the information of the fundamentalists through price function $Q_{k-1} : \Delta \to \mathbb{R}$. We say that she has *level*-k *reasoning* if, at prices q, she uses information $\mathcal{I} = Q_{k-1}^{-1}(q)$ in her choice of an optimal portfolio. Her optimal demand for the first security can thus be denoted as

$$Y_k^S(q) = Y^S(q; Q_{k-1}^{-1}(q));$$

it gives raise to a new pricing function, $Q_k : \Delta \to \mathbb{R}$, defined by

$$\mu^F \cdot Y^F(Q_k(\pi); \pi) + \mu^S \cdot Y_k^S(Q_k(\pi)) = 0,$$

assuming that such function exists.

Importantly, level-k reasoning is a form of bounded rationality: the speculators do *not* realize that their usage of function Q_{k-1} changes the equilibrium prices at each value of π—namely, that it induces a new mapping Q_k.

4.3 *Rational expectations again*

Let $\mathcal{Q}$ be the space of functions $Q : \Delta \to \mathbb{R}$. Note that, starting from Q_0, the definition of level-k reasoning recursively constructed a sequence $\langle Q_k \rangle_{k \in \mathbb{N}}$ in $\mathcal{Q}$. Let us denote by R the mapping that defined the recursion. By construction, any REE is a fixed point of R. The first question that will occupy us is whether there exist conditions that ensure that Q_k converges to $\bar{Q}$ as $k \to \infty$.

5 A Fully Solvable Example: Betting with Log Preferences

In order to be able to compute a closed form solution, suppose that both types of trader have the same Bernoulli index, $\upsilon(x) = \ln x$, that in both states $\omega_\sigma^F = \Omega > \omega = \omega_\sigma^S$, and $\mu^F = \mu^S = 1$. Also, let the Π have expectation $\mathbb{E}(\Pi) = 1/2$ and support $\Delta = [\underline{\pi}, \bar{\pi}] \subset (\omega^2/2\Omega^2, 1 - \omega^2/2\Omega^2)$.

5.1 *Rational expectations*

Let us conjecture that the rational expectations mapping $\bar{Q}$ is bijective, so that Eq. (6) holds true. The individual demands then are

$$Y^F(q;\pi) = \frac{\pi}{q}(q \cdot \omega_1^F + \omega_2^F) - \omega_1^F = \left(\pi \cdot \frac{q+1}{q} - 1\right) \cdot \Omega, \tag{7}$$

and

$$Y^S(q, \{q\}) = \frac{\pi}{q}(q \cdot \omega_1^S + \omega_2^S) - \omega_1^S = \left(\pi \cdot \frac{q+1}{q} - 1\right) \cdot \omega.$$

Substituting into Eq. (4), and using that $\mu^F = \mu^S = 1$, one can solve for $\bar{Q} : (0,1) \to \mathbb{R}_{++}$ as

$$\bar{Q}(\pi) = \frac{\pi}{1-\pi} \cdot \frac{\omega_2^F + \omega_2^S}{\omega_1^F + \omega_1^S} = \frac{\pi}{1-\pi}. \tag{8}$$

Since $\bar{Q}$ is bijective, we confirm that the REE is fully revealing. Its inverse, which we denote as $\bar{\Pi} : \mathbb{R}_{++} \to (0,1)$, is

$$\bar{\Pi}(q) = \frac{q \cdot (\omega_1^F + \omega_1^S)}{q \cdot (\omega_1^F + \omega_1^S) + \omega_2^F + \omega_2^S} = \frac{q}{q+1}. \tag{9}$$

5.2 *Level-k reasoning*

Assuming that mapping Q_{k-1} is bijective, we can use its inverse function to pin down the beliefs of level-k speculators upon observation of price q, implicitly, by $\{\hat{\Pi}_k(q)\} = Q_{k-1}^{-1}(q)$. In this case, the demand of the fundamentalists is still given by Eq. (7), while for the speculators it is

$$Y_k^S(q) = \frac{\hat{\Pi}_k(q)}{q}(q \cdot \omega_1^S + \omega_2^S) - \omega_1^S.$$

Since $\mu^F = \mu^S = 1$, Eq. (4) allows us to define the next pricing function, Q_k, implicitly, by

$$Y^F(Q_k(\pi);\pi) = -Y_k^S(Q_k(\pi)).$$

Upon substitution, that is

$$\frac{\pi}{Q_k(\pi)}[Q_k(\pi) \cdot \omega_1^F + \omega_2^F] - \omega_1^F = \omega_1^S - \frac{\hat{\Pi}_k(Q_k(\pi))}{Q_k(\pi)}[Q_k(\pi) \cdot \omega_1^S + \omega_2^S]. \tag{10}$$

Unfortunately, solving for Q_k explicitly is not easy. Under the further simplification given by the assumption that $\omega_1^F = \omega_2^F = \Omega > \omega = \omega_1^S = \omega_2^S$, Eq. (10) becomes

$$\left[\pi \cdot \frac{Q_k(\pi) - 1}{Q_k(\pi)} - 1\right] \cdot \Omega = \left[1 - \hat{\Pi}_k(Q_k(\pi)) \cdot \frac{Q_k(\pi) + 1}{Q_k(\pi)}\right] \cdot \omega \tag{11}$$

Starting from $\hat{\Pi}_0(q) = \mathbb{E}(\Pi) = 1/2$, and therefore from

$$Q_0(\pi) = \frac{2 \cdot \pi \cdot \Omega + \omega}{2 \cdot (1 - \pi) \cdot \Omega + \omega},$$

and using mathematical induction, one has that, for all natural k

$$Q_k(\pi) = \frac{2 \cdot \pi \cdot \Omega^{k+1} + (-1)^k \cdot \omega^{k+1}}{2 \cdot (1 - \pi) \cdot \Omega^{k+1} + (-1)^k \cdot \omega^{k+1}}, \tag{12}$$

for any $\pi \in \Delta$. Importantly, this mapping is bijective.

5.3 *Rational expectations again*

We can re-write Eq. (12) as

$$Q_k(\pi) = \frac{2 \cdot \pi + (-1)^k \cdot \left(\frac{\omega}{\Omega}\right)^{k+1}}{2 \cdot (1 - \pi) + (-1)^k \cdot \left(\frac{\omega}{\Omega}\right)^{k+1}},$$

and then it follows that, since $\Omega > \omega > 0$, $Q_k(\pi) \to \bar{Q}(\pi)$ for all $\pi \in \Delta$. Moreover,

$$Q_k(\pi) - \bar{Q}(\pi) = \frac{(-1)^k \cdot (1 - 2\pi) \cdot \left(\frac{\omega}{\Omega}\right)^{k+1}}{2 \cdot (1 - \pi)^2 + (-1)^k \cdot (1 - \pi) \cdot \left(\frac{\omega}{\Omega}\right)^{k+1}}, \tag{13}$$

and for $k \geqslant 2$ this difference is continuous on π over Δ, which is compact, so $Q_k \to \bar{Q}$ not just pointwise but uniformly. Interestingly, pointwise convergence is *not* monotonic: because of the term $(-1)^k$ on the right-hand side of Eq. (13), the sign of the difference $Q_k(\pi) - \bar{Q}(\pi)$ will oscillate between even and odd k's. Only at $\pi = \mathbb{E}(\Pi)$ is that sign constant, and the difference there is null.

5.4 *Introducing background risk*

As mentioned before, Eq. (10) is difficult to solve without the assumptions that simplified it to Eq. (11). Still, maintaining the assumption that both types have logarithmic Bernoulli indices, Eq. (10) implies that, if Q_k is bijective, its inverse must satisfy

$$\frac{\hat{\Pi}_{k+1}(q)}{q}(q \cdot \omega_1^F + \omega_2^F) - \omega_1^F = \omega_1^S - \frac{\hat{\Pi}_k(q)}{q}(q \cdot \omega_1^S + \omega_2^S). \tag{14}$$

Eq. (14) is critical for our analysis. Upon simplification, it gives us the difference equation that governs the beliefs of speculators, at given prices, as their level of reasoning evolves:

$$\hat{\Pi}_k(q) = \min\left\{\max\left\{\frac{q \cdot (\omega_1^F + \omega_1^S)}{q \cdot \omega_1^F + \omega_2^F} - \frac{q \cdot \omega_1^S + \omega_2^S}{q \cdot \omega_1^F + \omega_2^F} \cdot \hat{\Pi}_{k-1}(q), \underline{\pi}\right\}, \bar{\pi}\right\}. \tag{15}$$

It follows that, so long as the nominal wealth of the fundamentalists is higher than the one of the speculators, which was true in the simpler case with no background risk, the sequence of level-k beliefs converges to the rational expectations beliefs even if such convergence is non-monotonic. That is,

Proposition 1. *For any price* q *for which*

$$\frac{q \cdot \omega_1^S + \omega_2^S}{q \cdot \omega_1^F + \omega_2^F} < 1,$$

one has that

$$\hat{\Pi}_k(q) \to \frac{q \cdot (\omega_1^F + \omega_1^S)}{q \cdot (\omega_1^F + \omega_1^S) + \omega_2^F + \omega_2^S} = \bar{\Pi}(q),$$

Proof. To simplify notation, let

$$A(q) = \frac{q \cdot (\omega_1^F + \omega_1^S)}{q \cdot \omega_1^F + \omega_2^F} \quad \text{and} \quad B(q) = \frac{q \cdot \omega_1^S + \omega_2^S}{q \cdot \omega_1^F + \omega_2^F}.$$

Also, define five cutoff prices $q_1, \ldots, q_5$ implicitly as follows:

$$\begin{aligned} A(q_1) - B(q_1) \cdot \underline{\pi} &= \underline{\pi} \\ A(q_2) - B(q_2) \cdot \mathbb{E}(\Pi) &= \underline{\pi} \\ A(q_3) - B(q_3) \cdot \mathbb{E}(\Pi) &= \mathbb{E}(\Pi) \\ A(q_4) - B(q_4) \cdot \mathbb{E}(\Pi) &= \bar{\pi} \\ A(q_5) - B(q_5) \cdot \bar{\pi} &= \bar{\pi} \end{aligned}$$

and let $A(q_i)$ and $B(q_i)$ be denoted as A_i and B_i. Note that $A(q) - B(q) \cdot \pi$ is a strictly increasing function of q, for any $\pi \in [\underline{\pi}, \bar{\pi}]$, when $B(q) < 1$. Therefore, we can rank these prices as $q_1 < q_2 < q_3 < q_4 < q_5$, since we have $\underline{\pi} < \mathbb{E}(\pi) < \bar{\pi}$.

The level-1 mapping is

$$\hat{\Pi}_1(q) = \min\{\max\{A(q) - \mathbb{E}(\Pi)B(q), \underline{\pi}\}, \bar{\pi}\}.$$

Applying our definition, we could write the level-1 mapping as the follow:

$$\hat{\Pi}_1(q) = \begin{cases} \underline{\pi}; & q_1 \leqslant q \leqslant q_2 \\ A(q) - \mathbb{E}(\Pi) \cdot B(q); & q_2 < q < q_4 \\ \bar{\pi}; & q_4 \leqslant q \leqslant q_5 \end{cases}.$$

According to (15), if $q_1 \leqslant q \leqslant q_2$, the level-2 mapping is

$$\hat{\Pi}_2(q) = \min\{\max\{A - B \cdot \underline{\pi}, \underline{\pi}\}, \bar{\pi}\}.$$

By monotonicity of $A(q) - B(q) \cdot \pi$ in q, we have $A(q) - B(q) \cdot \underline{\pi} \geqslant A_1 - B_1 \cdot \underline{\pi} = \underline{\pi}$. In addition,

$$A - B \cdot \underline{\pi} \leqslant A_2 - B_2 \cdot \underline{\pi} = B_2\mathbb{E}(\Pi) + (1 - B_2)\underline{\pi} < \mathbb{E}(\Pi).$$

Now consider the case when $q_4 \leqslant q \leqslant q_5$, we have

$$\hat{\Pi}_2(q) = \min\{\max\{A - B \cdot \bar{\pi}, \underline{\pi}\}, \bar{\pi}\}.$$

With identical arguments, we can show $\mathbb{E}(\Pi) < A - B \cdot \bar{\pi} < \bar{\pi}$.

When price $q_2 \leqslant q \leqslant q_4$, we have

$$\hat{\Pi}_2(q) = \min\{\max\{A - B \cdot (A - B \cdot \mathbb{E}(\Pi)), \underline{\pi}\}, \bar{\pi}\}.$$

By the monotonicity $A(q) - B(q) \cdot \pi$, we again have

$$A - B \cdot (A - B \cdot \mathbb{E}(\Pi)) < A - B \cdot \bar{\pi} \leqslant \bar{\pi}$$

and

$$A - B \cdot (A - B \cdot \mathbb{E}(\Pi)) > A - B \cdot \underline{\pi} \geqslant \underline{\pi}.$$

As a result, the level-2 mapping is

$$\hat{\Pi}_2(q) = \begin{cases} A(q) - B(q) \cdot \underline{\pi}; & q_1 \leqslant q \leqslant q_2 \\ A(q) - B(q) \cdot \mathbb{E}(\Pi); & q_2 < q < q_4 \\ A(q) - B(q) \cdot \bar{\pi}; & q_4 \leqslant q \leqslant q_5 \end{cases}.$$

By induction,

$$\hat{\Pi}_k(q) = \begin{cases} A \cdot \sum_{j=0}^{k-1}(-B)^j + \underline{\pi} \cdot (-B)^k; & q_1 \leqslant q \leqslant q_2 \\ A \cdot \sum_{j=0}^{k-1}(-B)^j + \mathbb{E}(\Pi) \cdot (-B)^k; & q_2 < q < q_4 \\ A \cdot \sum_{j=0}^{k-1}(-B)^j + \bar{\pi} \cdot (-B)^k; & q_4 \leqslant q \leqslant q_5 \end{cases}.$$

which is a continuous function with two kinks. So long as we have $B(q) < 1$, sequence $\langle \hat{\Pi}_k(q) \rangle_{k \in \mathbb{N}}$ converges. Eq. (15) must then hold with both $\hat{\Pi}_k(q)$ and $\hat{\Pi}_{k-1}(q)$ by replaced by the limit, which implies that $\hat{\Pi}_k(q) \to \bar{\Pi}(q)$. □

Figure 1 shows the convergence of level-k strategy.[2] It confirms our claim that convergence to REE occurs in a non-monotonic way.

Importantly, Eq. (15) shows that the premise of this result is also necessary: *at prices at which the two nominal wealths are equal, the sequence of level-k beliefs oscillates in a two-level cycle; and at prices at which the nominal wealth of the fundamentalists is lower, the sequence of level-k beliefs diverges non-monotonically*. Fig. 2 shows the level-k mapping when the nominal wealth of fundamentalists equals speculators. The mapping oscillates between two functions. In addition, Fig. 3 illustrates the situation when the nominal wealth of speculators is greater than the fundamentalists. For sufficient large k, the mappings at odd levels behave as step functions. Speculators believe that when price is below the cutoff value, state 1 occurs with probability $\underline{\pi}$. On the contrary, they believe that when price is above the cutoff, state 1 occurs with probability $\bar{\pi}$. The mapping at even levels describes the relation between

[2] We pick $\underline{\pi} = 0.1$, $\bar{\pi} = 0.9$ and $\mathbb{E}(\Pi) = 0.5$ in this simulation.

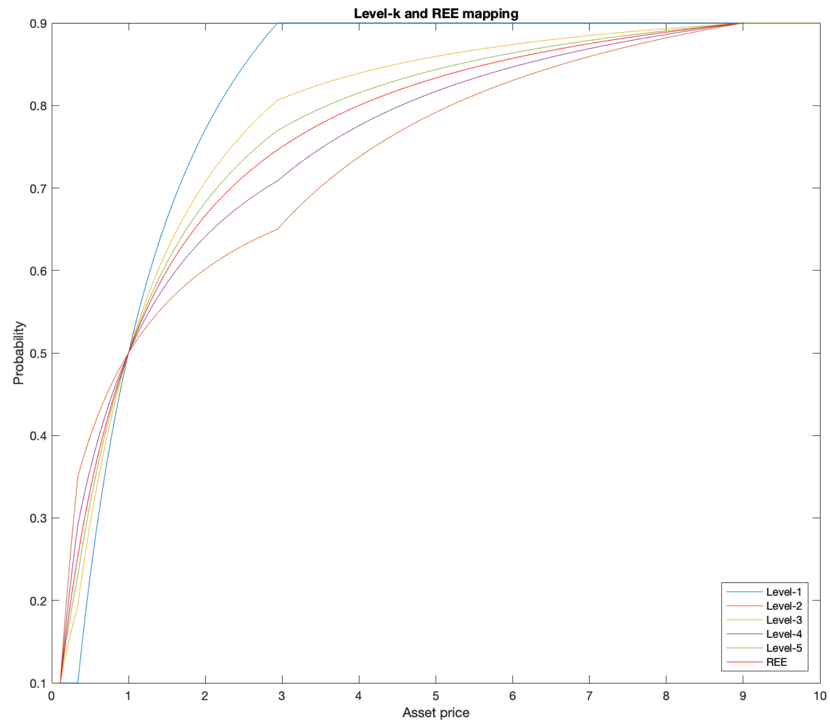

Figure 1: Convergence of Level-k mapping with logarithmic utility $(\omega_1^F, \omega_2^F) = (8, 8)$ and $(\omega_1^S, \omega_2^S) = (5, 4)$

true probabilities and asset prices when all the speculators hold the belief which is described by the mapping at odd levels.

6 Other Solvable Examples

We now study convergence for other prominent utility functions for which we can use analytic expressions. As with Eq. (14), since we cannot solve all expressions explicitly, we replace the recursion of price functions, $Q \mapsto R(Q)$ defined by

$$\mu^F \cdot Y^F(R(Q)(\pi); \pi) + \mu^S \cdot Y^S(R(Q)(\pi); Q^{-1}(R(Q)(\pi))) = 0,$$

by the recursion of belief functions $\hat{\Pi} \mapsto T(\hat{\Pi})$ defined by

$$\mu^F \cdot Y^F(q; T(\hat{\Pi})(q)) + \mu^S \cdot Y^S(q; \hat{\Pi}(q)) = 0. \tag{16}$$

As long as the price functions are injective, the latter simply results in the inverse mappings of the former.

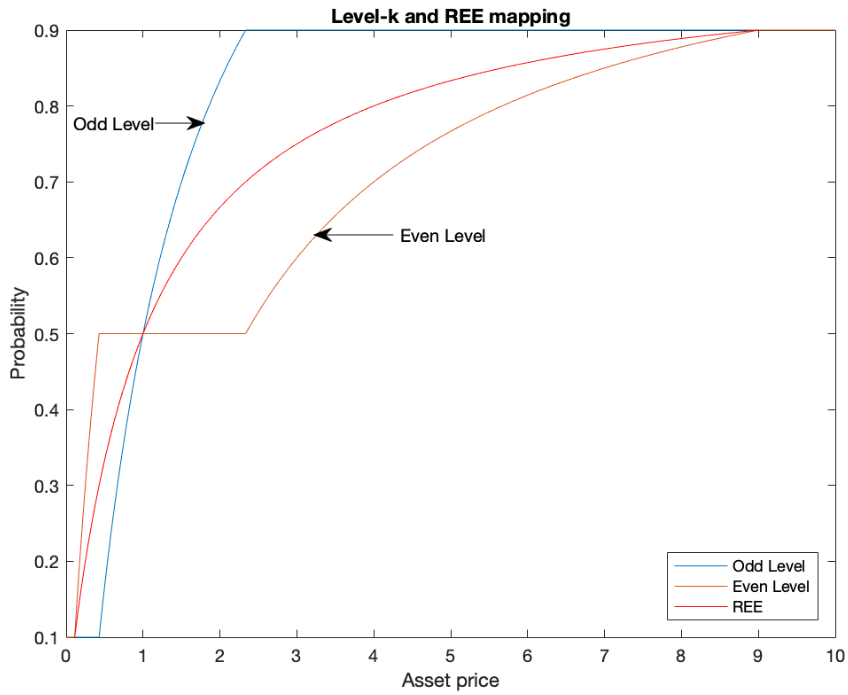

Figure 2: Oscillation of Level-k mapping with logarithmic utility
$\omega_1^F = \omega_1^S$ and $\omega_2^F = \omega_2^S$

6.1 *Other CRRA functions*

Let us assume now that both types have Bernoulli index $v(x) = x^{1-\gamma}/(1-\gamma)$ and have mass $\mu^F = \mu^S = 1$.

Proposition 2. *The sequence of belief functions $\langle \Pi_k \rangle_{k=1}^{\infty}$ converges to the rational expectation equilibrium $\bar{\Pi}$ pointwise at every q for which*

$$\frac{q \cdot \omega_1^S + \omega_2^S}{q \cdot \omega_1^F + \omega_2^F} < 1.$$

At prices that violate this inequality, the sequence of beliefs diverges in an oscillating manner.

Proof. The two types' optimal demands are

$$Y^F(q,\pi) = \frac{q\omega_1^F + \omega_2^F}{q + q^{\frac{1}{\gamma}} \left(\frac{1-\pi}{\pi}\right)^{\frac{1}{\gamma}}} - \omega_1^F \ \text{ and } \ Y_k^S(q) = \frac{q\omega_1^S + \omega_2^S}{q + q^{\frac{1}{\gamma}} \left[\frac{1-\hat{\Pi}_k(q)}{\hat{\Pi}_k(q)}\right]^{\frac{1}{\gamma}}} - \omega_1^S.$$

Market clearing requires the sum of these two values to be null. Substitution of π by $\hat{\Pi}_{k+1}(q)$ yields, after some algebra, the equality

$$\left\{ q + q^{\frac{1}{\gamma}} \left[\frac{1-\hat{\Pi}_{k+1}(q)}{\hat{\Pi}_{k+1}(q)}\right]^{\frac{1}{\gamma}} \right\}^{-1} = \frac{\omega_1^F + \omega_1^S}{q\omega_1^F + \omega_2^F} - \frac{q\omega_1^S + \omega_2^S}{q\omega_1^F + \omega_2^F} \cdot \left\{ q + q^{\frac{1}{\gamma}} \left[\frac{1-\hat{\Pi}_k(q)}{\hat{\Pi}_k(q)}\right]^{\frac{1}{\gamma}} \right\}^{-1}.$$

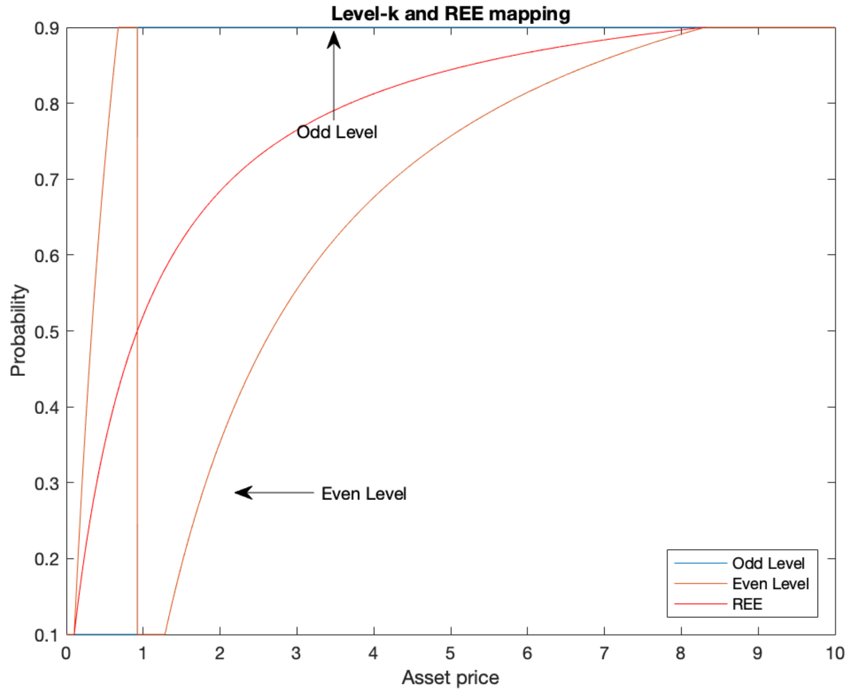

Figure 3: Divergence of Level-k mapping with logarithmic utility $(\omega_1^F, \omega_2^F) = (5, 5)$ and $(\omega_1^S, \omega_2^S) = (8, 7)$

This recursion implies that

$$\left\{ q + q^{\frac{1}{\gamma}} \left[\frac{1 - \hat{\Pi}_k(q)}{\hat{\Pi}_k(q)} \right]^{\frac{1}{\gamma}} \right\}^{-1}$$

converges as $k \to \infty$ if, and only if, the condition of the proposition holds true.

Since mapping $\pi \mapsto (1-\pi)/\pi$ is monotonic, the latter convergence is equivalent to the convergence of $\hat{\Pi}_k(q)$ as $k \to \infty$. The limit of the sequence is defined by

$$q^{\frac{1}{\gamma}} \left[\frac{1 - \lim_{k\to\infty} \hat{\Pi}_k(q)}{\lim_{k\to\infty} \hat{\Pi}_k(q)} \right]^{\frac{1}{\gamma}} = \frac{q(\omega_1^F + \omega_1^S) + \omega_2^F + \omega_2^S}{\omega_1^F + \omega_1^S} - q = \frac{\omega_2^F + \omega_2^S}{\omega_1^F + \omega_1^S}$$

The solution to the latter equation gives

$$\lim_{k\to\infty} \hat{\Pi}_k(q) = \frac{q(\omega_1^F + \omega_1^S)^\gamma}{q(\omega_1^F + \omega_1^S)^\gamma + (\omega_2^F + \omega_2^S)^\gamma}, \tag{17}$$

which is, indeed, $\bar{\Pi}(q)$. □

Corollary 1. *The sequence of belief functions* $\langle \Pi_k \rangle_{k=1}^{\infty}$ *converges to the rational expectation equilibrium* $\bar{\Pi}$ *uniformly if* $\omega^F > \omega^S$.

Proof. It suffices to observe that $\omega^F > \omega^S$ implies the condition of the proposition, and that the domain of prices is compact, since so is the support Δ and all the belief functions are continuous. $\square$

6.2 CARA functions

Assume now that both types of agents share the exponential Bernoulli index, $\upsilon(x) = -\exp\{-\rho x\}$.

Proposition 3. *The sequence of belief functions $\langle \Pi_k \rangle_{k=1}^{\infty}$ converges to the rational expectation equilibrium $\bar{\Pi}$ uniformly if, and only if, $\mu^F > \mu^S$.*

Proof. The demand of each fundamentalist is

$$Y^F(q, \pi) = \frac{q\omega_1^F + \omega_2^F}{1+q} - \frac{\ln q}{\rho(1+q)} - \frac{1}{\rho(1+q)} \ln\left(\frac{\pi}{1-\pi}\right) - \omega_1^F.$$

Similarly each speculator demands

$$Y_k^S(q) = \frac{q\omega_1^F + \omega_2^F}{1+q} - \frac{\ln q}{\rho(1+q)} - \frac{1}{\rho(1+q)} \ln\left[\frac{\hat{\Pi}_k(q)}{1-\hat{\Pi}_k(q)}\right] - \omega_1^S$$

Substituting π with $\hat{\Pi}_{k+1}$ in the market clearing equation, one gets

$$\mu^F \ln\left[\frac{\hat{\Pi}_{k+1}(q)}{1-\hat{\Pi}_{k+1}(q)}\right] + \mu^S \ln\left[\frac{\hat{\Pi}_k(q)}{1-\hat{\Pi}_k(q)}\right] = \rho[\mu^F(\omega_2^F - \omega_1^F) + \mu^S(\omega_2^S - \omega_1^S)] - (\mu^F + \mu^S)\ln q,$$

or, equivalently,

$$\ln\left[\frac{\hat{\Pi}_{k+1}(q)}{1-\hat{\Pi}_{k+1}(q)}\right] = -\frac{\mu^S}{\mu^F} \ln\left[\frac{\hat{\Pi}_k(q)}{1-\hat{\Pi}_k(q)}\right] + \rho\left[\frac{\mu^F}{\mu^S}(\omega_2^F - \omega_1^F) + \omega_2^S - \omega_1^S\right] - \frac{\mu^F + \mu^S}{\mu^S} \ln q.$$

If, and only if, $\mu^F > \mu^S$, the sequence of mappings

$$\ln\left[\frac{\hat{\Pi}_k(q)}{1-\hat{\Pi}_k(q)}\right]$$

converges uniformly. Since $\pi \mapsto \pi/(1-\pi)$ is a monotonic mapping, we have that $\hat{\Pi}_k(q)$ converges uniformly to $\bar{\Pi}(q)$ if, and only if, $\mu^F > \mu^S$.

To complete the argument, note that the rational expectation equilibrium, $\bar{\Pi}(q) = \gamma/(q+\gamma)$, with

$$\gamma = \exp\left\{\rho \cdot (\frac{\mu^S}{\mu^S + \mu^F}) \cdot \left[\frac{\mu^F}{\mu^S}(\omega_2^F - \omega_1^F) + (\omega_2^S - \omega_1^S)\right]\right\}$$

is bijective. $\square$

7 Conditions on General Demand Functions

Going back to the general case, assume that both types of agent have $\mathcal{C}^2$, strictly increasing and strictly concave Bernoulli indexes and that price functions Q_k are injective. Recall from Eq. (16) that level-k reasoning is the implicit recursion

$$\mu^F \cdot Y^F(q; \hat{\Pi}_k(q)) + \mu^S \cdot Y^S(q; \hat{\Pi}_{k-1}(q)) = 0, \tag{18}$$

starting from the "most naive" beliefs $q \mapsto \hat{\Pi}_0(q) = \mathbb{E}[\Pi]$.

Lemma 1. *Suppose that there is a unique, fully-revealing REE, $\bar{Q}$. When the realized π happens to be $\mathbb{E}[\Pi]$, for all levels of reasoning the price that clears the markets is the same and equals the price that clears them under rational expectations. That is, $Q_k(\mathbb{E}[\Pi]) = \bar{Q}(\mathbb{E}[\Pi])$ for all k.*

Proof. By definition, the beliefs of level-0 speculators are the constant mapping $q \mapsto \hat{\Pi}_0(q) = \mathbb{E}(\Pi)$. Suppose now π equals $\mathbb{E}[\Pi]$. The market clearing condition is

$$0 = \mu^F \cdot Y^F(q, \mathbb{E}[\Pi]) + \mu^S \cdot Y_0^S(q),$$

while the market clearing condition under fully-reveling rational expectations is

$$0 = \mu^F \cdot Y^F(q, \mathbb{E}[\Pi]) + \mu^S \cdot Y^S(q, \mathbb{E}[\Pi]).$$

Since the latter has a unique solution $q = \bar{Q}(\mathbb{E}[\Pi])$, it follows that $Q_0(\mathbb{E}[\Pi]) = \bar{Q}(\mathbb{E}[\Pi])$.

Now, suppose that $Q_{k-1}(\mathbb{E}[\Pi]) = \bar{Q}(\mathbb{E}[\Pi])$ for some natural number k. Again, when $\pi = \mathbb{E}[\Pi]$ the market clearing condition requires that

$$0 = \mu^F \cdot Y^F(q, \mathbb{E}[\Pi]) + \mu^S \cdot Y_k^S(q) = \mu^F \cdot Y^F(q, \mathbb{E}[\Pi]) + \mu^S \cdot Y^S(q; Q_{k-1}^{-1}(q)).$$

Under the assumption that $Q_{k-1}^{-1}(\bar{Q}(\mathbb{E}[\Pi])) = \mathbb{E}[\Pi]$, the only solution has $q = \bar{Q}(\mathbb{E}[\Pi])$, so $Q_k(\mathbb{E}[\Pi]) = \bar{Q}(\mathbb{E}[\Pi])$.

The lemma, hence, follows by mathematical induction. □

Corollary 2. *Suppose that there is a unique, fully-revealing REE and let $\bar{q}$ be the price that clears the markets when the realized π happens to be $\mathbb{E}[\Pi]$ under rational expectations. For all levels of reasoning, when the speculators observe $\bar{q}$, they believe that the realized π is indeed $\mathbb{E}[\Pi]$. That is, $\hat{\Pi}_k(\bar{q}) = \mathbb{E}[\Pi]$ for $\bar{q} = \bar{Q}(\mathbb{E}[\Pi])$ and all k.*

Theorem 1. *Suppose that there is a unique, fully-revealing REE. If*

$$\frac{\mu^F \frac{\partial Y^F}{\partial q}(q, \pi) + \mu^S \frac{\partial Y^S}{\partial q}(q, \{\pi\})}{\mu^F \frac{\partial Y^F}{\partial \pi}(q, \pi)}$$

is bounded and

$$\sup_{\pi, q} \left\{ \frac{\mu^S}{\mu^F} \cdot \frac{\partial Y^S}{\partial \pi}(q, \pi) \cdot \left[\frac{\partial Y^F}{\partial \pi}(q, \pi) \right]^{-1} \right\} < 1,$$

then the sequence of level-k price functions $\langle Q_k \rangle_{k=1}^{\infty}$ converges uniformly to the REE.

Proof. It suffices to show that the sequence $\langle \hat{\Pi}_k \rangle_{k=1}^{\infty}$ converges uniformly to $\bar{\Pi} = \bar{Q}^{-1}$, for which, given Corollary 2, it suffices that the sequence $\langle \hat{\Pi}_k' \rangle_{k=1}^{\infty}$ converges uniformly to $\bar{\Pi}'$.[3]

Define the functions

$$A_k(q) = -\frac{\mu^F \frac{\partial Y^F}{\partial q}(q, \hat{\Pi}_k(q)) + \mu^S \frac{\partial Y^S}{\partial q}(q, \hat{\Pi}_{k-1}(q))}{\mu^F \frac{\partial Y^F}{\partial \pi}(q, \hat{\Pi}_k(q))}$$

and

$$B_k(q) = \frac{\mu^S}{\mu^F} \frac{\frac{\partial Y^S}{\partial \pi}(q, \hat{\Pi}_{k-1}(q))}{\frac{\partial Y^F}{\partial \pi}(q, \hat{\Pi}_k(q))}$$

both of which take only positive values. Differentiating Eq. (18) implicitly, we have that

$$\hat{\Pi}_k' = A_k - B_k \hat{\Pi}_{k-1}', \tag{19}$$

with all the functions evaluated at q. By recursive substitution, that is

$$\hat{\Pi}_k' = \sum_{j=1}^{k} \left[(-1)^{k-j} A_j \prod_{\ell=j+1}^{k} B_\ell \right] + (-1)^k \prod_{j=1}^{k} B_j \hat{\Pi}_0'.$$

Since each A_k is bounded and each B_k is bounded above strictly below 1, the first of these two summands converges uniformly.[4], By the assumption on each B_k, the second term converges uniformly to the constant function at 0. □

8 Price Volatility

Equation (13) implies that even in the most well behaved setting, the level-k learning process is not monotonic. This feature has implications on the volatility of market equilibrium prices. In this section, we work on the simple case with logarithmic utility and no background risk to derive the closed form solution for price volatility for any level.

Equation (12) characterizes the closed form solution of the price function $Q_k(\pi)$. We have

$$Q_k(\pi) = \frac{2 \cdot \pi \cdot \Omega^{k+1} + (-1)^k \cdot \omega^{k+1}}{2 \cdot (1 - \pi) \cdot \Omega^{k+1} + (-1)^k \cdot \omega^{k+1}}.$$

For simplicity, denote $a_k = 2\Omega^{k+1}$ and $b_k = (-1)^k \omega^{k+1}$ and let the support of Π to be $\Delta = [\underline{\pi}, \bar{\pi}] \subsetneq [\omega^2/2\Omega^2, 1 - \omega^2/2\Omega^2]$.

[3] See Theorem 7.17 in Rudin (1976).

[4] See Theorem 7.10 in Rudin (1976).

Define y_0^k and y_1^k as:

$$y_0^k = a_k \cdot (1-\underline{\pi}) + b_k \quad \text{and} \quad y_1^k = a_k \cdot (1-\bar{\pi}) + b_k.$$

The expected value for $Q_k(\pi)$ is

$$\mathbb{E}\left(Q_k\right) = \int_\Delta \frac{a_k x + b_k}{a_k(1-x)+b_k} \cdot \frac{1}{\bar{\pi}-\underline{\pi}}\, dx$$

which, after some algebra, equals

$$\mathbb{E}\left(Q_k\right) = \frac{1}{\bar{\pi}-\underline{\pi}} \cdot \frac{a_k + 2b_k}{a_k} \cdot \left(\ln y_0^k - \ln y_1^k\right) - 1$$

The second moment of the price equals

$$\mathbb{E}(Q_k^2) = \int_\Delta \left(\frac{a_k x + b_k}{a_k(1-x)+b_k}\right)^2 \cdot \frac{1}{\bar{\pi}-\underline{\pi}}\, dx.$$

Again, after some algebra we have

$$\mathbb{E}(Q_k^2) = \frac{1}{\bar{\pi}-\underline{\pi}} \cdot \left[\frac{(a_k+2b_k)^2}{a_k}\left(\frac{1}{y_1^k} - \frac{1}{y_0^k}\right) + \frac{2a_k+4b_k}{a_k}\left(\ln y_1^k - \ln y_0^k\right)\right] + 1$$

Price variance $\mathrm{Var}(Q_k) = \mathbb{E}(Q_k^2) - \mathbb{E}(Q_k)^2$ is thus a function of Ω and ω and of the volatility of π, which also depends on ω and Ω. Instead of looking at the actual price variance, we define the relative volatility as $V_k = \mathrm{Var}(Q_k)/\mathrm{Var}(\bar{Q})$, where $\bar{Q}(\pi)$ is the rational expectation mapping. Equation (8) implies that the price volatility at REE equals

$$\mathrm{Var}(\bar{Q}) = \mathbb{E}\left[\left(\frac{\pi}{1-\pi}\right)^2\right] - \left[\mathbb{E}\left(\frac{\pi}{1-\pi}\right)\right]^2$$

with

$$\mathbb{E}\left[\left(\frac{\pi}{1-\pi}\right)^2\right] = 1 + \frac{2}{\bar{\pi}-\underline{\pi}} \cdot \log\left(\frac{1-\bar{\pi}}{1-\underline{\pi}}\right) + \frac{1}{(1-\bar{\pi})\cdot(1-\underline{\pi})}$$

and

$$\mathbb{E}\left(\frac{\pi}{1-\pi}\right) = \frac{\log(1-\underline{\pi}) - \log(1-\bar{\pi})}{\bar{\pi}-\underline{\pi}} - 1$$

Figure 4 plots the relative price volatility, which evolves in an oscillating manner. In addition, the relative volatility peaks when all the speculators are at level-1.[5] The intuition is: when the speculators believe that asset prices are insensitive to the true probabilities, the resulting price volatility will be small due to only minor change in the speculators' asset demand. However, the level-k mapping evolves in a way such that speculators over-correct their previous beliefs. For instance, the level-0 speculators believe that asset prices transmit no information on the true probabilities, so prices will be

[5] In this simulation, we choose $\Delta = [0.3, 0.7]$.

quite sensitive to true probabilities in the level-1 mapping since changes in demand for assets are mostly driven by the fundamentalists. As a result, price volatility spikes because level-1 agents are speculating intensely on the information transmitted through prices. This intuition also works for higher levels.

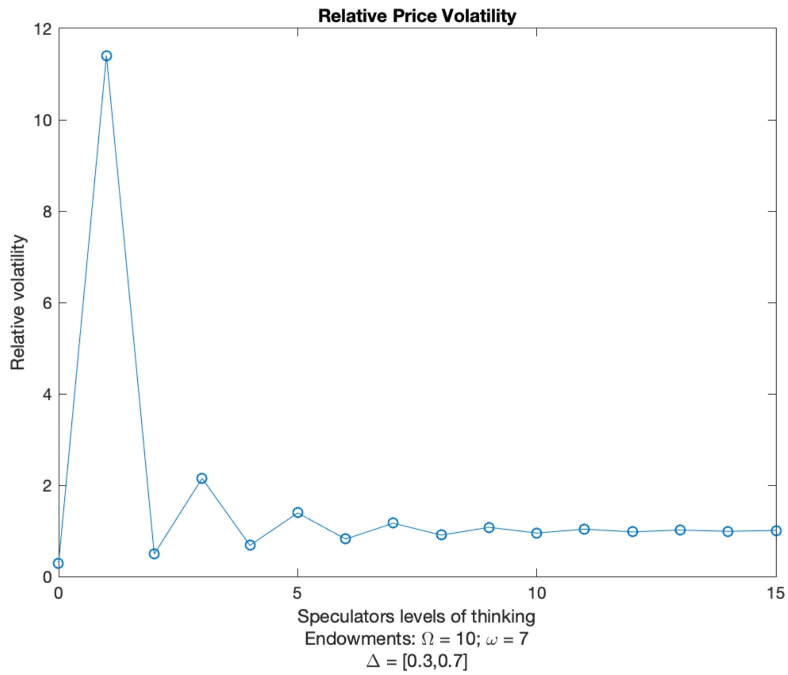

Figure 4: Relative price volatility for level-k mapping

9 Conclusion

Information revelation through asset prices remains an important question in the literature, and focus is on understanding this feature in the case of rational expectations equilibrium. In particular, how the agents may learn the (fully revealing, generically) rational expectations mapping remains unanswered in the literature. Our paper provides an answer to this question by incorporating level-k thinking model to general equilibrium. We show that convergences to REE requires that the informed traders' aggregate asset demand be more responsive to asset prices than the one of the uninformed speculators.

In addition, our results shed new light on the problem of information revelation with bounded rational agents. We show that the level-k mapping, which links asset prices with market fundamentals, evolves in an oscillating manner. When the unsophisticated speculators' asset demand is insensitive to prices, the market clearing prices become informative since market variations are mostly driven by the fundamentalists. Thus, the asset demand for sophisticated traders will be sensitive to prices, which leads to excessive speculation and weakens the informativeness of asset prices. Our mechanism generates oscillating behavior of price volatility as well.

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