On the sophistication of financial investors and the information revealed by prices

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Abstract

Does increasing the traders' understanding of the functioning of financial markets lead to rational expectations equilibrium? We provide an answer by studying a simple exchange economy with complete markets and asymmetric information. Each trader is either a fundamentalist, who knows the probability distributions of random shocks, or a speculator trying to infer those probability distributions from asset prices. Starting with the naïve beliefs that asset prices transmit no information, the speculators learn the mapping from asset prices to probabilities through adaptive observation. Our results are that: (1) convergence to rational expectations requires that speculators have less market impact than fundamentalists; (2) convergence, when it takes place, occurs in an oscillating manner; and (3) asset prices can be more volatile than at rational expectations equilibrium when speculators display low sophistication. We characterize the necessary conditions on convergence to rational expectations equilibrium for some specific utility functions (CRRA and CARA) and discuss the evolution of wealth, through simulations.

Keywords: rational expectations, level-k reasoning, information revealed by prices, bounded reasoning, Radner equilibrium.

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1 Introduction

Rational expectations equilibrium (REE, henceforth) has been an essential idea in economic theory, widely used in both micro and macroeconomics. The seminal work by Muth (1961) introduces rational expectations to study how agents predict price movements, while Radner (1979) models REE as self-fulling beliefs, where agents maximize their utility based on their beliefs and the market-clearing outcome confirms these beliefs. In particular, Radner models REE as a mapping from the set of states of the world to the set of commodity prices, and proves the generic existence and invertibility of a REE; prices are hence fully revealing, and all agents can figure out all the private information in the economy once they observe them.

Two limitations of Radner (1979), however, are the implicit assumption that the rational mapping from states to prices (and its inverse) are common knowledge among all agents and that the model is silent on how the agents would learn this mapping. To address these limitations, we investigate general conditions on which the mappings from states to prices converge to the REE. Our paper incorporates adaptive learning to a general equilibrium setting and defines an iterative process where the agents' mappings evolve as they become "more sophisticated."

The mathematics of our model resemble game theoretical models of level-k reasoning. In those models, level-k reasoning is an alternative to Nash equilibrium that describes how strategic sophistication determines players' strategies.[1] Here, in spite of the resemblance we do not interpret the evolution of beliefs as an issue of reasoning. Instead, we imagine the agents as learning through the observation of reality: after trading, the less informed agents discover the information they lacked and use this observation to try to figure out the structure of the mapping that determines prices as a function of the available information. If these agents were fully rational, they would realize that just by learning about such a mapping, the relation between the two variables may change; indeed, REE is the formalization of the idea that the equilibrium mapping is one that is consistent with itself, in the sense that it is the mapping that results when all agents are in effect using it. Our agents, on the other hand, follow an adaptive learning process: once they learn a mapping, they start using it and it takes them some time to observe the resulting relation between prices and information, namely the next mapping they will use.

Other researchers have applied the level-k thinking model to macroeconomics. For instance, Farhi and Werning (2019) explain the forward guidance puzzle with level-k thinking agents, whereas Angeletos and Lian (2017) argue that level-k thinking explains the slow adjustments of general equilibrium effects. These studies, however, do not determine conditions that guarantee convergence, and Farhi and Wern-

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[1] Importantly, this approach is supported by ample experimental evidence—see Nagel (1995), Stahl and Wilson (1994) and Stahl and Wilson (1995). Crawford et al. (2013) provides a thorough review on level-k thinking models and several supporting experimental evidence. Other experimental evidence on level-k thinking includes Arad and Rubinstein (2012), Costa-Gomes and Crawford (2006), Crawford and Iriberri (2007a), Crawford and Iriberri (2007b) and Bosch-Domenech et al. (2002).

[ing](2019) in fact suggest that convergence to rational expectations occurs in a monotonic way. Our paper investigates the convergence of the "level-k" mapping in a general equilibrium setting, and our results are that convergence to REE (1) is not guaranteed, and (2) if it occurs, occurs in a non-monotonic way. In addition, we study the effects of the learning process on the volatility of asset prices and the real cost that the uninformed agents bear while they learn the equilibrium.

2   A SIMPLE RADNER ECONOMY

While our argument can be made in a more general setting, for simplicity we consider a minimal, two-period exchange economy with uncertainty, where the state space for the future period is $\Sigma = \{1, 2\}$. There is only one commodity in the economy, and consumption takes place only in the second period.

There are two types of agent in the market. *Fundamentalists* know that the probability that state of the world $\sigma = 1$ will realize in the future is $\pi \in \Delta$. *Speculators* don't know this, and initially only understand that π is a realization of random variable Π, whose support is Δ. We will use the super-index $a \in \{F, S\}$ to denote the agents' types.

Besides their information, the two types may differ in their future wealth. In state σ, agents of type a will be endowed with a wealth ω_σ^a. We assume that there is a continuum of agents of each type, with respective masses μ^F and μ^S.

In the present, the agents trade the elementary securities corresponding to the two states of nature. They have expected-utility preferences with type-dependent Bernoulli index $\upsilon^a : \mathbb{R} \to \mathbb{R}$. Denoting the holdings by agents of type a of the security that pays in state σ as y_σ^a, the ex-ante utility of a fundamentalist is

$$\pi \cdot \upsilon^F(\omega_1^F + y_1^F) + (1 - \pi) \cdot \upsilon^F(\omega_2^F + y_2^F); \tag{1}$$

for a speculator, if she receives or discerns information $\mathcal{I} \subseteq \Delta$, it is

$$\mathrm{E}\left[\Pi \cdot \upsilon^S(\omega_1^S + y_1^S) + (1 - \Pi) \cdot \upsilon^S(\omega_2^S + y_2^S) \mid \mathcal{I}\right]. \tag{2}$$

What information the speculators use will depend, of course, on their understanding of the market.

We normalize the price of the security for $\sigma = 2$ to unity, and denote by q the price of the security for $\sigma = 1$. When an agent of type a chooses her portfolio, the only constraint she faces is that

$$q \cdot y_1^a + y_2^a = 0. \tag{3}$$

Note that one can use Eq. (3) to solve for the holdings of the second security and then rewrite Eqs. (1) and (2) in terms of the first security only. Using this simplification, we can write the optimal demands for the first security as $Y^F(q; \pi)$ for the fundamentalists, and as $Y^S(q; \mathcal{I})$ for the speculators.

Market clearing requires that the aggregate of the two types' demands vanish:

$$\mu^F \cdot Y^F(q; \pi) + \mu^S \cdot Y^S(q; \mathfrak{I}) = 0. \tag{4}$$

3 RATIONAL EXPECTATIONS

A *rational expectations equilibrium* is a function $\bar{Q} : \Delta \to \mathbb{R}$ such that, for all π, Eq. (4) is satisfied for $q = \bar{Q}(\pi)$ and

$$\mathfrak{I} = \bar{Q}^{-1}(q); \tag{5}$$

the equilibrium is *fully revealing* if it is injective, namely if

$$\bar{Q}^{-1}(\bar{Q}(\pi)) = \{\pi\} \tag{6}$$

for all π.

Eq. (5) requires that speculators discern all the information transmitted by prices *correctly at equilibrium*. Eq. (6) further requires that prices transmit *all* of the information available to the fundamentalists. It has been known, since Radner (1979), that mild conditions on preferences guarantee the existence of fully-revealing REE, generically on the agents' wealth levels.

4 ADAPTIVE LEARNING BY THE SPECULATORS

The concept of REE assumes implicitly that the speculators know the equilibrium itself and use it to infer information they lack. It is, in that loose sense, analogous to the concept of Nash equilibrium in game theory. There, an alternative approach is provided by the model of level-k reasoning: one starts by stipulating what the most naïve behavior of a player is, and proceeds inductively to define higher levels of sophistication as the reasoning of a player who thinks that everybody else in the game is one level below. We adapt this mathematical apparatus to the current setting by assuming that the speculators need to learn the mapping from fundamental to prices through observation, starting from the behavior of a trader who understand this mapping "the least."[2]

4.1 *Clueless speculators*

The least sophisticated a speculator can be is to fail to realize that the price depends on the information received by the fundamentalists. We call this level of naïveté *level-0 understanding*, and define the

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[2] We will continue to use the language of the level-k model, but insist that we maintain the Walrasian interpretation that the traders in a market like the one we are modeling need not engage in higher-order reasoning about the beliefs or rationality of others.

corresponding demand of the speculator as $Y_0^S(q) = Y^S(q; \Delta)$. This clueless trader is one who uses unconditional expectations on her portfolio problem (Eq. (2) with $\mathcal{I} = \Delta$, the full support of Π) regardless of q. By linearity, that is

$$Y_0^S(q) = \arg\max_y \left\{ E[\Pi] \cdot \upsilon(\omega_1^S + y) + (1 - E[\Pi]) \cdot \upsilon(\omega_2^S - q \cdot y) \right\}.$$

Assuming that it exists, we define the market-clearing pricing function $Q_0 : \Delta \to \mathbb{R}$ by the solution of Eq. (4) with level-0 understanding by the speculators; explicitly

$$\mu^F \cdot Y^F(Q_0(\pi); \pi) + \mu^S \cdot Y_0^S(Q_0(\pi)) = 0,$$

for all π.

4.2 Learning

For any natural number k, suppose that a speculator understands the dependence of prices on the information of the fundamentalists through price function $Q_{k-1} : \Delta \to \mathbb{R}$. We say that she has *level-k understanding* if, at prices q, she uses information $\mathcal{I} = Q_{k-1}^{-1}(q)$ in her choice of an optimal portfolio. Her optimal demand for the first security can thus be denoted as

$$Y_k^S(q) = Y^S(q; Q_{k-1}^{-1}(q));$$

it gives raise to a new pricing function, $Q_k : \Delta \to \mathbb{R}$, defined by

$$\mu^F \cdot Y^F(Q_k(\pi); \pi) + \mu^S \cdot Y_k^S(Q_k(\pi)) = 0,$$

assuming that such function exists.

Importantly, the speculators do *not* realize that their usage of function Q_{k-1} changes the equilibrium prices at each value of π—namely, that it induces the new mapping Q_k.

4.3 Rational expectations again

Let $\mathcal{Q}$ be the space of functions $Q : \Delta \to \mathbb{R}$. Note that, starting from Q_0, the definition of level-k understanding recursively constructs a sequence in $\mathcal{Q}$. Let us denote by R the mapping that defined the recursion $Q_{k-1} \mapsto R(Q_{k-1}) = Q_k$. By construction, any REE is a fixed point of R. The first question that will occupy us is whether there exist conditions that ensure that sequence $\langle Q_k \rangle_{k \in \mathbb{N}}$ converges to $\bar{Q}$.

5 AN ILLUSTRATIVE EXAMPLE: BETTING WITH LOG PREFERENCES

Suppose for the moment that both types of trader have the same Bernoulli index, $\upsilon(x) = \ln x$, that in both states $\omega_\sigma^F = \Omega > \omega = \omega_\sigma^S$, and that $\mu^F = \mu^S = 1$. Also, let random variable Π have expectation $E(\Pi) = 1/2$ and support $\Delta = [\underline{\pi}, \bar{\pi}] \subset (\omega^2/2\Omega^2, 1 - \omega^2/2\Omega^2)$.

5.1 Rational expectations

Let us conjecture that the rational expectations mapping $\bar{Q}$ is bijective, so that Eq. (6) holds true. The individual demands then are

$$Y^F(q; \pi) = \frac{\pi}{q}(q \cdot \omega_1^F + \omega_2^F) - \omega_1^F = \left(\pi \cdot \frac{q+1}{q} - 1\right) \cdot \Omega, \tag{7}$$

and[3]

$$Y^S(q; \pi) = \frac{\pi}{q}(q \cdot \omega_1^S + \omega_2^S) - \omega_1^S = \left(\pi \cdot \frac{q+1}{q} - 1\right) \cdot \omega. \tag{8}$$

Substituting into Eq. (4), and using that $\mu^F = \mu^S = 1$, one can solve for

$$\bar{Q}(\pi) = \frac{\pi}{1-\pi} \cdot \frac{\omega_2^F + \omega_2^S}{\omega_1^F + \omega_1^S} = \frac{\pi}{1-\pi}. \tag{9}$$

Since $\bar{Q}$ is bijective, we confirm that the REE is fully revealing. Its inverse, which we denote as

$$\bar{\Pi} : \left[\frac{\underline{\pi}}{1-\underline{\pi}}, \frac{\bar{\pi}}{1-\bar{\pi}}\right] \to \Delta,$$

is

$$\bar{\Pi}(q) = \frac{q \cdot (\omega_1^F + \omega_1^S)}{q \cdot (\omega_1^F + \omega_1^S) + \omega_2^F + \omega_2^S} = \frac{q}{q+1}. \tag{10}$$

5.2 Learning

Assuming that mapping Q_{k-1} is bijective, we can use its inverse function to pin down the beliefs of level-k speculators upon observation of price q: $\hat{\Pi}_k(q) = Q_{k-1}^{-1}(q)$. In this case, the demand of the fundamentalists is still given by Eq. (7), while for the speculators it is

$$Y_k^S(q) = \frac{\hat{\Pi}_k(q)}{q}(q \cdot \omega_1^S + \omega_2^S) - \omega_1^S.$$

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[3] Strictly speaking, the following expression should read $Y^S(q; \{\pi\})$ on its right-hand side. We exclude the braces for simplicity.

Since $\mu^F = \mu^S = 1$, Eq. (4) allows us to define the next pricing function, Q_k, implicitly, by

$$Y^F(Q_k(\pi); \pi) = -Y_k^S(Q_k(\pi)).$$

Upon substitution, that is

$$\frac{\pi}{Q_k(\pi)}[Q_k(\pi) \cdot \omega_1^F + \omega_2^F] - \omega_1^F = \omega_1^S - \frac{\hat{\Pi}_k(Q_k(\pi))}{Q_k(\pi)}[Q_k(\pi) \cdot \omega_1^S + \omega_2^S]. \tag{11}$$

Unfortunately, solving for Q_k explicitly is not easy. Under the further simplification given by the assumption that $\omega_1^F = \omega_2^F = \Omega > \omega = \omega_1^S = \omega_2^S$, Eq. (11) becomes

$$\left[\pi \cdot \frac{Q_k(\pi) - 1}{Q_k(\pi)} - 1\right] \cdot \Omega = \left[1 - \hat{\Pi}_k(Q_k(\pi)) \cdot \frac{Q_k(\pi) + 1}{Q_k(\pi)}\right] \cdot \omega \tag{12}$$

Starting from $\hat{\Pi}_0(q) = E(\Pi) = 1/2$, and therefore from

$$Q_0(\pi) = \frac{2 \cdot \pi \cdot \Omega + \omega}{2 \cdot (1 - \pi) \cdot \Omega + \omega},$$

and using mathematical induction over k, one has that

$$Q_k(\pi) = \frac{2 \cdot \pi \cdot \Omega^{k+1} + (-1)^k \cdot \omega^{k+1}}{2 \cdot (1 - \pi) \cdot \Omega^{k+1} + (-1)^k \cdot \omega^{k+1}}, \tag{13}$$

for any $\pi \in \Delta$. Importantly, this mapping is bijective.

5.3 *Asymptotics of learning*

We can re-write Eq. (13) as

$$Q_k(\pi) = \frac{2 \cdot \pi + (-1)^k \cdot \left(\frac{\omega}{\Omega}\right)^{k+1}}{2 \cdot (1 - \pi) + (-1)^k \cdot \left(\frac{\omega}{\Omega}\right)^{k+1}},$$

and then it follows that, since $\Omega > \omega > 0$, $Q_k(\pi) \to \bar{Q}(\pi)$ for all $\pi \in \Delta$. Moreover,

$$Q_k(\pi) - \bar{Q}(\pi) = \frac{(-1)^k \cdot (1 - 2\pi) \cdot \left(\frac{\omega}{\Omega}\right)^{k+1}}{2 \cdot (1 - \pi)^2 + (-1)^k \cdot (1 - \pi) \cdot \left(\frac{\omega}{\Omega}\right)^{k+1}}, \tag{14}$$

and for $k \geqslant 2$ this difference is continuous on π over Δ, which is compact, so $Q_k \to \bar{Q}$ not just pointwise but uniformly. Interestingly, pointwise convergence is *not* monotonic: because of the term $(-1)^k$ on the right-hand side of Eq. (14), the sign of the difference $Q_k(\pi) - \bar{Q}(\pi)$ will oscillate between

successive levels of understanding. Only at $\pi = E(\Pi)$ is that sign constant, and the difference there is null.

5.4 Introducing background risk

As mentioned before, Eq. (11) is difficult to solve without the assumptions that simplified it to Eq. (12). Still, maintaining the assumption that both types have logarithmic Bernoulli indices, Eq. (11) implies that, if Q_k is bijective, its inverse must satisfy

$$\frac{\hat{\Pi}_{k+1}(q)}{q}(q \cdot \omega_1^F + \omega_2^F) - \omega_1^F = \omega_1^S - \frac{\hat{\Pi}_k(q)}{q}(q \cdot \omega_1^S + \omega_2^S). \tag{15}$$

Eq. (15) is critical for our analysis. Upon simplification, it gives us the difference equation that governs the beliefs of speculators, at given prices, as their level of understanding evolves:

$$\hat{\Pi}_k(q) = \min\left\{\max\left\{\frac{q \cdot (\omega_1^F + \omega_1^S)}{q \cdot \omega_1^F + \omega_2^F} - \frac{q \cdot \omega_1^S + \omega_2^S}{q \cdot \omega_1^F + \omega_2^F} \cdot \hat{\Pi}_{k-1}(q), \underline{\pi}\right\}, \bar{\pi}\right\}. \tag{16}$$

It follows that, so long as the nominal wealth of the fundamentalists is higher than the one of the speculators, which was true in the simpler case with no background risk, the sequence of level-k beliefs converges to the rational expectations beliefs even if such convergence is non-monotonic. That is,

Proposition 1. *Under the assumptions of this example, for any price q for which*

$$\frac{q \cdot \omega_1^S + \omega_2^S}{q \cdot \omega_1^F + \omega_2^F} < 1,$$

one has that

$$\hat{\Pi}_k(q) \to \frac{q \cdot (\omega_1^F + \omega_1^S)}{q \cdot (\omega_1^F + \omega_1^S) + \omega_2^F + \omega_2^S} = \bar{\Pi}(q),$$

The proof of this result is highly algebraic, so we defer it to the appendix. Instead, Fig. 1 shows the sequence of level-k mappings when the hypothesis of the proposition is satisfied.[4] It confirms our claim that convergence to REE occurs in a non-monotonic way.

Importantly, Eq. (16) shows that the premise of this result is also necessary: *at prices at which the two nominal wealths are equal, the sequence of level-k beliefs oscillates in a two-level cycle; and at prices at which the nominal wealth of the fundamentalists is lower, the sequence of level-k beliefs diverges non-monotonically.* Fig. 2 shows the level-k mappings when the real endowments of fundamentalists and speculators are equal: the mapping oscillates between two functions.

Figure 3 illustrates the situation when the real wealth of the speculators is greater than the one of the

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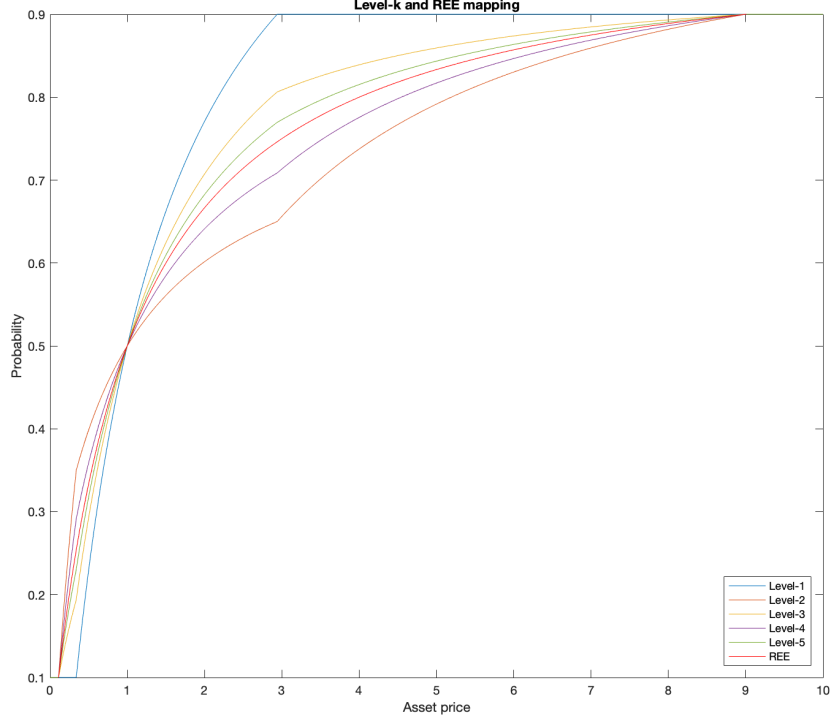[4] We pick $\underline{\pi} = 0.1$, $\bar{\pi} = 0.9$ and $E(\Pi) = 0.5$ in this simulation.

Figure 1: Convergence of the level-k mappings with logarithmic utility, $(\omega_1^F, \omega_2^F) = (8, 8)$ and $(\omega_1^S, \omega_2^S) = (5, 4)$

fundamentalists. For sufficiently large k, the mappings at odd levels behave as step functions. Speculators believe that when the price is below the cutoff value, state $\sigma = 1$ occurs with probability $\underline{\pi}$; and that when the price is above the cutoff, state $\sigma = 1$ occurs with probability $\bar{\pi}$. The mapping at even levels describes the relation between true probabilities and asset prices when all the speculators hold the belief which is described by the mapping at odd levels.

6   Conditions on General Demand Functions

Going back to the general case, assume that both types of agent have $\mathbb{C}^2$, strictly increasing and strictly concave Bernoulli indexes that yield demand functions $Y^a(q; \pi)$ that are decreasing in q and increasing in π. Suppose, moreover, that the price functions Q_k are injective, and recall that, as with Eq. (15), we can replace the recursion of price functions, $Q \mapsto R(Q)$ defined by

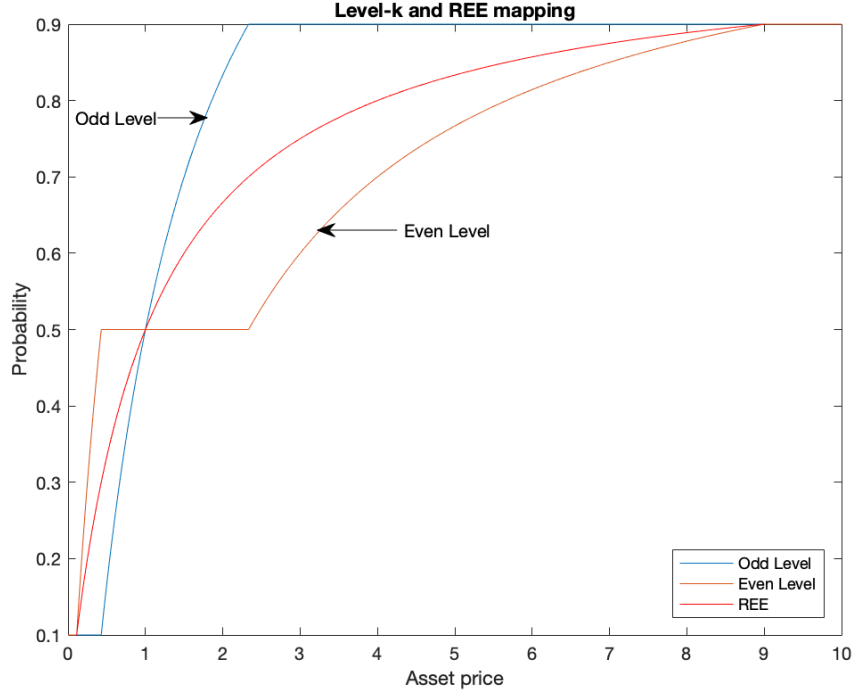$$\mu^F \cdot Y^F(R(Q)(\pi); \pi) + \mu^S \cdot Y^S(R(Q)(\pi); Q^{-1}(R(Q)(\pi))) = 0,$$

Figure 2: Oscillation of the level-k mappings with
logarithmic utility, $\omega_1^F = \omega_1^S$ and $\omega_2^F = \omega_2^S$

by the recursion of belief functions $\hat{\Pi} \mapsto T(\hat{\Pi})$ defined by

$$\mu^F \cdot Y^F(q; T(\hat{\Pi})(q)) + \mu^S \cdot Y^S(q; \hat{\Pi}(q)) = 0. \tag{17}$$

(As long as the price functions are injective, the latter simply results in the inverse mappings of the former.) More explicitly, Eq. (17) tells us that level-k learning is the implicit recursion

$$\mu^F \cdot Y^F(q; \hat{\Pi}_k(q)) + \mu^S \cdot Y^S(q; \hat{\Pi}_{k-1}(q)) = 0, \tag{18}$$

starting from the "most naïve" beliefs $q \mapsto \hat{\Pi}_0(q) = E(\Pi)$.

If we further assume that for all π there exist $\underline{q}$ and $\bar{q}$ such that

$$\mu^F \cdot Y^F(\underline{q}; \pi) + \mu^S \cdot Y^S(\underline{q}; \pi) > 0 > \mu^F \cdot Y^F(\bar{q}; \pi) + \mu^S \cdot Y^S(\bar{q}; \pi),$$

then the existence of a unique, fully-revealing REE is guaranteed by the intermediate value theorem, given the assumptions on monotonicity of the demand functions. We henceforth take this result for granted, and denote by $\bar{Q}$ the unique, fully-revealing REE.

Lemma 1. *When the realized π happens to be $E[\Pi]$, for all levels of learning the price that clears the markets is the same and equals the price that clears them under rational expectations. That is, $Q_k(E[\Pi]) = \bar{Q}(E[\Pi])$*
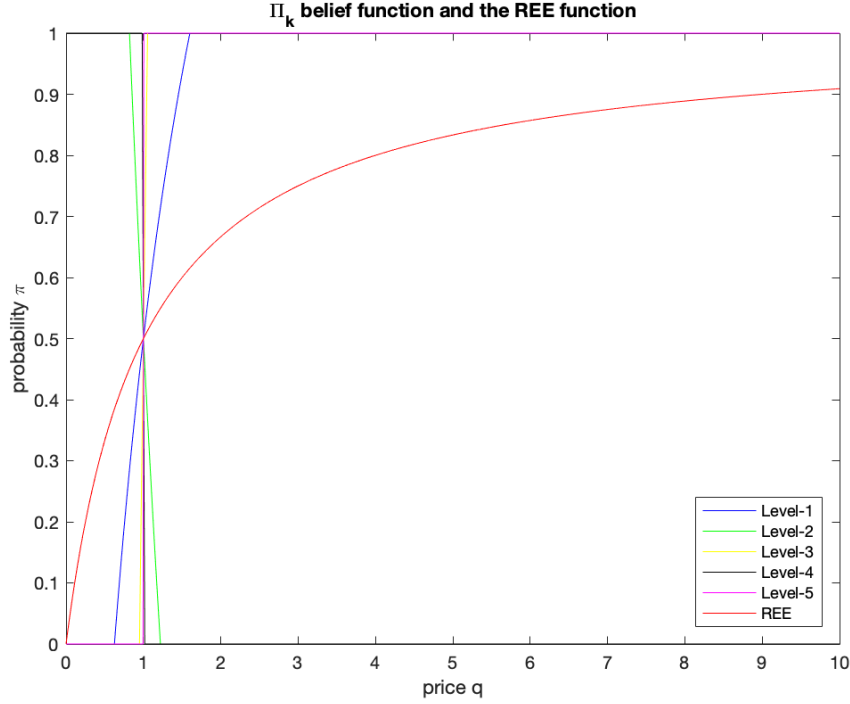
Figure 3: Divergence of the level-k mappings with
logarithmic utility, $(\omega_1^F, \omega_2^F) = (5,5)$ and $(\omega_1^S, \omega_2^S) = (8,7)$

for all k.

The proof of this lemma is deferred to the appendix. An immediate implication is the following corollary:

Corollary 1. *Let $\bar{q}$ be the price that clears the markets when the realized π happens to be $E[\Pi]$ under rational expectations. For all levels of learning, when the speculators observe $\bar{q}$, they believe that the realized π is indeed $E[\Pi]$. That is, $\hat{\Pi}_k(\bar{q}) = E[\Pi]$ for $\bar{q} = \bar{Q}(E[\Pi])$ and all k.*

We can now state and proof our main result:

Theorem 1. *If*

$$\frac{\mu^F \frac{\partial Y^F}{\partial q}(q,\pi) + \mu^S \frac{\partial Y^S}{\partial q}(q,\{\pi\})}{\mu^F \frac{\partial Y^F}{\partial \pi}(q,\pi)}$$

is bounded and

$$\sup_{\pi,q} \left\{ \frac{\mu^S}{\mu^F} \cdot \frac{\partial Y^S}{\partial \pi}(q,\pi) \cdot \left[\frac{\partial Y^F}{\partial \pi}(q,\pi) \right]^{-1} \right\} < 1,$$

then the sequence of level-k price functions $\langle Q_k \rangle_{k=1}^{\infty}$ converges uniformly to the REE.

Proof. It suffices to show that the sequence $\langle \hat{\Pi}_k \rangle_{k=1}^{\infty}$ converges uniformly to $\bar{\Pi} = \bar{Q}^{-1}$, for which, given

Corollary 1, it suffices that the sequence $\langle \hat{\Pi}_k' \rangle_{k=1}^\infty$ converges uniformly to $\bar{\Pi}'$.[5]

Define the functions

$$A_k(q) = -\frac{\mu^F \frac{\partial Y^F}{\partial q}(q, \hat{\Pi}_k(q)) + \mu^S \frac{\partial Y^S}{\partial q}(q, \hat{\Pi}_{k-1}(q))}{\mu^F \frac{\partial Y^F}{\partial \pi}(q, \hat{\Pi}_k(q))}$$

and

$$B_k(q) = \frac{\mu^S}{\mu^F} \frac{\frac{\partial Y^S}{\partial \pi}(q, \hat{\Pi}_{k-1}(q))}{\frac{\partial Y^F}{\partial \pi}(q, \hat{\Pi}_k(q))}$$

both of which take only positive values. Differentiating Eq. (18) implicitly, we have that

$$\hat{\Pi}_k'(q) = A_k(q) - B_k(q)\hat{\Pi}_{k-1}'(q). \tag{19}$$

By recursive substitution, that is

$$\hat{\Pi}_k'(q) = \sum_{j=1}^k \left[(-1)^{k-j} A_j(q) \prod_{\ell=j+1}^k B_\ell(q) \right] + (-1)^k \prod_{j=1}^k B_j(q)\hat{\Pi}_0'(q).$$

Since each A_k is bounded and each B_k is bounded above strictly below 1, the first of these two summands converges uniformly: see Theorem 7.10 in Rudin (1976). By the assumption on each B_k, the second term converges uniformly to the function constant at 0. $\qquad\square$

7 OTHER SOLVABLE EXAMPLES

We now study convergence for other prominent utility functions for which we can use analytic expressions. For simplicity, assume that $\mu^F = \mu^S = 1$.

7.1 Other CRRA functions

Let us assume first that both types have Bernoulli index $v(x) = x^{1-\gamma}/(1-\gamma)$.

Proposition 2. *The sequence of belief functions $\langle \Pi_k \rangle_{k=1}^\infty$ converges to the rational expectation equilibrium $\bar{\Pi}$ pointwise at every q for which*

$$\frac{q \cdot \omega_1^S + \omega_2^S}{q \cdot \omega_1^F + \omega_2^F} < 1.$$

At prices that violate this inequality, the sequence of beliefs diverges in an oscillating manner.

Note that the assumption that $\omega^F > \omega^S$ implies the condition of the proposition. Since, in addition, the domain of prices is compact, as so is the support Δ and all the belief functions are continuous, the following result follows immediately.

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[5] See Theorem 7.17 in Rudin (1976).

Corollary 2. *The sequence of belief functions* $\langle \Pi_k \rangle_{k=1}^{\infty}$ *converges to the rational expectation equilibrium* $\bar{\Pi}$ *uniformly if* $\omega^F > \omega^S$.

7.2 CARA functions

Assume now that both types share the exponential Bernoulli index $\upsilon(x) = -\exp\{-\rho x\}$.

Proposition 3. *The sequence of belief functions* $\langle \Pi_k \rangle_{k=1}^{\infty}$ *converges to the rational expectation equilibrium* $\bar{\Pi}$ *uniformly if, and only if,* $\mu^F > \mu^S$.

Verifying Theorem 1

Of course, we can apply Theorem 1 to both CRRA and CARA utility functions to get simple arguments for Propositions 2 and 3. For the former, $\mu^F = \mu^S$ and take the partial derivative of the CRRA demand function with respect to π yield

$$
\sup_{\pi,q} \left\{ \frac{\mu^S}{\mu^F} \cdot \frac{\partial Y^S}{\partial \pi}(q,\pi) \cdot \left[\frac{\partial Y^F}{\partial \pi}(q,\pi) \right]^{-1} \right\} = \sup_{\pi,q} \left\{ \frac{q\omega_1^S + \omega_2^S}{q\omega_1^F + \omega_2^F} \cdot \frac{f(q,\hat{\pi})}{f(q,\pi)} \right\}
$$

$$
= \sup_{q} \left\{ \frac{q\omega_1^S + \omega_2^S}{q\omega_1^F + \omega_2^F} \right\} \cdot \sup_{\pi,q} \left\{ \frac{f(q,\hat{\pi})}{f(q,\pi)} \right\}
$$

with

$$
f(q,\pi) = \left[q + q^{1/\gamma} \left(\frac{1-\pi}{\pi} \right)^{1/\gamma} \right]^{-2} \cdot \left(\frac{1-\pi}{\pi} \right)^{1-\gamma/\gamma} \cdot \frac{1}{\pi^2}
$$

being a continuous function. Proposition 2 hence follows from Theorem 1, because if

$$
\frac{q\omega_1^S + \omega_2^S}{q\omega_1^F + \omega_2^F} < 1
$$

for all q, then function $f(q,\hat{\pi})$ converges to $f(q,\pi)$ uniformly.

Similarly, the CARA demand function yields

$$
\sup_{\pi,q} \left\{ \frac{\mu^S}{\mu^F} \cdot \frac{\partial Y^S}{\partial \pi}(q,\pi) \cdot \left[\frac{\partial Y^F}{\partial \pi}(q,\pi) \right]^{-1} \right\} = \sup_{\pi,q} \left\{ \frac{\mu^S}{\mu^F} \cdot \frac{\pi}{\hat{\pi}} \right\}.
$$

And it is straightforward to see that Proposition 3 follows from Theorem 1.

8 PRICE VOLATILITY

Equation (14) implies that even in the most well behaved setting, the level-k learning process is not monotonic. This feature has implications on the volatility of market equilibrium prices. In this section, we work on the simple case with logarithmic utility and no background risk to derive the closed form solution for price volatility for any level.

Equation (13) characterizes the closed form solution of the price function $Q_k(\pi)$. We have

$$Q_k(\pi) = \frac{2 \cdot \pi \cdot \Omega^{k+1} + (-1)^k \cdot \omega^{k+1}}{2 \cdot (1 - \pi) \cdot \Omega^{k+1} + (-1)^k \cdot \omega^{k+1}}.$$

For simplicity, denote $a_k = 2\Omega^{k+1}$ and $b_k = (-1)^k \omega^{k+1}$ and let the support of Π to be $\Delta = [\underline{\pi}, \bar{\pi}] \subsetneq [\omega^2/2\Omega^2, 1 - \omega^2/2\Omega^2]$.

Define y_0^k and y_1^k as:

$$y_0^k = a_k \cdot (1 - \underline{\pi}) + b_k \quad \text{and} \quad y_1^k = a_k \cdot (1 - \bar{\pi}) + b_k.$$

The expected value for $Q_k(\pi)$ is

$$E(Q_k) = \int_\Delta \frac{a_k x + b_k}{a_k(1 - x) + b_k} \cdot \frac{1}{\bar{\pi} - \underline{\pi}} \, dx = \frac{1}{\bar{\pi} - \underline{\pi}} \cdot \frac{a_k + 2b_k}{a_k} \cdot \left(\ln y_0^k - \ln y_1^k \right) - 1$$

and its second moment is

$$E(Q_k^2) = \frac{1}{\bar{\pi} - \underline{\pi}} \cdot \left[\frac{(a_k + 2b_k)^2}{a_k} \left(\frac{1}{y_1^k} - \frac{1}{y_0^k} \right) + \frac{2a_k + 4b_k}{a_k} \left(\ln y_1^k - \ln y_0^k \right) \right] + 1.$$

Price variance $\operatorname{Var}(Q_k) = E(Q_k^2) - E(Q_k)^2$ is thus a function of Ω and ω and of the volatility of π, which also depends on ω and Ω. Instead of looking at the actual price variance, we define the relative volatility as $V_k = \operatorname{Var}(Q_k)/\operatorname{Var}(\bar{Q})$, where $\bar{Q}(\pi)$ is the rational expectation mapping.[6]

Figure 4 plots the relative price volatility, which evolves in an oscillating manner. Besides, the relative

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[6] Equation (9) implies that the price volatility at REE equals

$$\operatorname{Var}(\bar{Q}) = E\left[\left(\frac{\pi}{1 - \pi} \right)^2 \right] - \left[E\left(\frac{\pi}{1 - \pi} \right) \right]^2,$$

with

$$E\left[\left(\frac{\pi}{1 - \pi} \right)^2 \right] = 1 + \frac{2}{\bar{\pi} - \underline{\pi}} \cdot \log\left(\frac{1 - \bar{\pi}}{1 - \underline{\pi}} \right) + \frac{1}{(1 - \bar{\pi}) \cdot (1 - \underline{\pi})}$$

and

$$E\left(\frac{\pi}{1 - \pi} \right) = \frac{\log(1 - \underline{\pi}) - \log(1 - \bar{\pi})}{\bar{\pi} - \underline{\pi}} - 1.$$
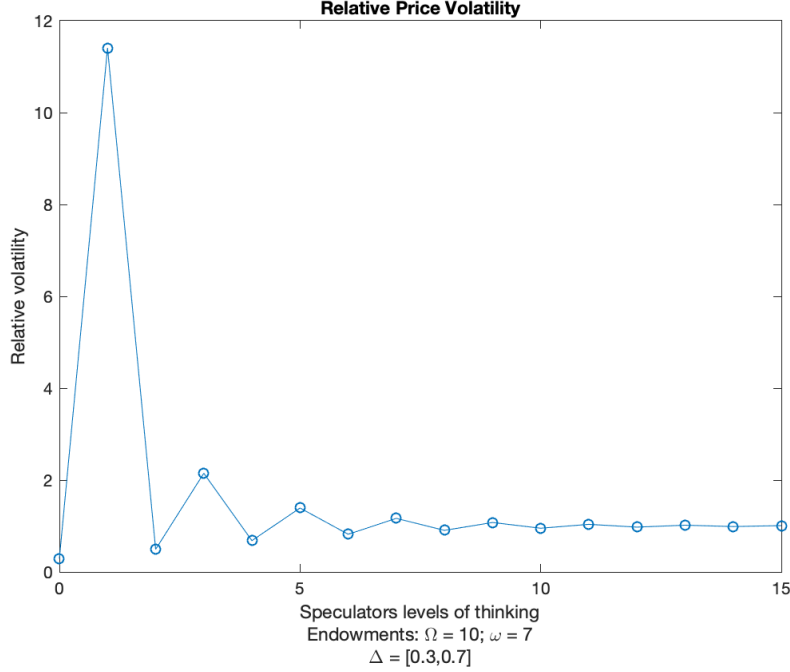
Figure 4: Relative price volatility for level-k mappings

volatility peaks when all the speculators are at level-1.[7] The intuition is: when the speculators believe that asset prices are insensitive to the true probabilities, the resulting price volatility will be small due to only minor changes in the speculators' asset demand. However, the level-k mapping evolves in a way such that speculators over-correct their previous beliefs. For instance, the level-0 speculators believe that asset prices transmit no information on the true probabilities, so prices will be quite sensitive to true probabilities in the level-1 mapping since it is the fundamentalists that mostly drive the changes in the demand for assets. As a result, price volatility spikes because level-1 agents speculate intensely on the information transmitted through prices. This intuition also works for higher levels.

9 Wealth Dynamics

Our results so far cast doubt on appropriateness of REE modelling in situations where (or for prices at which) the aggregate market impact of the speculators dominates the aggregate impact of the fundamentalists. In the context of Section 5, Proposition 1 implies that at prices at which the aggregate nominal wealth of the speculators is larger than the one of the fundamentalists, the sequence of level-k beliefs diverges.

Sandroni (2005) and Blume and Easley (2006), on the other hand, have shown that agents who hold wrong beliefs about future risks accumulate dynamic losses that in the long run lead them to bankruptcy. While our model is not genuinely dynamic, we can give a first assessment of the interaction between

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[7] In this simulation, we choose $\Delta = [0.3, 0.7]$.

level-k learning and market losses by introducing to the setting of Section 5 the wealth dynamics that would result from the losses that, on average, a level-k speculator sustains as a consequence of her use of her beliefs, as opposed to the rational expectations (correct) beliefs.

That is, suppose that both types again have Bernoulli index $v(x) = \ln x$, and that and Π follows the uniform distribution over $[\underline{\pi}, \bar{\pi}]$, with $\underline{\pi} + \bar{\pi} = 1$, so that $E(\Pi) = 0.5$. Adapting our previous notation, let

$$Y^S(q; \hat{\pi}, \omega) = \arg\max_y \{\hat{\pi} \cdot v(\omega + y) + (1 - \hat{\pi}) \cdot v(\omega - q \cdot y)\},$$

which is the optimal demand, at price q, of a speculator with degenerate beliefs $\hat{\pi}$ and wealth ω in both states of the world. If the actual probability of state $\sigma = 1$ is π, the difference

$$M^S(q; \pi, \hat{\pi}, \omega) = Y^S(q; \pi, \omega) - Y^S(q; \hat{\pi}, \omega) \tag{20}$$

is the quantity that the speculator under-demands of the elementary security for the first state. The expected cost of this mistake is $\pi \cdot M^S(q; \pi, \hat{\pi}, \omega)$, measured in units of wealth of state $\sigma = 1$.

In order to introduce the effect of these losses on the sequence of level-k mappings, we must enrich the setting of Section 5 with a dynamic process for the endowments of the two types of trader. We do this, recursively, as follows. Let ω_k be the endowment, in both states, of level-k speculators. At probability π, the mistake of the speculators at equilibrium is the difference

$$M^S\left(Q_k(\pi); \pi, Q_{k-1}^{-1}(Q_k(\pi)), \omega_k\right),$$

where Q_k denotes the market-clearing price function when the speculators use level-k understanding.[8] In expectation, their wrong beliefs cost the speculators

$$L_k^S = E\left[\Pi \cdot M^S\left(Q_k(\Pi); \Pi, Q_{k-1}^{-1}(Q_k(\Pi)), \omega_k\right)\right]$$

units of wealth in each state of the world.[9] Thus, starting from endowments ω_0 for the speculators and Ω_0 for the fundamentalists (in both states), we assume that their endowments evolve according to the process

$$\omega_{k+1} = \omega_k - L_k^S \quad \text{and} \quad \Omega_{k+1} = \Omega_k + L_k^S,$$

for speculators and fundamentalists, respectively.

The computations of Section 5 give us that, for the case of logarithmic utilities,

$$Y^S(q; \hat{\pi}, \omega) = \left(\pi \cdot \frac{q+1}{q} - 1\right) \cdot \omega,$$

---

[8] Strictly speaking, this mapping also depends on the distribution of wealth. While our computations below take account of this dependence, here we avoid it for notational simplicity.

[9] By symmetry, the expected losses in state $\sigma = 2$ equal L_k^S too.

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from Eq. (8), whereas

$$Q_k(\pi) = \frac{2 \cdot \pi + (-1)^k \cdot \left(\frac{\omega_k}{\Omega_k}\right)^{k+1}}{2 \cdot (1 - \pi) + (-1)^k \cdot \left(\frac{\omega_k}{\Omega_k}\right)^{k+1}},$$

from Eq. (13), *mutatis mutandis*. Level-$k+1$ agent infers π from price q by inverting this mapping

$$\hat{\Pi}_{k+1}(q) = \min \left\{ \max \left\{ \frac{2q + (q-1) \cdot (-1)^k \cdot \left(\frac{\omega_k}{\Omega_k}\right)^{k+1}}{2 \cdot (1 + q)}, \underline{\pi} \right\}, \bar{\pi} \right\}.$$
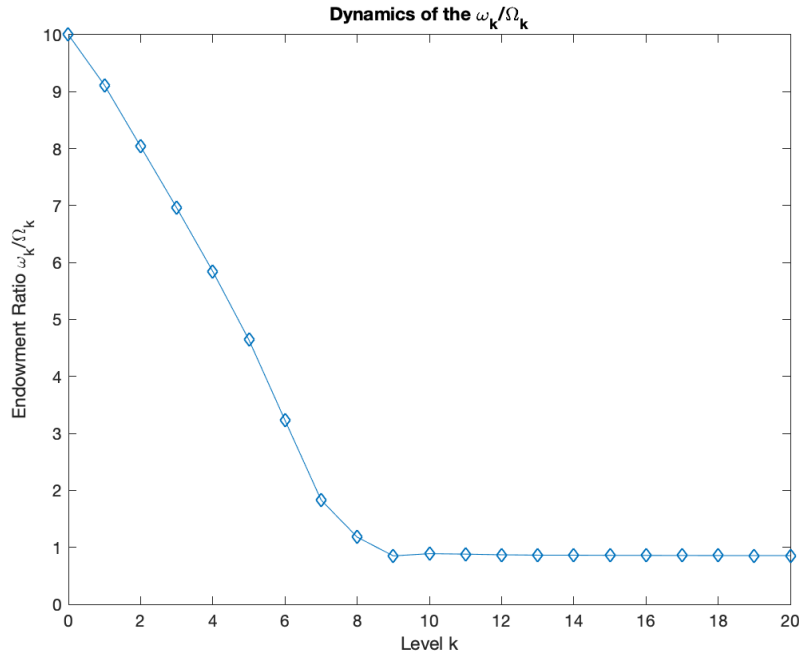

Figure 5: Evolution of the endowment ratio, with $\frac{\omega_0}{\Omega_0} = 10$

We numerically simulate this process, with $\omega_0/\Omega_0 > 1$ as the initial condition. Fig. 5 shows that as k increases, the speculators accumulate losses and the endowment ratio falls below 1, which places it in the convergence domain. Interestingly, the speculators do not lose all their wealth and there is a fixed point for this endowment process. As a consequence, the sequence of level-k mappings converges to the REE, as in Fig. 6.

To be sure, Eq. (20) is not the only way in which we can define the speculators mistake. When using Eq. (20) we compute this mistake at the equilibrium prices $Q_k(\pi)$. Alternatively, we could have defined it as the difference

$$Y^S(\bar{Q}(\pi); \pi, \omega) - Y^S(Q_k(\pi); \hat{\Pi}_k(Q_k(\pi)), \omega),$$

where the "correct" portfolio is computed at REE prices. This alternative would not change the results
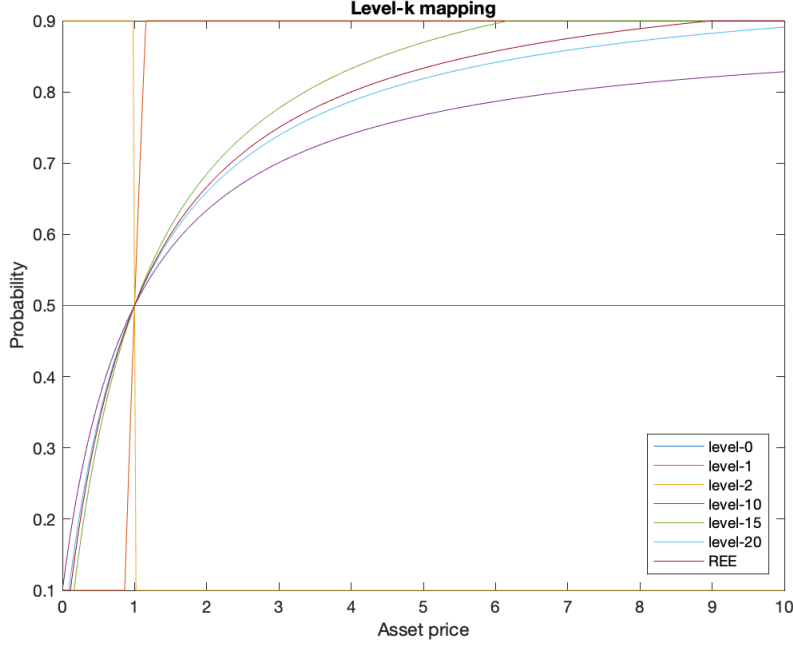
Figure 6: Evolution of the level-k mapping, with endowment ratio given in Fig. 5.

qualitatively, and only would the convergence process faster, as Fig. 7 shows.

The intuition for these results is the following. Starting with wrong beliefs, speculators gradually lose their wealth. However, as their wealth level falls, their impact on the market becomes less significant. As a result, the equilibrium prices reflect more the actual fundamentals, which makes the level-k mapping more informative. Eventually, the speculators stop losing wealth and the belief function converges to the REE uniformly.

10   Concluding remarks

Information revelation through asset prices remains an essential question in the literature, and the focus has been on understanding this feature in the case of rational expectations equilibrium. In particular, how the agents may learn the (fully revealing, generically) rational expectations mapping remains mostly unanswered in the literature.

Addressing the same difficulty of REE as here, McAllister (1990) incorporates rational expectations equilibrium in a decision-making framework and constructs the rational expectation mapping at the individual level. He considers the space of uncertainty for each agent to be the product of the state space and the set of all possible asset positions of other traders. An REE consists of an admissible prior, a price vector, and asset positions such that traders are optimizing while market clears for all states.

Later, Dutta and Morris (1997) generalize McAllister (1990) by relaxing the assumptions of common
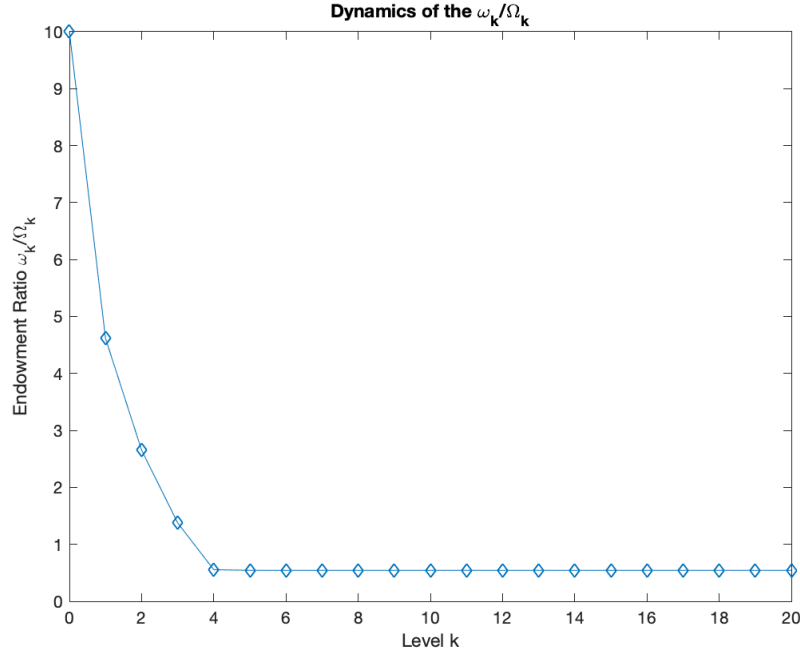
Figure 7: Evolution of the endowment ratio, with losses computed at REE prices

knowledge of consensus beliefs and degeneracy of expectations. They introduce the concepts of *belief equilibrium*, where agents might have disagreements on their prior beliefs, and of *common belief equilibrium*, where agents hold the same beliefs. There, an REE is a restriction to common belief equilibrium where the agents only consider the exogenous states of nature, and the mapping from states to prices is consensus between all agents. In this line of research, Ben-Porath and Heifetz (2011) is, to the best of our knowledge, the most general result on the epistemic foundation of rational expectations equilibrium literature. They show that common knowledge of rationality and market clearing (CKRMC) is sufficient to yield REE.

However, Dubey et al. (1987) criticizes the REE approach in general equilibrium models with asymmetric information since it fails to explain how information gets encoded into asset prices. Unlike the literature above, this paper regards the question of how agents learn the rational expectations mapping. To be sure, there have been many discussions on convergence to rational expectations in both the macro and the micro literature. Shiller (1978), for instance, studies convergence to the rational expectation forecast in Muth (1961), while DeCanio (1979) studies convergence to rational expectations in a linear forecast model with general autocorrelation structure. Closer to our paper, Bray (1982) studies a setting where agents are learning the relation between asset returns and prices using OLS estimation. Bray's results suggest the learning process could converge to rational expectations even if agents are using misspecified models. Blume and Easley (1984) study a dynamic market process in which agents learn a payoff relevant parameter by conditioning on past endogenously generated data. They define REE as the limit of the learning process once all the agents' beliefs converge to the true parameter almost

surely.

Instead of studying the convergence to a single parameter, as this past literature did, our paper provides an answer to this question by incorporating the level-k thinking model to general equilibrium. We show that convergence to REE requires that the informed traders' aggregate asset demand be more responsive to asset prices than the one of the uninformed speculators.

Also, our results shed new light on the problem of information revelation with boundedly rational agents. We show that the level-k mapping, which links asset prices with market fundamentals, evolves in an oscillating manner. When the unsophisticated speculators' asset demand is insensitive to prices, the market-clearing prices become informative since the fundamentalists mostly drive market variations. Thus, the asset demand for sophisticated traders will be sensitive to prices, which leads to excessive speculation and weakens the informativeness of asset prices. Our mechanism generates oscillating behavior of price volatility as well.

APPENDIX: PROOFS

Proof of Proposition 1: To simplify notation, let

$$A(q) = \frac{q \cdot (\omega_1^F + \omega_1^S)}{q \cdot \omega_1^F + \omega_2^F} \quad \text{and} \quad B(q) = \frac{q \cdot \omega_1^S + \omega_2^S}{q \cdot \omega_1^F + \omega_2^F}.$$

Also, define the following five cutoff prices, implicitly:

$$\begin{aligned}
A(q_1) - B(q_1)\underline{\pi} &= \underline{\pi} \\
A(q_2) - B(q_2)\,E(\Pi) &= \underline{\pi} \\
A(q_3) - B(q_3)\,E(\Pi) &= E(\Pi) \\
A(q_4) - B(q_4)\,E(\Pi) &= \bar{\pi} \\
A(q_5) - B(q_5)\bar{\pi} &= \bar{\pi}
\end{aligned}$$

and let $A(q_i)$ and $B(q_i)$ be denoted as A_i and B_i. Note that $A(q) - B(q) \cdot \pi$ is a strictly increasing function of q, for any $\pi \in [\underline{\pi}, \bar{\pi}]$, when $B(q) < 1$. Therefore, we can rank these prices as $q_1 < q_2 < q_3 < q_4 < q_5$, since we have $\underline{\pi} < E(\pi) < \bar{\pi}$.

The level-1 mapping is

$$\hat{\Pi}_1(q) = \min\left\{\max\left\{A(q) - E(\Pi)B(q), \underline{\pi}\right\}, \bar{\pi}\right\}$$

$$= \begin{cases}
\underline{\pi}, & q_1 \leqslant q \leqslant q_2; \\
A(q) - E(\Pi)B(q), & q_2 < q < q_4; \\
\bar{\pi}, & q_4 \leqslant q \leqslant q_5.
\end{cases}$$

According to (16), if $q_1 \leqslant q \leqslant q_2$, the level-2 mapping is

$$\hat{\Pi}_2(q) = \min\left\{\max\left\{A(q) - B(q)\underline{\pi}, \underline{\pi}\right\}, \bar{\pi}\right\}.$$

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By monotonicity of $A(q) - B(q)\pi$ in q, we have that $A(q) - B(q)\underline{\pi} \geqslant A_1 - B_1\underline{\pi} = \underline{\pi}$. In addition,

$$A(q) - B(q)\underline{\pi} \leqslant A_2 - B_2\underline{\pi} = B_2\,E(\Pi) + (1 - B_2)\underline{\pi} < E(\Pi).$$

Now consider the case when $q_4 \leqslant q \leqslant q_5$, we have

$$\hat{\Pi}_2(q) = \min\{\max\{A(q) - B(q)\bar{\pi}, \underline{\pi}\}, \bar{\pi}\}.$$

With identical arguments, we can show that $E(\Pi) < A(q) - B(q)\bar{\pi} < \bar{\pi}$.

When $q_2 \leqslant q \leqslant q_4$, we have

$$\hat{\Pi}_2(q) = \min\{\max\{A(q) - B(q)[A(q) - B(q)\,E(\Pi)], \underline{\pi}\}, \bar{\pi}\}.$$

By monotonicity of $A(q) - B(q)\pi$, we again have that

$$A(q) - B(q)[A(q) - B(q)\,E(\Pi)] < A(q) - B(q)\bar{\pi} \leqslant \bar{\pi}$$

and

$$A(q) - B(q)[A - B(q)\,E(\Pi)] > A(q) - B(q)\underline{\pi} \geqslant \underline{\pi}.$$

As a result, the level-2 mapping is

$$\hat{\Pi}_2(q) = \begin{cases} A(q) - B(q)\underline{\pi}, & q_1 \leqslant q \leqslant q_2; \\ A(q) - B(q)\,E(\Pi), & q_2 < q < q_4; \\ A(q) - B(q)\bar{\pi}, & q_4 \leqslant q \leqslant q_5. \end{cases}$$

By induction,

$$\hat{\Pi}_k(q) = \begin{cases} A(q) \cdot \sum_{j=0}^{k-1} [-B(q)]^j + \underline{\pi} \cdot [-B(q)]^k, & q_1 \leqslant q \leqslant q_2; \\ A(q) \cdot \sum_{j=0}^{k-1} [-B(q)]^j + E(\Pi) \cdot [-B(q)]^k, & q_2 < q < q_4; \\ A(q) \cdot \sum_{j=0}^{k-1} [-B(q)]^j + \bar{\pi} \cdot [-B(q)]^k, & q_4 \leqslant q \leqslant q_5; \end{cases}$$

which is a continuous function with two kinks. So long as we have $B(q) < 1$, sequence $\langle \hat{\Pi}_k(q) \rangle_{k \in \mathbb{N}}$ converges. Eq. (16) must then hold with both $\hat{\Pi}_k(q)$ and $\hat{\Pi}_{k-1}(q)$ by replaced by the limit, which implies that $\hat{\Pi}_k(q) \to \bar{\Pi}(q)$. $\qquad\square$

Proof of Lemma 1: The beliefs of level-0 speculators are the constant mapping $q \mapsto \hat{\Pi}_0(q) = E(\Pi)$. Suppose now π equals $E[\Pi]$. The market clearing condition is

$$0 = \mu^F \cdot Y^F(q; E[\Pi]) + \mu^S \cdot Y_0^S(q),$$

while the market clearing condition under fully-reveling rational expectations is

$$0 = \mu^F \cdot Y^F(q; E[\Pi]) + \mu^S \cdot Y^S(q; E[\Pi]).$$

Since the latter has a unique solution $q = \bar{Q}(E[\Pi])$, it follows that $Q_0(E[\Pi]) = \bar{Q}(E[\Pi])$.

Now, suppose that $Q_{k-1}(E[\Pi]) = \bar{Q}(E[\Pi])$ for some natural number k. Again, when $\pi = E[\Pi]$ the market clearing

condition requires that

$$0 = \mu^F \cdot Y^F(q; E[\Pi]) + \mu^S \cdot Y_k^S(q) = \mu^F \cdot Y^F(q; E[\Pi]) + \mu^S \cdot Y^S(q; Q_{k-1}^{-1}(q)).$$

Under the assumption that $Q_{k-1}^{-1}(\bar{Q}(E[\Pi])) = E[\Pi]$, the only solution has $q = \bar{Q}(E[\Pi])$, so $Q_k(E[\Pi]) = \bar{Q}(E[\Pi])$. The lemma, hence, follows by mathematical induction. $\square$

Proof of Proposition 2: The two types' optimal demands are

$$Y^F(q, \pi) = \frac{q\omega_1^F + \omega_2^F}{q + q^{\frac{1}{\gamma}}\left(\frac{1-\pi}{\pi}\right)^{\frac{1}{\gamma}}} - \omega_1^F \quad \text{and} \quad Y_k^S(q) = \frac{q\omega_1^S + \omega_2^S}{q + q^{\frac{1}{\gamma}}\left[\frac{1-\hat{\Pi}_k(q)}{\hat{\Pi}_k(q)}\right]^{\frac{1}{\gamma}}} - \omega_1^S.$$

Market clearing requires the sum of these two values to be null. Substitution of π by $\hat{\Pi}_{k+1}(q)$ yields, after some algebra, the equality

$$\left\{ q + q^{\frac{1}{\gamma}}\left[\frac{1-\hat{\Pi}_{k+1}(q)}{\hat{\Pi}_{k+1}(q)}\right]^{\frac{1}{\gamma}}\right\}^{-1} = \frac{\omega_1^F + \omega_1^S}{q\omega_1^F + \omega_2^F} - \frac{q\omega_1^S + \omega_2^S}{q\omega_1^F + \omega_2^F}\cdot\left\{ q + q^{\frac{1}{\gamma}}\left[\frac{1-\hat{\Pi}_k(q)}{\hat{\Pi}_k(q)}\right]^{\frac{1}{\gamma}}\right\}^{-1}.$$

This recursion implies that

$$\left\{ q + q^{\frac{1}{\gamma}}\left[\frac{1-\hat{\Pi}_k(q)}{\hat{\Pi}_k(q)}\right]^{\frac{1}{\gamma}}\right\}^{-1}$$

converges as $k \to \infty$ if, and only if, the condition of the proposition holds true.

Since mapping $\pi \mapsto (1 - \pi)/\pi$ is monotonic, the latter convergence is equivalent to the convergence of $\hat{\Pi}_k(q)$ as $k \to \infty$. The limit of the sequence is defined by

$$q^{\frac{1}{\gamma}}\left[\frac{1 - \lim_{k\to\infty}\hat{\Pi}_k(q)}{\lim_{k\to\infty}\hat{\Pi}_k(q)}\right]^{\frac{1}{\gamma}} = \frac{q(\omega_1^F + \omega_1^S) + \omega_2^F + \omega_2^S}{\omega_1^F + \omega_1^S} - q = \frac{\omega_2^F + \omega_2^S}{\omega_1^F + \omega_1^S}$$

The solution to the latter equation gives

$$\lim_{k\to\infty}\hat{\Pi}_k(q) = \frac{q(\omega_1^F + \omega_1^S)^\gamma}{q(\omega_1^F + \omega_1^S)^\gamma + (\omega_2^F + \omega_2^S)^\gamma}, \tag{21}$$

which is, indeed, $\bar{\Pi}(q)$. $\square$

Proof of Theorem 3: The demand of each fundamentalist is

$$Y^F(q, \pi) = \frac{1}{\rho(1+q)}\ln\left(\frac{\pi}{1-\pi}\right) - \frac{\ln q}{\rho(1+q)} - \frac{\omega_1^F - \omega_2^F}{1+q}.$$

Similarly each speculator demands

$$Y_k^S(q) = \frac{1}{\rho(1+q)}\ln\left[\frac{\hat{\Pi}_k(q)}{1-\hat{\Pi}_k(q)}\right] - \frac{\ln q}{\rho(1+q)} - \frac{\omega_1^S - \omega_2^S}{1+q}.$$

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Substituting π with $\hat{\Pi}_{k+1}$ in the market clearing equation, one gets

$$\mu^F \ln \left[\frac{\hat{\Pi}_{k+1}(q)}{1 - \hat{\Pi}_{k+1}(q)} \right] + \mu^S \ln \left[\frac{\hat{\Pi}_k(q)}{1 - \hat{\Pi}_k(q)} \right] = \rho[\mu^F(\omega_2^F - \omega_1^F) + \mu^S(\omega_2^S - \omega_1^S)] + (\mu^F + \mu^S) \ln q,$$

or, equivalently,

$$\ln \left[\frac{\hat{\Pi}_{k+1}(q)}{1 - \hat{\Pi}_{k+1}(q)} \right] = -\frac{\mu^S}{\mu^F} \ln \left[\frac{\hat{\Pi}_k(q)}{1 - \hat{\Pi}_k(q)} \right] + \rho \left[\omega_2^F - \omega_1^F + \frac{\mu^S}{\mu^F}(\omega_2^S - \omega_1^S) \right] + \frac{\mu^F + \mu^S}{\mu^F} \ln q.$$

If, and only if, $\mu^F > \mu^S$, the sequence of mappings

$$\ln \left[\frac{\hat{\Pi}_k(q)}{1 - \hat{\Pi}_k(q)} \right]$$

converges uniformly. Since $\pi \mapsto \pi/(1 - \pi)$ is a monotonic mapping, we have that $\hat{\Pi}_k(q)$ converges uniformly to $\bar{\Pi}(q)$ if, and only if, $\mu^F > \mu^S$.

To complete the argument, note that the rational expectation equilibrium,

$$\bar{\Pi}(q) = \frac{e^\gamma \cdot q}{1 + e^\gamma \cdot q},$$

with

$$\gamma = \rho \left[\left(\frac{\mu^S}{\mu^S + \mu^F} \right) \cdot (\omega_2^S - \omega_1^S) + \left(\frac{\mu^F}{\mu^S + \mu^F} \right) \cdot (\omega_2^F - \omega_1^F) \right]$$

is bijective. $\qquad\square$

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