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Equilibrium behavior in markets and games: testable restrictions and identification☆

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Abstract

We provide a selective survey of the recent literature on the empirical implications of individually rational behavior in markets and games. We concentrate on work that develops empirical implications while making as few parametric assumptions as possible. We focus on two major themes: 1. the testable restrictions on the equilibrium manifold and the identification of economic fundamentals from the equilibrium manifold; and 2. the implications of the revealed preference theory of individual behavior for aggregated data.

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1. Introduction

The standard for what is to be considered scientific knowledge has been a prominent topic of debate in epistemology. Karl Popper argued that scientists should actively try to prove their theories wrong, rather than merely attempt to verify them through inductive reasoning. The Popperian postulate thus states that a scientific discovery ought to distinguish the theory from its empirical implications and that the empirical implications should be contrasted to

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reality, in order for the theory to be corroborated (however, not verified) or refuted. If a theory fails a test, and there exists no reasonable excuse that can itself be tested, then the theory should be abandoned.

This "empiricist" position, often referred to as "falsificationism", had been previously exposed by Poincaré who, in 1908, wrote that[3] ". . . when a theory has been established, we have first to look for cases in which the rule stands the best chance of being found at fault." This principle was introduced to economics by Paul Samuelson, for whom "meaningful theorems" are hypotheses ". . . about empirical data which could conceivably be refuted" (Samuelson, 1947, p. 4). It seems desirable to obtain testable implications from the equilibrium concepts in economics, even if one considers the views of Popper to be an extreme.

General equilibrium models provide a way to analyze the interactions of individuals and firms in a market economy. The basic components of a general equilibrium model are fairly simple: agents maximize utility subject to a budget constraint, firms maximize profits, and markets clear. It is clearly of interest to see how these essential components of a general equilibrium model restrict market behavior; that is, one would like to know what the empirical implications or the testable restrictions of this theory are.

The seminal work by Sonnenschein, Mantel, Debreu and others has had quite negative implications for the existence of testable restrictions of competitive equilibrium (Shafer and Sonnenschein, 1982). In particular, with a few qualifications, any compact set of prices can be the set of equilibrium prices for some economy (Mas-Colell, 1977). A popular textbook in microeconomic theory (Mas-Colell et al., 1995) categorizes these results as that ". . . anything satisfying. . . " the very mild restrictions of the Sonnenschein–Mantel–Debreu theorem ". . . can actually occur." Recent research, however, has obtained more positive results regarding the empirical implications of general equilibrium theory. This work examines different constructs, most notably, the equilibrium manifold rather than the market excess demand function, and uses different techniques to obtain comparative statics, for example, tools from the revealed preference theory. From a theoretical perspective, these results establish that general equilibrium theory is refutable, and thus has scientific meaning in the Popperian sense. From an economic policy perspective, these results suggest that we do not have to rely on empirical work that uses parametric specifications of preferences or technologies, or other restrictive assumptions on the parameters of the model.

A related question is whether observable data allows for the identification of the unobservable fundamentals of the economy. The transfer paradox clearly illustrates the relevance of this question: without any knowledge of the fundamentals, the actual effects of economic policy may be opposite to the conjecture of policy makers and to their policy goals.

In this paper, we review some of the recent developments in this literature. We limit our discussion to three distinct but closely related lines of research. In Section 2, we describe the existence and derivation of testable restrictions of competitive equilibrium over finite data sets. Section 3 addresses the issue of identification of fundamentals from the equilibrium outcomes. In Section 4, we describe testable restrictions of game-theoretic models. This is not an exhaustive survey of all the topics related to observable implications of equilibrium models. We rather hope to develop some of the important themes of the literature and expose some interesting avenues for further research.

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[3] See Zahar (2001).

2. Testable restrictions on data

Influential work by Sonnenschein, Mantel, Debreu, and others suggests that the theory of competitive equilibrium imposes essentially no testable restrictions on data (see Shafer and Sonnenschein (1982), for an overview of this literature). This work has generally looked for restrictions on the market excess demand function, which treats prices as exogenous variables and endowments as fixed.

One of the major innovations of Brown and Matzkin (1996) was to search for testable restrictions on the equilibrium manifold, i.e., the graph of the Walrasian correspondence, as defined by Balasko (1975). The equilibrium manifold is the set of ordered pairs of individual endowments and prices such that the market excess demand function is zero for a given set of utility functions. One could interpret the variation as an economy with changing endowments over time (where each agent has a one-period decision making horizon), or as a cross-section of different groups of traders with the same utility functions. Given the logic of comparative statics, the equilibrium manifold is the appropriate construct to use to study the testable restrictions of general equilibrium: the system's exogenous variables—the individual endowments—are allowed to vary to derive restrictions on the system's endogenous variables—the equilibrium prices. This is also the approach that Arrow and Hahn, and in fact Walras, took in developing comparative statics results of the pure exchange model when equilibrium is unique (Arrow and Hahn, 1971). Importantly, studying the implications of the theory on the equilibrium manifold rather than on market excess demand expands the potential data set, and thereby increases the likelihood of the existence of nonvacuous testable restrictions.

Another major innovation of Brown and Matkzin was to search for testable restrictions of the competitive equilibrium model over finite data sets. It is well known that restrictions such as the Weak Axiom of Revealed Preference (WARP) do not aggregate; in addition, Andreu (1982) shows that WARP alone will result in no restrictions on market demand with a fixed income distribution on finite domains. Brown and Matzkin, however, use the finiteness of data to motivate a set of different techniques for analyzing the complete set of testable propositions a model can possess. They apply semialgebraic theory rather than traditional comparative statics tools such as calculus. The methods are straightforward to apply, lead to direct nonparametric tests applicable to potentially observable data sets, and, importantly, do not rely on properties of the equilibria such as uniqueness or stability.

As a result of these innovations, Brown and Matzkin describe the complete set of testable propositions of the pure exchange model on finite observations of the equilibrium manifold and prove that these tests are nonvacuous. For the case of two agents and two observations they derive the tests in the form of a finite set of polynomial inequalities over the data alone. They also derive similar tests for the case of homothetic utility. Adding production to the model, they describe the testable propositions of a Robinson Crusoe economy. In contrast, Brown and Shannon (2000) find that local uniqueness and stability of equilibrium impose no further restrictions on finite data sets beyond the competitive equilibrium restrictions. That is, while equilibrium might be testable, local qualitative features of equilibrium are not.

The results of Brown and Matzkin are based on classical revealed-preference theory and, therefore, implicitly assume that individual preferences are invariant. Experimental evidence has convinced many psychologists of the importance of allowing preferences to vary randomly, in an unobservable manner. Carvajal (2003b) has proved that even under random utilities, the general equilibrium model has testable restrictions, whenever individual income and probabilistic distributions of prices are observed.

The above results are based on multiple observations of static, one-period models. Kübler (2003a) explores testable restrictions in a dynamic model and finds that if all individuals have time-separable, expected utility preferences, the general equilibrium hypothesis is falsifiable given aggregate data only. He also offers evidence that the restrictions are likely to become vacuous if time-separability is not assumed.

There also exists applied theory on nonparametric testable restrictions. Snyder (1999) derives testable restrictions of a model of efficient public good provision on aggregate-level variables and individual income. Once again restrictions exist though key individual data remain unobserved. In efficient public good provision there are price variables that are inherently unobservable: Lindahl prices, which correspond to the marginal benefits of each public good.

Carvajal (2003a) analyzes the testable restrictions of a Nash–Walras equilibria in which there exist consumption externalities. He finds that restrictions do not exist unless the individual consumptions of the externality good are observed and that even in that case, the restrictions are very weak.

Various models of collective decision making within the household have been proposed. A general approximation is given by Chiappori's (1988, 1992) collective rationality model, which describes individually rational agents who achieve a Pareto efficient allocation within the household. Issues concerning the testable restrictions of such a model are similar to those of the competitive equilibrium model, as data is most likely to be observed at the aggregate level (here, the household level). He finds testable restrictions can exist when some individual level data, such as individual labor supplies, can be observed.

In this section, we describe in greater detail the above results. We first describe some required results from the revealed preference literature and semialgebraic theory in Section 2.1. Brown and Matzkin's results are presented in Section 2.2. We describe Carvajal's results on random utilities and Kübler's work on dynamic models in Sections 2.3 and 2.4 respectively. We then turn to the testable restrictions of more applied models: efficient public good provision in Section 2.5; externalities in the Nash–Walras model in Section 2.6; and finally, the collective rationality model of the household in Section 2.7.

2.1. Preliminaries

Definition 1. Suppose there are observations on prices and quantities $\langle p^r, x^r \rangle_{r=1}^R$. Then a utility function $U(x)$ *rationalizes the data* if $U(x^r) \geq U(x)$ for all x such that $p^r \cdot x^r \geq p^r \cdot x$.

Theorem 1 (Afriat). *Given a data set* $\langle p^r, x^r \rangle_{r=1}^R$, *the following conditions are equivalent*:[4]

- *There exists a non-satiated utility function that rationalizes the data.*

[4] This statement of the theorem is due to Varian (1982).

- *The data satisfy "cyclical consistency"*: $p^r \cdot x^r \geq p^r \cdot x^s$, $p^s \cdot x^s \geq p^s \cdot x^u, \ldots,$ *and* $p^q \cdot x^q \geq p^q \cdot x^r$ *together imply* $p^r \cdot x^r = p^r \cdot x^s$, $p^s \cdot x^s = p^s \cdot x^u, \ldots,$ *and* $p^q \cdot x^q = p^q \cdot x^r$.
- *There exist numbers* V^r, $\lambda^r > 0$, $r = 1, \ldots, R$ *that satisfy the "Afriat inequalities"*: $V^r \leq V^s + \lambda^s p^s \cdot (x^r - x^s)$ for $r, s = 1, \ldots, R$.
- *There exists a concave, monotonic, continuous, non-satiated utility function that rationalizes the data.*

Varian (1982) showed that cyclical consistency is equivalent to the data satisfying the Generalized Axiom of Revealed Preference (GARP), a generalized version of WARP. Thus, we can also say that the data is rationalizable if and only if GARP is satisfied. Like cyclical consistency, GARP is a condition on prices and quantities, and does not involve the inherently unobservable V's or λ's that appear in the Afriat inequalities.

The polynomial form of the restrictions results entirely from the assumption of finiteness of data, not from any assumptions on the functional form of utility. There are versions of Afriat's theorem for stronger restrictions on utility functions as well. Matzkin and Richter (1991) describe a modification of the Afriat inequalities that is equivalent to Houthakker's Strong Axiom of Revealed Preference (SARP), the transitive version of WARP, which restricts demand to be single-valued; Chiappori and Rochet (1987) give restrictions for smooth demand. Varian (1983) describes restrictions for separability and homotheticity. There are also analogous conditions for profit maximization and cost minimization; e.g., the Weak Axiom of Profit Maximization (WAPM) provides necessary and sufficient conditions for profit maximizing behavior in the form of polynomial inequalities defined over a discrete series of data.[5]

Thus, many of the basic building blocks of general equilibrium models have the feature that testable restrictions will be polynomial in form. The finite polynomial form of these conditions is important because it means semialgebraic theory can be applied to describe the data that satisfy the restrictions.

A semialgebraic set is defined by a finite system of polynomial inequalities.[6]

Definition 2. A subset Φ of $\mathbb{R}^m$ is a semialgebraic set if it is the finite union of sets of the form: $\{x \in \mathbb{R}^m : f_i(x) = 0, i = 1, \ldots, k; \quad g_j(x) > 0, j = 1, \ldots, p\}$ where f_i and g_j are polynomials with real coefficients.

Semialgebraic sets in $\mathbb{R}^m$ can also be defined in terms of propositional algebraic sentences, which are composed of: real constants, m real variables, arithmetic operators $(+, -, \cdot, /)$, binary relation operators $(=, \neq, >, <, \geq, \leq)$, and Boolean connectives $(\neg, \Rightarrow, \wedge, \vee)$. If we add universal and existential quantifiers, $(\forall, \exists)$, ranging over the real numbers, we can compose *Tarski sentences*, which define subsets of $\mathbb{R}^m$ called *Tarski sets*.

The following theorem shows that given a Tarski set, we can always find an equivalent semialgebraic set, i.e., the quantifiers can be eliminated.

[5] For origins of WAPM see Varian (1984).

[6] These definitions and results are found in Mishra (1993), also see van den Dries (1988).

Theorem 2 (Tarski–Seidenberg). *Every Tarski set is a semialgebraic set and the quantifier-free definition is derivable in finite time.*

This process of deriving the quantifier-free definition referred to in the Tarski–Seidenberg theorem is called *quantifier elimination*. An implicit application of quantifier elimination is seen in Afriat's theorem when prices and consumptions are observed: the unobserved V's and λ's in the Afriat inequalities do not appear in the cyclical consistency condition, yet both conditions describe equivalent restrictions on the observed p's and x's.

2.2. Testable restrictions of pure exchange

Consider a pure exchange economy with N agents each with preferences representable by a nonsatiated utility function U_n over ℓ commodities. Each agent has an initial endowment $\omega_n \in \mathbb{R}^{\ell}_{++}$. A feasible allocation is a set of consumption vectors $\{x_n\}_{n=1}^N$, with $x_n \in \mathbb{R}^{\ell}_{+}$, such that $\sum_{n=1}^N x_n = \sum_{n=1}^N \omega_n$. A competitive equilibrium is a feasible allocation $\{x_n\}_{n=1}^N$ and prices $p \in \mathbb{R}^{\ell}_{++}$ such that each x_n maximizes U_n subject to the budget constraint $p \cdot x_n \leq p \cdot \omega_n$.[7]

The following result is an adaptation of Brown and Matzkin's result which describes the restrictions the competitive equilibrium model places on discrete observations of prices and individual endowments (the equilibrium manifold).

Theorem 3 (Brown and Matzkin, 1996). *Let $\langle p^r, \{\omega_n^r\}\rangle_{r=1}^R$ be given. Then there exists a set of continuous, monotone, and concave utility functions $\{U_n\}$ such that each p^r is an equilibrium price vector for the exchange economy $\langle\{U_n\}, \{\omega_n^r\}\rangle$ if and only if the following system is satisfied.*

There exist numbers V_n^r, λ_n^r, and vectors $\{x_n^r\}$ such that:

- *markets clear*: $\sum_{n=1}^N x_n^r = \sum_{n=1}^N \omega_n^r$, *for* $r = 1, \ldots, R$;
- *budget constraints are satisfied*: $p^r \cdot x_n^r = p^r \cdot \omega_n^r$, *for* $r = 1, \ldots, R$, *and* $n = 1, \ldots, N$;
- $x_n^r \geq 0$, $\lambda_n^r \geq 0$, *for* $r = 1, \ldots, R$, *and* $n = 1, \ldots, N$.
- *the Afriat inequalities are satisfied for all agents*: $V_n^r - V_n^s - \lambda_n^s\, p^s \cdot (x_n^r - x_n^s) \leq 0$, *for* $r, s = 1, \ldots, R$, *and* $n = 1, \ldots, N$.

The conditions can easily be rewritten for the case when individual incomes, $I_n^r = p^r \cdot \omega_n^r$, and the aggregate endowment, ω^r, are observed, rather than individual endowments. Also note that these are the same restrictions we get on market demand when prices, aggregate consumption, and individual endowments or incomes are observed, if we replace the sum of individual endowments with the observed aggregate consumption vector. The difference that prices are endogenous in a competitive equilibrium model and exogenous in a market demand model turns out not to matter.[8]

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[7] For the rest of the paper, $\{x_n\}$ will be understood to mean the sequence $\{x_n\}_{n=1}^N$.

[8] Compare this theorem also to Propositions 2 and 3 of Chiappori (1988), in which nonparametric testable restrictions of household behavior are defined over (aggregate) household behavior and individual labor supplies.

Theorem 3 describes competitive equilibrium behavior in terms of a finite set of polynomial inequalities. Thus, Brown and Matzkin characterize equilibrium behavior as a semialgebraic subset of the price-individual endowment space, in contrast to traditional comparative statics, where one would derive local properties on a market demand/excess demand function or an equilibrium mapping. For any given model and any given set of observables, either the semialgebraic set defined by the testable restrictions would contain all potential data, meaning the model was irrefutable for that set of observables, or it would be empty, meaning equilibrium could never be obtained, or it would contain a strict subset of all potential data. In the last case, we say the model is testable, or nonvacuous, given the potential data set, and one could (theoretically) derive the testable restrictions on observable variables.

Theorem 4 (Brown and Matzkin, 1996). *The pure exchange model of competitive equilibrium is testable on R observations of prices and individual endowments* $\langle p^r, \omega_n^r \rangle$.

The proof of this theorem depends on Brown and Matzkin's counterexample showing that there exist data that do not satisfy the conditions of Theorem 3, as shown in Fig. 1. There are two observations of a two-good, two-agent exchange economy. The figure shows an Edgeworth box for each period, with the vertex agent 1's origin. The feasible consumption bundles in period 1 and period 2 are along the segment of the budget line between a and b, and along the segment of the budget line between c and d, respectively. Every consumption bundle agent 1 could have chosen in period 1 is revealed preferred to every consumption bundle agent 1 could have chosen in period 2, and vice versa. Thus, there are no feasible consumption bundles that could satisfy GARP for this agent.

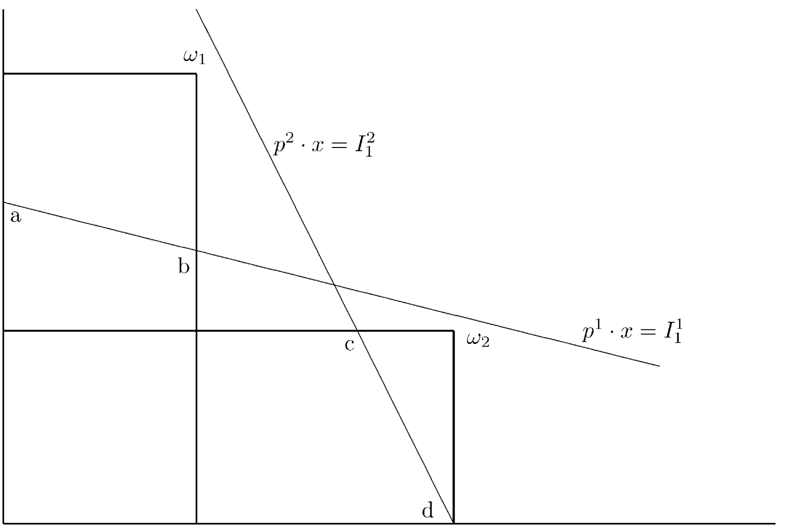

Fig. 1. Data not consistent with competitive equilibria.

To apply the test in Theorem 3 one must determine whether there is a solution to the system given a realization of data. The Tarski–Seidenberg algorithm can be used to derive the testable restrictions in terms of the observable variables alone. The tests would then be in the form of checking the signs of a finite number of polynomial inequalities in the data. The algorithm is doubly-exponential in practice, however, and thus not feasible for many problems. Brown and Matzkin use the Tarski–Seidenberg algorithm to derive the testable restrictions in terms of the observable variables alone for two observations of a two-agent economy. Thus, data could be directly checked against a set of polynomial inequalities, which they call the Weak Axiom of Revealed Equilibrium (WARE).

Given these positive results, we next ask what else is testable. Brown and Shannon (2000) further consider which features of an equilibrium are refutable given a finite data set.

Theorem 5 (Brown and Shannon, 2000). *Let $\langle p^r, \omega^r, \{I_n^r\}\rangle_{r=1}^{R}$ be given. Then there exists a set of smooth, monotone, strictly quasi-concave utility functions $\{U_n\}$ and $\{\omega_n^r\}$ such that each p^r is an equilibrium price vector for the exchange economy $\langle\{U_n\}, \{\omega_n^r\}\rangle$ if and only if there exists a set of smooth, strictly quasiconcave monotone utility functions $\{U_n\}$ such that each p^r is a locally unique and locally stable under tatonnement equilibrium price vector for the exchange economy $\langle\{U_n\}, \{\omega_n^r\}\rangle$, and for which the equilibrium correspondence $\langle p^r, \omega^r\rangle$ is locally monotone for each r.*

Theorem 5 is a striking result: while the assumptions needed to ensure local uniqueness and stability are quite strong, if the data are consistent with any equilibrium, they are consistent with an equilibrium with those features. Once again, it is key that data are finite: finite data cannot be used to refute local characterizations of equilibrium.[9]

2.3. Random preferences

Consider an economy with N consumers and ℓ commodities, which must be consumed in nonnegative amounts. Prices are restricted to lie in the $(\ell - 1)$-dimensional simplex, $\mathcal{S}_+^{\ell-1}$. In order to define probability distributions on prices, the simplex is endowed with a σ-algebra, Σ. For a set $\Omega \subseteq \mathbb{R}_{++}^{\ell N}$ of profiles of endowments, and for each $\omega \in \Omega$, a probability measure over prices $\chi_\omega : \Sigma \to [0, 1]$ is observed.

Let $\mathcal{U}$ be the class of all continuous, strongly concave and strictly monotone utility functions (on $\mathbb{R}_+^{\ell}$) and denote by $W_{U,\omega}$ the set of Walrasian equilibrium prices of the economy $\langle\{U_n\}, \{\omega_n\}\rangle$, for each $U = \{U_n\} \in \mathcal{U}^N$ and each $\omega = \{\omega_n\} \in \mathbb{R}_{++}^{\ell N}$.

A first contribution of Carvajal (2003a) is the definition of consistency between a data set $\langle\Omega, (\chi_\omega)_{\omega\in\Omega}\rangle$ and general equilibrium under random utility. Since preferences and prices are random, one has to explain them via joint probability measures. Since endowments matter for equilibrium prices, these joint distributions must be indexed by the observed profiles of endowments. A weak definition of consistency of data and theory would then require that for each observed ω there exists a joint probability measure over preferences and prices,[10] $\pi_\omega : \mathcal{P}(\mathcal{U}^N) \times \Sigma \to [0, 1]$, such that:

[9] See also Balasko and Tvede (2002) and Snyder (2003).

[10] The power set of any set A is denoted by $\mathcal{P}(A)$.

- observed probabilities on prices are explained by the marginals (for prices) of the joint theoretical probabilities: for each ω and each measurable set of prices $C \in \Sigma$, $\chi_\omega(C) = \pi_\omega(\mathcal{U}^N, C)$;
- the mechanism by which consistency holds indeed corresponds to Walrasian equilibrium, in the sense that only Walrasian equilibrium prices can have positive probability: for each set of profiles of preferences, $\mathcal{V} \in \mathcal{P}(\mathcal{U}^N)$, and each measurable set of prices, $C \in \Sigma$, $\pi_\omega(\mathcal{V}, C) > 0 \Rightarrow \exists U \in \mathcal{V} : C \cap W_{U,\omega} \neq \varnothing$.

In general, for a theory to impose empirical restrictions, one must assume some independence of the fundamentals of the theory with respect to observed data. Otherwise, anomalies found in the predictions of the theory could be explained by changes in the fundamentals. In the cases of Brown and Matzkin (1996), Snyder (1999) or Carvajal (2003a) it is assumed that preferences do not depend on income. With the previous conditions only, no analogous independence would hold here: for different observed profiles of endowments, ω and ω', marginal distributions for U, $\pi_\omega(\cdot, \mathcal{S}_+^{\ell-1})$ and $\pi_{\omega'}(\cdot, \mathcal{S}_+^{\ell-1})$, which ought to be an invariant fundamental, may differ. In order to rule out this possibility, Carvajal requires that, in addition to the previous two conditions, all the joint distributions have a common marginal for utility: there exists a probability measure on preferences, $\vartheta : \mathcal{P}(\mathcal{U}^N) \to [0, 1]$, such that for all observed ω, and every set of preferences, $\mathcal{V} \in \mathcal{P}(\mathcal{U}^N)$, $\vartheta(\mathcal{V}) = \pi_\omega(\mathcal{V}, \mathcal{S}_+^{\ell-1})$.

Under this stronger requirement, conditionals for prices are "random selectors" as in Allen (1985). Also, this assumption allows us to rewrite the first condition as: for each $C \in \Sigma$, $\chi_\omega(C) = \int_{\mathcal{U}^N} \pi_{\omega|U}(C) d\vartheta(\{U\})$.

For his results, Carvajal (2003b) considers the case in which there is a finite number of states of nature, there is perfect observability of prices, $\Sigma = \mathcal{P}(\mathcal{S}_+^{\ell-1})$, and all observed distributions of prices have finite support. Formally, let Ξ be the set of states of the world and let F be the set of all probability measures on $\mathcal{S}$ (over Σ). Then, we have the following definition.

Definition 3. A data set $\langle \Omega, (\chi_\omega)_{\omega \in \Omega} \rangle$ is rationalizable if there exists a probability measure over states of nature, $\delta : \mathcal{P}(\Xi) \to [0, 1]$, a function that assigns profiles of preferences to states of nature, $U : \Xi \to \mathcal{U}^N$ and a function assigning random selectors to economies, $\varphi : U[\Xi] \times \Omega \to \mathsf{F}$, such that:

- observed probabilities are explained by theoretical probabilities: $\chi_\omega(C) = \sum_{\xi \in \Xi} \delta(\xi)\varphi(U(\xi), \omega)(C)$, for $\omega \in \Omega$, and $C \in \Sigma$;
- this explanation is given via Walrasian equilibria: $\varphi(U(\xi), \omega)(W_{U(\xi),\omega}) = 1$, for $\xi \in \Xi$, and $\omega \in \Omega$.

Carvajal's main result is that, given a set of states of nature, the hypothesis that a data set is rationalizable can be refuted. As in Brown and Matzkin, this result is obtained by means of two other results: one that says that there exist conditions that are equivalent to rationalizability and one that shows that these conditions are not tautologies.

Theorem 6 (Carvajal, 2003b). *Suppose that Ξ and $\Omega \subseteq \mathbb{R}_+^{\ell N}$ are given. Suppose also that for each $\omega \in \Omega$ the set* $\operatorname{Supp}(\chi_\omega) \subseteq \mathcal{S}_+^{\ell-1}$ *is fixed and let Ψ be the set of* *vectors* $((\chi_{\omega,p})_{p\in \mathrm{Supp}(\chi_\omega)})_{\omega\in\Omega} \in \prod_{\omega\in\Omega}[0,1]^{\#\mathrm{Supp}(\chi_\omega)}$ *such that the data set* $\left\{\Omega, \left(C \mapsto \sum_{p\in C}\chi_{\omega,p}\right)_{\omega\in\Omega}\right\}$ is *Ξ-rationalizable.*[11] *Ψ is a semialgebraic set.*

To understand this result, note that the observed variables are profiles of individual endowments and one distribution of prices for each profile of endowments. This constitutes the data set. The unobservable variables are profiles of individual preferences at each state of the world, one probability for each state of the world, and one random selector for each profile of endowments and each profile of utilities. This is the set of concepts whose existence (subject to the requirements for rationalizability) one is trying to determine. There exists, however, a third category of variables: those which would be observable in a laboratory where one could observe individual data and even run, repeatedly, counterfactual experiments. These variables are a profile of individual demands for each profile of endowments, each price vector and each state of the world; a joint distribution of individual demands for each profile of endowments and each vector of prices; and a probability for each observed price vector, for each profile of endowments and each state of the world. The result is that these "observable-but-unobserved" variables must exist such that: the first one satisfies SARP, individually, for each state of the world, across budgets; the second one satisfies a version of SARP for random utilities, the Axiom of Revealed Stochastic Preferences, first proposed by McFadden and Richter (1990), and generalized by Carvajal (2003a) for the case of collective decision problems; and the third one satisfies a random version of market clearing, for every profile of endowments and state of the world (prices that do not clear markets have zero probability).

Carvajal uses Tarski–Seidenberg quantifier elimination theory to prove the previous theorem. As in cases mentioned before, this method does not imply the testability of the theory, as it does not rule out the possibility that the condition is a tautology. Two independent examples given by Carvajal show that, indeed, this is not the case.

Example 1 (Carvajal, 2003b). Consider Fig. 1 and assume that $p^1, p^2 \in \mathcal{S}_+^{\ell-1}$. Suppose that individual endowments consistent with ω_1 and ω_2 have been observed. Suppose further that distributions χ_{ω_1} and χ_{ω_2} have been observed such that $\mathrm{Supp}(\chi_{\omega_1}) \subset \{p \in \mathcal{S}_+^{\ell-1} : p_2 < p_2^1\}$, and $\mathrm{Supp}(\chi_{\omega_2}) \subset \{p \in \mathcal{S}_+^{\ell-1} : p_1 < p_1^2\}$. This implies that all the prices that have been observed to occur when the Edgeworth box is ω^1 give a steeper budget line for agent 1 than p^1, while all the prices that have been observed under ω^2 give this individual a flatter budget line than p^2. By the same arguments of the deterministic case, it follows that all the prices in $\mathrm{Supp}(\chi_{\omega_1})$ are inconsistent with all those in $\mathrm{Supp}(\chi_{\omega_2})$ in the sense that, even for a given state of nature, no consumption bundles for individual 1 can be consistent with the standard axioms of revealed preferences. No data set with these features could be rationalized, regardless of the number of states of the world.

Example 1 illustrates a case in which it is impossible to find the first set of "observable-but-unobserved" variables satisfying the conditions of Theorem 6. Carvajal (2003b) also

[11] The notation $C \mapsto \sum_{p\in C}\chi_{\omega,p}$ means that the function $\chi_\omega : \Sigma \to [0,1]$ is constructed as: $\forall C \in \Sigma \chi_\omega(C) = \sum_{p\in C}\chi_{\omega,p}$.

provides an example illustrating a similar impossibility for the other two sets of “observable-but-unobserved” variables.

2.4. Dynamic problems

To make the problem dynamic, suppose that a tree Ξ describes the evolution of nature. Nodes of this tree will be denoted by ξ, and can be understood as representing date-events. The root of the tree will be denoted by ξ_0 and the set of terminal nodes by Ξ^{T}. For each $\xi \in \Xi\backslash\{\xi_0\}$, let $\xi_- \in \Xi$ denote the predecessor node and let $F(\xi)$ denote the set of its (immediate) successors.[12] We will denote by X the number of nodes in Ξ.

At each $\xi \in \Xi$, spot markets for all ℓ commodities are open. In addition, there are J long-lived assets that can be traded at every node. Let $\mathcal{S}^{\ell-1}_{++} = \{p \in \mathbb{R}^{\ell}_{++} : p_1 = 1\}$. A pricing rule for commodities is a function $p : \Xi \to \mathcal{S}^{\ell-1}_{++}$, where $p_l(\xi)$ represents the price of commodity l at node ξ. Restricting prices to lie in $\mathcal{S}^{\ell-1}_{++}$ imposes the first commodity as numeraire at every node. A pricing rule for assets is $q : \Xi \to \mathbb{R}^J$, where $q_j(\xi)$ denotes the price of asset j at node ξ. The (real) return of asset j will be represented by a function $r_j : \Xi \to \mathbb{R}$, so that $r_j(\xi)$ represents the return of the asset, in units of the numeraire, at node ξ. Given a node ξ, $r(\xi)$ will denote the vector $(r_1(\xi), \dots, r_J(\xi)) \in \mathbb{R}^J$.

Individual n has preferences represented by utility function $U_n : \mathbb{R}^{\ell X}_{+} \to \mathbb{R}$ and his endowment is denoted by $\omega_n : \Xi \to \mathbb{R}^{\ell}_{+}$. For an economy with preferences $\{U_n\}$, endowments $\{\omega_n\}$ and asset returns $\{r_j\}_{j=1}^{J}$, Kübler (2003a) defines an equilibrium as: prices for commodities, p, and assets, q; individual consumptions $\{x_n : \Xi \to \mathbb{R}^{\ell}_{+}\}$; and individual portfolios $\{\theta_n : \Xi \to \mathbb{R}^J\}$ that are consistent with market clearing and individual rationality.

Definition 4. Given an economy $\langle\{U_n, \omega_n\}, \{r_j\}\rangle$, an equilibrium is $(p, q, \{x_n, \theta_n\})$ such that:

- all markets for assets and commodities clear: $\sum_{n=1}^{N} x_n(\xi) = \sum_{n=1}^{N} \omega_n(\xi)$, for $\xi \in \Xi$; $\sum_{n=1}^{N} \theta_n(\xi) = (0, \dots, 0)$, for $\xi \in \Xi$;
- all individuals are maximizing their utilities: for every n, (x_n, θ_n) solves the problem: $\max U_n((x(\xi))_{\xi\in\Xi})$ subject to the constraint $p(\xi) \cdot x(\xi) + q(\xi) \cdot \theta(\xi) \leq p(\xi) \cdot \omega(\xi) + (q(\xi) + r(\xi)) \cdot \theta(\xi_-)$, for every ξ, with the convention that $\theta(\xi_-) = 0$ when $\xi = \xi_0$.

Given an asset structure $(q, \{r_j\})$, there exist no arbitrage opportunities if it is impossible to find a portfolio of assets that strictly increases wealth at some node, without decreasing it at some other node. Formally, we have the following definition.

Definition 5. Asset prices and returns $(q, \{r_j\})$ allow no arbitrage opportunities if there does not exist a portfolio $\theta : \Xi \to \mathbb{R}^J$ such that:

- at no node does wealth strictly decrease: $\theta(\xi_-) \cdot (q(\xi) + r(\xi)) - \theta(\xi) \cdot q(\xi) \geq 0$ for every $\xi \in \Xi$;

[12] Of course, $\xi \in \Xi^{\mathrm{T}}$ if and only if $F(\xi) = \varnothing$.

- at some node wealth strictly increases: $\theta(\xi_-) \cdot (q(\xi) + r(\xi)) - \theta(\xi) \cdot q(\xi) > 0$ for some $\xi \in \Xi$.

It is well known that given an economy $\langle \{U_n\}, \{\omega_n\}, \{r_j\} \rangle$ where preferences are monotone, a necessary condition for $(p, q, \{x_n, \theta_n\})$ to be an equilibrium is that $(q, \{r_j\})$ allow no arbitrage opportunities; otherwise, individuals would find a portfolio with which they could increase their levels of utility unboundedly. It is remarkable that equilibrium imposes no further restrictions for a broad class of utility functions, even when individual endowments and demands for commodities and assets are observed: given $\langle p, q, \{\omega_n, x_n, \theta_n\}, \{r_j\} \rangle$, if $(q, \{r_j\})$ allow no arbitrage opportunities (and all individual demands satisfy the budget constraints at a binding level) then there exists, for each individual n, a strictly monotone and strongly concave utility function U_n such that $(p, q, \{x_n, \theta_n\})$ is an equilibrium for $\langle \{U_n\}, \{\omega_n\}, \{r_j\} \rangle$. This result (Kübler, 2003a) can be easily understood: when only one $\{\omega_n\}$ is observed, revealed preference lacks all power.[13] The fact that we are considering a dynamic problem turns out to be irrelevant, given that it is not being used to restrict the class of preferences.

The main contribution of Kübler (2003a) is to show that the result holds because the class of preferences allowed in the rationalization is very broad. Indeed, when economists deal with dynamic problems under uncertainty it is customary to make assumptions on preferences that restrict, usually in a great manner, the class of preferences of individuals. Kübler's results are strong in the sense that they do not assume the observation of any individual data, not even nominal income. He finds restrictions on the process that prices may follow, given observed processes for returns and aggregate endowments. Kübler's test requires observation of whole stochastic processes, the values of all necessary variables on and off the realized path.

Definition 6. A utility function $U : \mathbb{R}_+^{\ell X} \to \mathbb{R}$ is a time-separable, expected-utility function if there exist functions $v : \mathbb{R}_+^{\ell} \to \mathbb{R}$, $\pi : \Xi \to [0, 1]$ and $\beta : \Xi \to \mathbb{R}_{++}$ such that:

- v is strictly increasing, strongly concave, differentiable on $\mathbb{R}_{++}^{\ell}$ and has interior contour sets;
- for every $\xi \in \Xi \backslash \Xi^{\mathrm{T}}$, $\sum_{\xi' \in F(\xi)} \pi(\xi') = 1$;
- for every consumption path x, there exists a function $V : \Xi \to \mathbb{R}$, such that $\forall \xi \in \Xi^{\mathrm{T}}$, $V(\xi) = v(x(\xi))$; $\forall \xi \in \Xi \backslash \Xi^{\mathrm{T}}$, $V(\xi) = v(x(\xi)) + \beta(\xi) \sum_{\xi' \in F(\xi)} \pi(\xi') V(\xi')$; and $U(x) = V(\xi_0)$.

Here v represents the state-independent instantaneous utility, π represents conditional probabilities for the nodes of the tree, given their predecessors, β represents

[13] Formally, Kübler (2003a) uses Farkas' lemma (Rockafellar, 1970, Theorem 22.3.1) recursively to show that if $(q, \{r_j\})$ allow no arbitrage, then there exists a function $\lambda : \Xi \to \mathbb{R}_{++}$ such that $q(\xi) = \sum_{\xi' \in F(\xi)} \lambda(\xi')(q(\xi') + r(\xi'))$ for every $\xi \in \Xi$. Then, this λ can be used to construct strongly concave and strictly monotone utility functions, whose gradients with respect to consumption of commodities are $\lambda(\xi) p(\xi)$, when evaluated at $x_n(\xi)$, at all ξ.

discounting and V, which can be described as "utility levels at nodes", is defined recursively.

Theorem 7 (Kübler, 2003a). *Let a joint process* $\langle p, q, \{r_j\}, \omega\rangle$ *be given, where* p, q, and $\{r_j\}$ *are defined as before and* $\omega : \Xi \to \mathbb{R}^{\ell}_{+}$ *represents aggregate endowments of commodities. There exist individual preferences* $\{U_n\}$*, all of which are time-separable, expected-utility functions, such that prices* (p, q) *are equilibrium prices of the economy* $\langle\{U_n, \omega_n\}, \{r_j\}\rangle$*, for some distribution* $\{\omega_n\}$ of ω*, if and only if, for each individual n there exist* $x_n : \Xi \to \mathbb{R}^{\ell}_{+}$, $V_n : \Xi \to \mathbb{R}$, $\lambda_n : \Xi \to \mathbb{R}_{++}$, $\pi_n : \Xi \to [0, 1]$ *and* $\beta_n : \Xi \to \mathbb{R}_{++}$ *such that*:[14]

- $\sum_{\xi' \in F(\xi)} \pi(\xi') = 1$*, for every n and* $\xi \in \Xi \backslash \Xi^{\mathrm{T}}$;
- $q(\xi)\lambda_n(\xi) = \beta_n(\xi) \sum_{\xi' \in F(\xi)} \pi_n(\xi')\lambda_n(\xi')(q(\xi') + r(\xi'))$*, for every n and* $\xi \in \Xi \backslash \Xi^{\mathrm{T}}$;
- $V_n(\xi) \leqslant V_n(\xi') + \lambda_n(\xi') p(\xi') \cdot (x_n(\xi) - x_n(\xi'))$*, for every n* and $\xi, \xi' \in \Xi$*; with strict inequality if* $(x_n(\xi) \neq x_n(\xi'))$;[15]
- $\sum_{n=1}^{N} x_n(\xi) = \omega(\xi)$*, for every* $\xi \in \Xi$.

Of course, the theorem alone does not make the case for falsifiability: there is, in principle, the possibility that the quantified variables always exist, which would render the equilibrium hypothesis unfalsifiable. The following example shows that it is not the case.

Example 2 (Kübler, 2003a). Suppose that $\ell = 1$, $J = 3$ and $X = 5$. Suppose further that all nodes but the root are terminal (as if there are two time periods, the second of which is the end of the horizon): $\Xi = \{\xi_0, \xi_1, \xi_2, \xi_3, \xi_4\}$, $\Xi^{\mathrm{T}} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$. For simplicity, since we will necessarily have that $q(\xi) = (0, 0, 0)$ for all nodes other than the root, we will denote $q = q(\xi_0)$. Suppose that the ranges for probabilities and discounts are restricted so that the following matrix is known:

$$\Delta = \begin{bmatrix} \beta_n(\xi_0)\pi_n(\xi_1)r_1(\xi_1) & \beta_n(\xi_0)\pi_n(\xi_1)r_2(\xi_1) & \beta_n(\xi_0)\pi_n(\xi_1)r_3(\xi_1) \\ \beta_n(\xi_0)\pi_n(\xi_2)r_1(\xi_2) & \beta_n(\xi_0)\pi_n(\xi_2)r_2(\xi_2) & \beta_n(\xi_0)\pi_n(\xi_2)r_3(\xi_2) \\ \beta_n(\xi_0)\pi_n(\xi_3)r_1(\xi_3) & \beta_n(\xi_0)\pi_n(\xi_3)r_2(\xi_3) & \beta_n(\xi_0)\pi_n(\xi_3)r_3(\xi_3) \\ \beta_n(\xi_0)\pi_n(\xi_4)r_1(\xi_4) & \beta_n(\xi_0)\pi_n(\xi_4)r_2(\xi_4) & \beta_n(\xi_0)\pi_n(\xi_4)r_3(\xi_4) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

[14] Formally, Kübler requires that the ranges of π_n and β_n be restricted, respectively, to a strict subset of $[0, 1]$ and a bounded subset of $\mathbb{R}_{++}$.

[15] When $\ell = 1$, this condition may be replaced by its equivalent: for every n and $\xi, \xi' \in \Xi$, $(x_n(\xi) - x_n(\xi'))(\lambda_n(\xi) - \lambda_n(\xi')) \leqslant 0$, with strict inequality if $(x_n(\xi) \neq x_n(\xi'))$ (see Rockafellar, 1970, Theorem 24.8).

(Notice that R is common to all individuals. This assumes some homogeneity.) Let $q = (3, 2, 4)$.[16] For any $\gamma \in \mathbb{R}^4_{++}$ such that $q = \Delta^\top \gamma$, it is true that $\gamma_1 = 1 + 2\gamma_2$. Now, take any ω such that $\omega(\xi_1) > \omega(\xi_2)$. By the fourth condition in the theorem, it must be that for some n, $x_n(\xi_1) > x_n(\xi_2)$. Suppose that $\lambda_n : \Xi \to \mathbb{R}_{++}$ is such that the first condition is satisfied. Let

$$\gamma = \begin{bmatrix} \dfrac{\lambda(\xi_1)}{\lambda(\xi_0)} \\ \dfrac{\lambda(\xi_2)}{\lambda(\xi_0)} \\ \dfrac{\lambda(\xi_3)}{\lambda(\xi_0)} \\ \dfrac{\lambda(\xi_4)}{\lambda(\xi_0)} \end{bmatrix}$$

Since $q = \Delta^\top \gamma$, it follows that $\lambda(\xi_1)/\lambda(\xi_0) = \gamma_1 = 1 + 2\gamma_2 > \gamma_2 = \lambda(\xi_2)/\lambda(\xi_0) > 0$, which implies that $\lambda(\xi_1) > \lambda(\xi_2)$ and, therefore, $(x_n(\xi) - x_n(\xi'))(\lambda_n(\xi) - \lambda_n(\xi')) > 0$ which is a violation of the last condition of the theorem.

The message therefore is the following: when the class of preferences that is allowed is “large” there are no restrictions, even if one observes individual consumptions; when the class is restricted so that only time-separable expected-utility functions are allowed, the general equilibrium hypothesis can be rejected on the basis of aggregate variables alone. Even natural enlargements of the class of preferences destroy almost all empirical restrictions. In particular, if one allows an aggregator of instantaneous utilities more general than the additive one required by expected utility,[17] then the restrictions of the previous theorem disappear (see Theorem 4.1 in Kübler, 2003a). Kübler (2003b) examines the restrictions imposed by Kreps-Porteus preferences, an alternative to time-separability, and finds similar results: there are no testable restrictions, even on individual-level data, without additional assumptions.

[16] To see that $(q, \{r_1, r_2, r_3\})$ allow no arbitrage opportunities, notice that

$$\Delta^\top \begin{bmatrix} 1.2 \\ 0.1 \\ 0.9 \\ 0.4 \end{bmatrix} = q$$

and using footnote 13 and $\beta_n(\xi_0)\pi_n(\xi) > 0$, we have

$$\begin{bmatrix} \beta_n(\xi_0)\pi_n(\xi_1)(1.2) \\ \beta_n(\xi_0)\pi_n(\xi_2)(0.1) \\ \beta_n(\xi_0)\pi_n(\xi_3)(0.9) \\ \beta_n(\xi_0)\pi_n(\xi_4)(0.4) \end{bmatrix} \in \mathbb{R}^4_{++}$$

[17] That is, if Definition 6 were relaxed to require that for some increasing and concave $f : \mathbb{R}^2 \to \mathbb{R}$, $\forall \xi \in \Xi^{\mathrm{T}}$, $V(\xi) = f(v(x(\xi)), 0)$, and $\forall \xi \in \Xi \backslash \Xi^{\mathrm{T}}$, $V(\xi) = f(v(x(\xi)), \sum_{\xi' \in F(\xi)} \pi(\xi')V(\xi'))$.

The existence of nonvacuous testable restrictions also depends crucially on assumptions concerning the stochastic structure of the individual income process. Krebs (2003) strengthens results by Constantinides and Duffie (1996) to show that in the presence of idiosyncratic individual risk there are no restrictions on either macro data or on the first N moments of the cross-sectional distribution of consumption growth, even when the one-period utility functions are known.

2.5. Public goods

A model of public competitive equilibrium Foley (1967, 1970) is used here. Consumers behave competitively with respect to private goods. The public sector chooses public goods and lump-sum taxes to finance them such that there is no other public sector proposal that would be preferred by every individual. Foley proves that public competitive equilibria are Pareto optimal, and that there exist prices and taxes such that, given endowments, a Pareto optimal allocation is a public competitive equilibrium.

Formally, each consumer n has preferences over public and private goods represented by a continuous, strictly monotonic, strictly quasi-concave utility function $U_n(x_n, y_n)$, $x_n \in \mathbb{R}^\ell_+$ (private goods), $y_n \in \mathbb{R}^K_+$ (public goods). There is no free disposal: $y_n = y$ for $n = 1, \dots, N$. These assumptions will be critical for finding nonvacuous restrictions, in contrast to Chiappori's (1990) findings on aggregate demand for public goods.

Consumers have endowments of private goods $\omega_n \in \mathbb{R}^\ell_+$. These endowments can be used to produce public goods and private goods, with net output of private goods represented by $z = x - \omega$.

Production technology is assumed to be constant returns to scale. The set of all technically possible production plans is $\mathcal{Z}$, a closed, convex cone. A feasible allocation is a set of vectors $(\{x_n\}, y)$ such that $(x - \omega, y) \in \mathcal{Z}$.

There exists a government that can purchase public goods for the use of the economy's members and that also has the power to tax members to pay for the public goods and to redistribute income. Each consumer makes a tax payment (or receives a subsidy) of τ_n.

It is straightforward to show that the equilibrium conditions of the model are the same as the equilibrium conditions of a Lindahl equilibrium with transfers. Without transfers, each individual must pay taxes based on their marginal benefit of the public good, the Lindahl price: q_n. In the following definition, note the distinction between monetary transfers (τ_n) and full income transfers ($T_n = q_n y_n - \tau_n$).

Definition 7. A public competitive equilibrium is a feasible allocation $(\{x_n\}, y)$, prices $(p, q) \gg 0$, and taxes $\{\tau_n\}$ in which:

- each consumer solves the problem: $\max_{(x_n, y_n)} U_n(x_n, y_n)$ such that $p\, x_n + q_n y_n \le p\, \omega_n + T_n$;
- producers solve the problem: $\max_{(z,y)} (p, q) \cdot (z, y)$ such that $(z, y) \in \mathcal{Z}$;
- the government chooses $\{\tau_n\}$ such that: $q\, y = \sum_{n=1}^N \tau_n$, and $\tau_n = q_n y_n - T_n$, for $n = 1, \dots, N$;
- public good restrictions are satisfied: $y_n = y$, for $n = 1, \dots, N$, and $\sum_{n=1}^N q_n = q$.

Snyder (1999) describes the model's complete set of testable propositions on finite data in the form of a finite set of polynomial inequalities in observed and unobserved variables. Thus, the data that satisfy the restrictions form a semialgebraic set. For two observations of a two agent economy (labelled a and b), the restrictions can be written in terms of the observables alone as follows.

Theorem 8 (Snyder, 1999). *Let the collection* $\langle x^r, y^r, p^r, q^r, \omega^r, I_a^r, I_b^r, \tau^r\rangle$ *of non-negative vectors of variables be given for* $r = 1, 2$. *Let* $\mathcal{D}_1 = \langle x^r, y^r, p^r, q^r, I_a^r, I_b^r\rangle$, $\mathcal{D}_2 = \langle x^r, y^r, p^r, q^r, \omega^r\rangle$, *and* $\mathcal{D}_3 = \langle y^r, p^r, q^r, \omega^r, I_a^r, I_b^r, \tau^r\rangle$. *Then, there exist continuous, strictly monotonic, strictly concave utility functions* $\{U_n\}$ *and a closed convex conical, negative monotonic production set* $\mathcal{Z}$ *such that this data is consistent with a series of Public Competitive Equilibria for the economy* $\langle\{U_n\}, \mathcal{Z}, \{\omega_n^r\}_{r=1}^R\rangle$, *if and only if*:

- $\mathcal{D}_1$ *satisfy the following conditions*: *for some* $r, s = 1, 2$, $r \neq s$, *either*, $[H_a^{rs} > I_a^r$, $H_b^{rs} > I_b^r$ *and* $(p^r, q^r) \cdot (x^s - x^r, y^s - y^r) > 0]$, or, $[H_a^{rs} > I_a^r$, $H_b^{sr} > I_b^s]$, *where*, $H_n^{rs} = \max_{\nu,\mu} p^r \nu + \mu \cdot (y^s - y^r)$ *s.t.* $p^s \nu = I_n^s, 0 \leq \nu \leq x^s, 0 \leq \mu \leq q^r$;
- $\mathcal{D}_2$ *satisfy profit maximization and technology restrictions*: $0 = (p^r, q^r) \cdot (x^r - \omega^r, y^r) \geq (p^r, q^r) \cdot (x^s - \omega^s, y^s)$, *for* $r, s = 1, 2$, $r \neq s$; *and if* $x^r - \omega^r \neq 0$, *then* $x^r - \omega^r \ngeq 0$, *for* $r = 1, 2$.
- $\mathcal{D}_3$ *satisfies the following constraints for* $r = 1, 2$: $\tau^r = q^r y^r$, *and* $I_a^r + I_b^r = p^r \omega^r - \tau^r$.

The restrictions are nonvacuous, as can be shown in the following counterexample, which relates only to the demand side of the model. Fig. 2 has the same interpretation as Fig. 1, but now with one private good and one public good. The agents have identical incomes, thus the picture could represent either consumer's constraints. We know the consumption vector

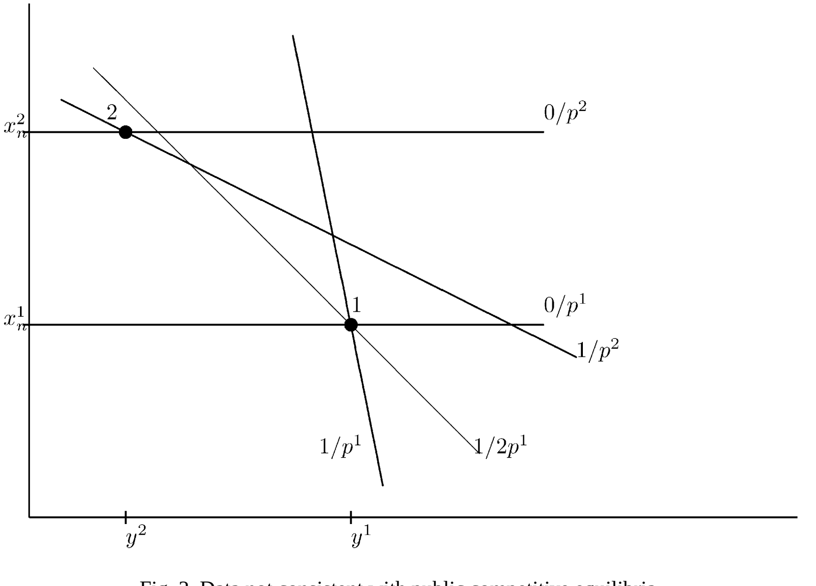

Fig. 2. Data not consistent with public competitive equilibria.

of each consumer each period, marked 1 and 2. We know the price of the public good and the private good, but we do not observe each agent's Lindahl price, thus we do not know the slope of the budget constraint each agent faces. Here the price of the public good is one, thus each Lindahl price is in the range (0, 1), and together they sum to 1.

The limit-cases of the budget constraints are drawn in the figure. Note that for all feasible Lindahl prices, the consumer's period 2 consumption is revealed preferred to her period 1 consumption. Thus, the only way this consumer can satisfy WARP is if we choose the Lindahl price in period 1 such that her period 1 consumption is not revealed preferred to her period 2 consumption. For example, a Lindahl price close to 0 would satisfy this requirement. This would imply the other consumer has a Lindahl price close to 1, however, which means that consumer would fail WARP. With the data illustrated here, a Lindahl price in period 1 of 1/2 is too high—at this price, period 1 consumption is revealed preferred to period 2 consumption. But because the Lindahl prices must sum to 1, this means one consumer will definitely fail WARP.

2.6. *Externalities*

Consider an economy with $N \geqslant 2$ consumers and $\ell + 1$ commodities, which must be consumed in nonnegative amounts. There exist externalities in the sense that each individual is affected not only by his own consumption, but also by the consumption of one of the commodities by all the other consumers. Formally, for each individual n, denote by $(x_n, y_n) \in \mathbb{R}^\ell_+ \times \mathbb{R}_+$ his own consumption and by $y_{-n} \in \mathbb{R}^{N-1}_+$ the consumption of the last commodity by the rest of the consumers. Each individual n has preferences represented by $U_n : \mathbb{R}^\ell_+ \times \mathbb{R}_+ \times \mathbb{R}^{N-1}_+ \to \mathbb{R}$.

Given the set of agents, an economy is described by preferences, $\{U_n\}$, endowments of commodities other than the externality, $\{\omega_n\}$, and endowments of the externality, $\{\kappa_n\}$, all of which are assumed to be strictly positive. Given an economy, $\langle\{U_n\}, \{w^r_n, \kappa^r_n\}\rangle$, a Nash–Walras equilibrium is a vector $(p, q, \{x_n, y_n\})$ such that, for each n, (x_n, y_n) maximizes $U_n(x, y, y_{-n})$ subject to $p \cdot x + qy \leqslant p \cdot \omega_n + q\kappa_n$, and all markets clear, i.e., $\sum_{n=1}^N (x_n, y_n) = \sum_{n=1}^N (\omega_n, \kappa_n)$.

The set of Nash–Walras equilibria of economy $\mathcal{E}$ is denoted by $NW(\mathcal{E})$. Carvajal studies whether or not the concept of Nash–Walras equilibrium imposes testable restrictions on equilibrium prices. Formally, let $\mathrm{NWPE}(\mathcal{E})$ denote the projection of the set of Nash–Walras equilibria of economy $\mathcal{E}$, $NW(\mathcal{E})$, into the space of prices and demands for the externality and let $NWP(\mathcal{E})$ denote the projection of the same set into the space of prices only.

Definition 8. Suppose that one observes a finite data set of prices, endowments and demands for the externality, $\langle p_r, q_r, \{\omega_{n,r}, \kappa^r_n, y^r_n\}\rangle$.[18] Such set is consistent with Nash–Walras equilibrium if for each n there exists U_n, continuous and satisfying that, everywhere in y_{-n}, $U_n(\cdot, \cdot, y_{-n})$ is strictly monotone and strongly concave, such that for each r, $(p_r, q_r, \{y^r_n\}) \in \mathrm{NWPE}(\langle\{U_n\}, \{w^r_n, \kappa^r_n\}\rangle)$. Suppose now that demands for the externality are not observed.

[18] Assuming that for each r, all prices are strictly positive, $\sum_n y^r_n = \sum_n \kappa^r_n$ and for each n, endowments are strictly positive, consumption of the externality is nonnegative and the budget constraint is not violated: $p_r \cdot \omega^r_n + q_r(\kappa^r_n - y^r_n) \geqslant 0$.

The data set $\langle p_r, q_r, \{\omega_n^r, \kappa_n^r\}\rangle_{r=1}^R$ is *consistent with Nash–Walras equilibrium* if $\{U_n\}$ with the same properties exist, such that $(p_r, q_r) \in NWP(\langle\{U_n\}, \{w_n^r, \kappa_n^r\}\rangle)$.

Carvajal's main result is that, unless further conditions are imposed, restrictions of the kind obtained by Brown and Matzkin (1996) do not survive the introduction of externalities.

Theorem 9 (Carvajal, 2003a). *When individual consumptions of the externality are observed, a data set $\langle p_r, q_r, \{\omega_n^r, \kappa_n^r, y_n^r\}\rangle$ is consistent with Nash–Walras equilibrium if and only if there exist $x_n^r \in \mathbb{R}_+^\ell$, $V_n^r \in \mathbb{R}$, and $\lambda_n^r \in \mathbb{R}_{++}$ such that*:

- *for each n and r, $p_r \cdot x_n^r = p_r \cdot \omega_n^r + q_r(\kappa_n^r - y_n^r)$;*
- *for each n and for each pair r and r' such that $y_{-n}^r = y_{-n}^{r'}$, $V_n^{r'} \leq V_n^r + \lambda_n^r p_r \cdot (x_n^{r'} - x_n^r) + \lambda_n^r q_r(y_n^{r'} - y_n^r)$, with strict inequality whenever $(x_n^{r'}, y_n^{r'}) \neq (x_n^r, y_n^r)$;*
- *for each r, $\sum_n x_n^r = \sum_n \omega_n^r$.*

Moreover, given a vector $d = (\{w_n^r, \kappa_n^r, y_n^r\})_{r=1}^R \in ((\mathbb{R}_{++}^\ell \times \mathbb{R}_{++} \times \mathbb{R}_+)^N)^R$ there exists a semialgebraic set P, a subset of $(\mathbb{R}_{++}^\ell \times \mathbb{R}_{++})^R$, such that $(p_r, q_r)_{r=1}^R \in P$ if and only if $\langle p_r, q_r, \{\omega_n^r, \kappa_n^r, y_n^r\}\rangle$ is consistent with Nash–Walras equilibrium. The set P may be a proper subset of $(\mathbb{R}_{++}^\ell \times \mathbb{R}_{++})^R$.

Also, if individual demands for the externality are not observed, every data set $\langle p_r, q_r, \{\omega_n^r, \kappa_n^r\}\rangle_{r=1}^R$ is consistent with Nash–Walras equilibrium.

In the second part of the theorem, the fact that the semialgebraic set P exists is obvious from the Tarski–Seidenberg theorem, except that it may be proper on $(\mathbb{R}_{++}^\ell \times \mathbb{R}_{++})^R$. Again, as in Brown and Matzkin (1996), this results is obtained by means of an example. In fact, one can easily find one such example using the one proposed by Brown and Matzkin, as presented in Section 2.2 here: suppose that endowments of commodities other than the externality are as in Fig. 1, that endowments of the externality are given such that the aggregate amount is constant, and that observed individual demands are such that agent 2 consumes all the externality. Prices are such that the projections of the budget sets for individual 1 are like the lines depicted in Fig. 1 and would be inconsistent with Nash–Walras equilibrium as agent 1 could not satisfy WARP given nonnegativity of consumption.

The example and, of course, the conditions themselves, however, show that the restrictions imposed by the equilibrium concept are extremely mild, since the conditioning of revealed-preference restrictions to the actions of other consumers severely weakens the predictive power of individual rationality. In a sense, these restrictions have zero measure: the probability that randomly generated data are inconsistent with the equilibrium concept is zero, whenever the measures used in the generation are nonatomic. From a practical perspective, the restrictions are so weak that the theory appears unfalsifiable.

Moreover, Carvajal (2003a) exploits this weakness to argue the last part of the theorem. When individual consumptions of the externality have not been observed, one cannot reject the hypothesis that for every pair of observations at least two individuals changed their demand for the externality (the argument given for this is constructive). Individual rationality becomes vacuous and preferences that rationalize the data set always exist.

Carvajal's results, in fact, hold for a more general case, namely strategic externalities, where the action that affects other people's well being need not be consumption and, hence, is not subject to budgetary considerations or aggregation. For this, it is assumed that these actions have compact domains and that preferences satisfy one additional smoothness condition (Lipschitz continuity).

All these results are subject to what class of preferences consumers are allowed to be endowed with. Under further conditions imposed on this class, the restrictions obtained in the first part of the theorem continue to be necessary, but sufficiency may fail, in which case neither the observation that these restrictions are weak or the unfalsifiability of the equilibrium hypothesis when individual demands are not observed need to be true. Indeed, Carvajal (2003a) exhibits an instance of this: if weak separability of preferences on the bundle of commodities other than the externality is imposed, restrictions a la Brown–Matzkin are imposed by the Nash–Walras equilibrium concept.

2.7. Household behavior

A household is made up of two individuals, a and b. For $n = a, b$, each member can supply some amount of labor, l_n, in a market outside the household. Let T represent the fixed amount of total time available to each a and b; $L_n = T - l_n$ defines leisure consumption for each individual. There is also a privately consumed good X; let x_n, a non-negative number, denote each member's consumption of the good. The price of the consumption good is normalized to one. Consumption and labor choices are made given nonzero wages, w_a, w_b, and non-labor household income I.

Assume first that each agent has preferences over only their own personal consumption. Each agent has preferences representable by a nonsatiated utility function $U_n(L_n, x_n)$.

The exact mechanism for determining household consumption is left unspecified. Let an *income-sharing rule* be some function η: $\eta(w_a, w_b, I) = (I_a, I_b)$ such that $I_a + I_b = I$. Then an allocation is Pareto efficient given $\{U_a, U_b, \eta\}$ if each agent maximizes utility subject to a budget constraint $x_n \leq I_n + w_n l_n$.

Theorem 10 (Chiappori, 1988). *Let $\langle X^r, I^r, \{w_n^r, l_n^r\}\rangle_{r=1}^R$ be given.*[19] *Then there exist strictly monotonic, strictly concave utility functions $\{U_n\}$ and an income-sharing rule $\{\eta\}$ such that the data are consistent with a Pareto optimal allocation within the household $\langle U_a(L_a, x_a), U_b(L_b, x_b)\rangle$ iff there exist numbers $\{x_n^r, I_n^r\}$ such that*:

- *household aggregation conditions are satisfied*: $x_a^r + x_b^r = X^r$, *and* $I_a^r + I_b^r = I^r$;
- *individual budget constraints are satisfied*: $x_n^r = I_n^r + w_n^r l_n^r$ for $n = a, b$;
- *individuals satisfy the Strong Axiom of Revealed Preference* (*SARP*);
- $x_a^r \geq 0, x_b^r \geq 0$.

SARP here is defined in terms of choosing the consumption good and leisure given full income. For two observations, SARP is: $x_n^2 + w_n^1 L_n^2 > x_n^1 + w_n^1 L_n^1$ or $x_n^1 + w_n^2 L_n^1 > x_n^2 + w_n^2 L_n^2$.

[19] Assume that $\{I^r, w_a^r, w_b^r\} \neq \{I^s, w_a^s, w_b^s\}$ for any $r \neq s$.

Chiappori shows that these are nonvacuous tests. The tests are in the form of solving a linear programming problem. Snyder (2000) derives conditions over the observable variables for the two-observation case by applying Fourier–Motzkin elimination. Fourier–Motzkin elimination (sometimes called Fourier elimination) is a technique similar to Gaussian elimination that can be applied to linear problems (see Dantzig and Eaves, 1973). In general, it is doubly-exponential in computational time, but can be useful for small problems such as this.

Theorem 11 (Snyder, 2000). *Let* $\langle X^r, I^r, \{w_n^r, \ell_n^r\}\rangle$ for $r = 1, 2$ *be given. Then there exist strictly monotonic, strictly concave utility functions* $\{U_n\}$ *and income-sharing rule* $\{\eta\}$ *such that the data are consistent with a Pareto optimal allocation within the household* $\langle U_a(L_a, x_a), U_b(L_b, x_b)\rangle$ *iff the collective rationality nonparametric restrictions are satisfied*:

- $\forall r = 1, 2,\ X^r = I^r + w_a^r \ell_a^r + w_b^r \ell_b^r$;
- $\exists r, s = 1, 2,\ r \neq s$ *such that either* $(X^s + w_a^r L_a^s > w_a^r L_a^r$ *and* $X^r + w_b^s L_b^r > w_b^s L_b^s)$ *or* $(X^s + w_a^r L_a^s + w_b^r L_b^s > X^r + w_a^r L_a^r + w_b^r L_b^r$ *and* $X^s + w_a^r L_a^s > w_a^r L_a^r$ *and* $X^s + w_b^r L_b^s > w_b^r L_b^r)$.

These restrictions are analogous to WARP as used to test the unitary model of utility maximization. Snyder applies both these tests to data from the National Longitudinal Survey and finds almost all households satisfy both tests—that is, there exist preferences such that households could be acting according to Chiappori's collective rationality model, but there also exist preferences such that households are acting as unitary rational agents.

One could interpret these results as satisfaction of specification tests: given that the data satisfy the collective rationality restrictions, one could proceed to further work with this data that makes functional form assumptions and estimates sharing rules within the household (see Diewert and Parkan, 1983). However, these results once again illustrate a general problem for nonparametric tests: while they may be nonvacuous, they are perceived to be very weak. There is no general theory of the power of nonparametric tests, though Bronars (1987) and Manser and McDonald (1988) suggest possible approaches. There is no stochastic element built in to these tests, thus it is not clear how to interpret rejections when we do see them. Varian (1985, 1990) discusses the interpretation of rejections when the stochastic element is assumed to be measurement error. Brown and Matzkin (1998) address estimation issues that arise in a random utility model in a nonparametric framework.

Chiappori (1988) also derives tests for the case when agents' utility depends on the other household member's consumption or leisure. In this model, individual consumption and leisure become public goods within the household. The tests are now in the form of solving a bilinear programming problem: both Lindahl prices and the public good levels (in particular, individual consumptions) are unobserved. He provides a counterexample to show that the conditions are nonvacuous. Snyder (1999) derives testable restrictions over the observables alone for two observations when agents are non-egoistic over individual consumption (while leisure remains a private good).

3. Identification

One interpretation of Brown and Matzkin's results is that some of the structure imposed by individual rationality is preserved upon aggregation. An ambitious task is the one of determining whether or not all the structure imposed by individual rationality is preserved upon aggregation; can one uniquely (up to observational equivalence) recover the preferences of all individuals from the observation of aggregate equilibrium outcomes? To address this issue of identification, we no longer assume finite data sets; we rather take complete functions or correspondences as given.

A first solution to this problem is provided by Balasko (1999), under the assumption that the whole equilibrium manifold is known. Without any assumptions on the topological properties of this set,[20] Balasko exploits the fact that individual demand depends on endowments only via the value of income, and shows that one can recover the aggregate demand function from the manifold.

A criticism of Balasko's approach is that knowledge of the whole manifold (in particular of its boundary, where there is information about individual decisions) may be too much to ask. Chiappori and Ekeland (1999a), Chiappori et al. (2000, 2002) and Kübler et al. (2002) solve the identification problem via differential analysis of relatively open subsets of the equilibrium manifold. These papers also build upon the insight of Brown and Matzkin that the variations in individual endowments lead to restrictions. As in Balasko, the main result of this approach is that, in general, the answer to the identifiability question is positive: under certain assumptions, one can uniquely recover (locally) individual preferences from local knowledge of the equilibrium manifold.

Three ideas are behind this result. First, suppose that, following Brown–Matzkin, one studies aggregate excess demand as a function of prices and individual endowments (or, prices, income distributions and aggregate endowments). Then, when one changes the endowment of only one individual, given that only his demand is to change, one can identify his income effect. This, by itself, is an obvious source of tests of the general equilibrium hypothesis. Second, under the assumption that income effects are significant enough, in the sense that they allow for distinction between commodities, one can use them to identify the individual demands via the solution of a system of partial differential equations. By well-known results, this identifies preferences up to ordinal equivalence. Third, if the equilibrium manifold only, and not the aggregate excess demand function, is known, the problem is more complicated; however, the previous ideas still work as a manifold can only be an equilibrium manifold if there exists an aggregate excess demand function that vanishes at every point of the manifold. Hence, (local) identification of individual income effects is still possible.

All the above results are obtained in the standard framework of an exchange economy under certainty. Using identification results from Geanakoplos and Polemarchakis (1990), Kübler et al. (2002) show that even with uncertainty and incomplete markets, one can identify individual preferences from the equilibrium correspondence, under the assumption on the significance conditions of income effects for commodities and also for assets (a

[20] For this reason, it is more accurately called the graph of the equilibrium correspondence.

condition that turns out to be stronger in this case than in the standard framework of exchange economies).

In what follows, we discuss the above mentioned literature in a greater detail. Section 3.1 introduces the results on global identification, while Section 3.2 develops the main results in the literature on local identification. The results for economies with uncertainty and incomplete markets are then discussed in Section 3.3.

3.1. Global identification

Balasko (1999) analyzes identification using the following logic. Characterize aggregate demand as a function of normalized prices and the profile of individual incomes. Then, for each pair consisting of a price vector and a profile of individual incomes, one only needs to find a point in the manifold that has that price vector and for which the profile of endowments generates that profile of individual incomes. The sum of those endowments is, by definition, equal to the value of the aggregate demand at the fixed prices and individual incomes.

Theorem 12 (Balasko, 1999). *The equilibrium manifold E identifies the aggregate demand function X.*

Notice that the above theorem determines aggregate demand but says nothing, at least explicitly, about individual demands.[21] The second result of Balasko shows that if one knows the aggregate demand function over the whole of $\mathcal{S}_+^{\ell-1} \times \mathbb{R}_+^N$, crucially including the boundary of $\mathbb{R}_+^N$, then one can identify each individual demand just by letting the incomes of all individuals but one, say the n-th, be fixed at zero, and evaluating the aggregate demand at all possible prices and incomes of n.

Theorem 13 (Balasko, 1999). *For each individual n, the aggregate demand function X identifies the individual demand function x_n.*

It has not been assumed that individual demands are derived from optimization; any process that generates demands satisfying homogeneity and Walras' law will be recoverable from the equilibrium manifold. However, if the process is derived from optimization, then one can recover individual preferences, up to ordinal equivalence, from the manifold, as individual demands are identifiable.

3.2. Local identification

Chiappori and Ekeland (1999a) consider the problem of the characterization of aggregate excess demand as a function of both prices and individual endowments. The question that they answer is whether or not one can uniquely recover individual demands, and hence

[21] Recall that nothing has been assumed about the topological properties of E. Balasko proves that if E is closed, then the associated aggregate demand function is continuous and that if E is smooth in the interior of $\mathcal{S}_+^{\ell-1} \times \left(\mathbb{R}_+^{\ell}\right)^N$, then the associated aggregate demand must be smooth in the interior of $\mathcal{S}_+^{\ell-1} \times \mathbb{R}_+^N$.

preferences, from the values that this function takes on the interior of its domain. To see that this problem is far from being trivial, consider the following example.

Example 3 (Chiappori et al., 2000). Suppose that $N = \ell = 2$ and that individual preferences are given by $U^1(x) = \ln(x_1 - \varepsilon) + \ln(x_2 + \varepsilon)$, and $U^2(x) = \ln(x_1 + \varepsilon) + \ln(x_2 - \varepsilon)$, where $\varepsilon = 0$ is possible. Individual endowments are assumed to be high enough, so each individual reaches the regions of $\mathbb{R}^2_{++}$ where his preferences are defined. Aggregate excess demand is then

$$Z(p, (\omega_1, \omega_2)) = \frac{1}{2}\begin{bmatrix} \dfrac{\sum_{n=1}^{2} p\omega_n}{p_1} \\ \dfrac{\sum_{n=1}^{2} p\omega_n}{p_2} \end{bmatrix} - \sum_{n=1}^{2} \omega_n$$

which does not depend on ε. In the interior of $\mathbb{R}^2_+$, one cannot distinguish between small enough values of ε (including $\varepsilon = 0$ as an irrefutable hypothesis).

It is necessary, therefore, to establish conditions on the preferences (demands) if a positive result is to be found. For this, suppose that one has a smooth function mapping normalized nonnegative prices and individual endowments into the space of commodities. As mentioned before, derivative of this function with respect to the income (endowments) of one individual identifies his income effect.

Formally, suppose that one observes a smooth aggregate excess demand function, which depends on prices and on endowments of all individuals, $Z : \mathcal{S}^{\ell-1}_+ \times (\mathbb{R}^\ell_+)^N \to \mathbb{R}^\ell$, defined as: $Z(p, \{\omega_n\}) = \sum_{n=1}^{N} x_n\,(p, p\cdot\omega_n) - \sum_{n=1}^{N} \omega_n$. By construction, the Jacobian of Z with respect to $\omega_{n'}$ identifies the income effects of agent n': $D_{\omega_{n'}} Z(p, \{\omega_n\}) = D_y x_{n'}(p, p \cdot \omega_{n'}) p^\top - \mathbb{I}$, where $\mathbb{I}$ denotes the identity matrix.

Lemma 1 (Chiappori and Ekeland, 1999a). *For every n', there exists a unique function $a_{n'} : \mathcal{S}^{\ell-1}_+ \times \mathbb{R}_+ \to \mathbb{R}^\ell$ such that $D_{\omega_{n'}} Z\,(p, \{\omega_n\}) = a_{n'}(p, p\cdot\omega_{n'}) p^\top - \mathbb{I}$, for each p and each $\{\omega_n\}$.*

Walras' law immediately imposes simple testable restrictions: by Engel aggregation, $p \cdot a_n(p, p\cdot\omega_n) = 1$. What is more important, though, is that these income effects and the conditions of homogeneity and Walras' law can be used to solve for an individual demand function that is unique up to a (restricted) function of prices only. That is, for individual n, a_n defines a system of partial differential equations whose solution identifies a zero-degree-homogeneous function $A_n : \mathcal{S}^{\ell-1}_+ \times \mathbb{R}_+ \to \mathbb{R}^\ell_+$, which satisfies $p \cdot A_n(p, I) = I$ and $x_n(p, I) = A_n(p, I) + B_n(p)$, for some zero-degree-homogeneous function $B_n : \mathcal{S}^{\ell-1}_+ \to \mathbb{R}^\ell$, for which $p \cdot B_n(p) = 0$.

The difficulty is that B_n need not be unique, and hence the previous equation still remains unidentified. However, under the null hypothesis of utility maximization, there must exist at least one such function B_n such that the individual demand satisfies the Slutsky restrictions: for every $j, k \in \{1, \ldots, \ell\}$, ignoring the arguments,

$$\frac{\partial A_{j,n}}{\partial p_k}+\frac{\partial B_{j,n}}{\partial p_k}+(A_{k,n}+B_{k,n})\frac{\partial A_{j,n}}{\partial I}=\frac{\partial A_{k,n}}{\partial p_j}+\frac{\partial B_{k,n}}{\partial p_j}+\left(A_{j,n}+B_{j,n}\right)\frac{\partial A_{k,n}}{\partial I}.$$

The only remaining problem, then, is whether one can pin down these B_n functions. Under the assumption of significance of the income effects, it is shown that these functions of prices can be identified from the Slutsky equation for the two commodities, which thereby identifies the individual demand. Note that first and second derivatives of the Slutsky equation with respect to income yields the system

$$\begin{bmatrix} \frac{\partial a_{j,n}}{\partial I}(p,I) & -\frac{\partial a_{k,n}}{\partial I}(p,I) \\ \frac{\partial^2 a_{j,n}}{\partial I^2}(p,I) & -\frac{\partial^2 a_{k,n}}{\partial I^2}(p,I) \end{bmatrix}\begin{bmatrix} B_{k,n}(p) \\ B_{j,n}(p) \end{bmatrix}=\begin{bmatrix} \gamma^n_{k,j}(p,I) \\ \frac{\partial \gamma^n_{k,j}}{\partial I}(p,I) \end{bmatrix}$$

where the function $\gamma^n_{k,j}:\mathcal{S}^{\ell-1}_{+}\times\mathbb{R}_{+}\to\mathbb{R}^2$ is known, as it depends on a_n but not on B_n. Now, assume that the following condition holds.

Condition 1 (Regularity). For every individual n, and vector price p, there exist $I\in\mathbb{R}_{+}$ and $j,k\in\{1,\ldots,\ell\}$ such that:

$$\begin{vmatrix} \frac{\partial^2 x_{j,n}}{\partial I^2}(p,I) & \frac{\partial^2 x_{k,n}}{\partial I^2}(p,I) \\ \frac{\partial^3 x_{j,n}}{\partial I^3}(p,I) & \frac{\partial^3 x_{k,n}}{\partial I^3}(p,I) \end{vmatrix}\neq 0$$

Then, one can solve for the function B_n in a unique manner, since

$$\begin{bmatrix} \frac{\partial a_{j,n}}{\partial I}(p,I) & -\frac{\partial a_{k,n}}{\partial I}(p,I) \\ \frac{\partial^2 a_{j,n}}{\partial I^2}(p,I) & -\frac{\partial^2 a_{k,n}}{\partial I^2}(p,I) \end{bmatrix}=\begin{bmatrix} \frac{\partial^2 x_{j,n}}{\partial I^2}(p,I) & -\frac{\partial^2 x_{k,n}}{\partial I^2}(p,I) \\ \frac{\partial^3 x_{j,n}}{\partial I^3}(p,I) & -\frac{\partial^3 x_{k,n}}{\partial I^3}(p,I) \end{bmatrix}$$

which identifies $B_{k,n}(p)$ and $B_{j,n}(p)$, whereas, for each $l\in\{1,\ldots\ell\}\backslash\{j,n\}$ one can find, from the Slutsky equations,

$$B_{l,n}(p)=\left(\frac{\partial a_{j,n}}{\partial I}(p,I)\right)^{-1}\left(\gamma^n_{l,j}(p,I)+\frac{\partial a_{l,n}}{\partial I}(p,I)B_{j,n}(p)\right)$$

Identification, of course, occurs if the result obtained for $B_n(p)$ is independent of the order in which its components are disentangled: that is, if there are more than one pair of commodities that satisfy regularity, the result should not depend on the order of these pairs used in the argument. This is guaranteed by a consistency condition imposed by Chiappori and Ekeland, which is necessary under the assumption that Z is the aggregate excess demand for some economy.

In order to determine the strength of this positive result, it only remains to establish the generality of the regularity assumption. Unless the demand system is really simplistic (rank 1), Chiappori and Ekeland argue that the assumption holds generically on prices and endowments, which suffices for identification, via continuity, at all points where it does not hold.

As it works with the aggregate excess demand function, this first approach by Chiappori and Ekeland (1999a) can be seen as a generalization of the Sonnenschein–Mantel–Debreu problem. This approach however does not avoid the criticism about the assumption that one can observe the aggregate excess demand everywhere in its domain, whereas under the hypothesis of general equilibrium one can only observe this function precisely where it vanishes.

This difficulty is overcome by focusing again on the equilibrium manifold. In Chiappori et al. (2000, 2002) this is done via the differential, local analysis of smooth manifolds.

Definition 9. A set $E \subseteq \mathcal{S}_{+}^{\ell-1} \times \mathbb{R}_{+}^{\ell N}$ is a smooth equilibrium manifold if for each n, there exists a smooth demand function $x_n : \mathcal{S}_{+}^{\ell-1} \times \mathbb{R}_{+} \to \mathbb{R}_{+}^{\ell}$, satisfying Walras' law, such that $E = \{(p, \{\omega_n\}) \in \mathcal{S}_{+}^{\ell-1} \times \mathbb{R}_{+}^{\ell N} : Z(p, \{\omega_n\}) = (0)_{l=1}^{\ell}\}$.

The problem now is whether one can uniquely recover the individual demands (preferences) from a smooth equilibrium manifold, without pegging some of the individual endowments at zero. Given the previous results, under the regularity assumption, it suffices to note that for each individual, the equilibrium manifold identifies the income effect $D_I x_n$, at least over some open subset of $\mathcal{S}_{+}^{\ell-1} \times \mathbb{R}_{++}^{\ell}$.

Theorem 14 (Chiappori et al., 2000). *Let $(\bar{p}, \{\bar{\omega}_n\})$ be a regular point of a smooth equilibrium manifold E. For each n, there exists a unique function a_n, defined on a neighborhood of $(\bar{p}, \bar{\omega}_n)$, such that $a_n(p, \omega_n) = D_I x_n(p, p \cdot \omega_n)$.*[22]

The point of this theorem is that for identification of $D_I x_n(p, p\cdot\omega_n)$, it suffices to show that $D_{\omega_n} Z_n(p, \omega_n)$ has rank $(\ell - 1)$.[23] For this, notice that $p^\top D_{\omega_n} Z_n(p, \omega_n) = p^\top (D_I x_n(p, p\cdot\omega_n) p^\top - \mathbb{I}) = 0$, while for q orthogonal to p, $D_{\omega_n} Z_n(p, \omega_n) q = (D_I x_n(p, p\cdot\omega_n) p^\top - \mathbb{I}) q = -q$. This implies that $rank(D_{\omega_n} Z_n(p, \omega_n)) = \ell - 1$.

Now, identification obtains as follows (given that $rank(D_{\omega_n} Z_n(p, \omega_n)) < \ell$):

- One can find a direction in which infinitesimal perturbations of ω_n, with prices fixed, do not change the excess demand of individual n.
- Since prices are fixed, individual excess demands for $n' \neq n$ do not change (only ω_n is being perturbed).
- If the fixed prices constitute equilibrium for the initial endowments, such perturbations must not (to a first order) take us out of the equilibrium manifold.
- We must be able to figure out one such direction of perturbation without knowing $D_I x_n$ or $D_{\omega_n} Z_n$. This is achievable from the manifold itself: one finds a direction of changes in ω_n in which the tangent space to the manifold is "flat" on prices (see Example 4 below).

[22] Specifically, function a_n has, as its domain, the projection of some open neighborhood of $(\bar{p}, \{\bar{\omega}_n\})$ into the space for (p, ω_n). It maps into $\mathbb{R}^\ell$. The last condition must hold for every (p, ω_n)in the domain of a_n.

[23] An alternative proof is given by Chiappori et al. (2002).

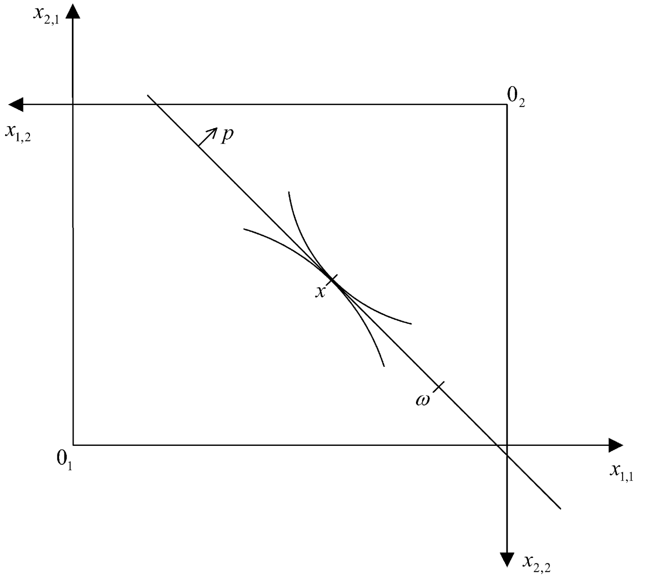

Fig. 3. Original equilibrium.

- This direction defines a linear subspace Δ_n, of dimension at least one, such that for $q \in \Delta_n$, $D_{\omega_n} Z_n(p, \omega_n)q = 0$. But, then, any $q \in \Delta_n \backslash \{\mathbf{0}\}$ implies $D_I x_n(p, p \cdot \omega_{n'}) = q/p^\top q$, where $p^\top q \neq 0$, because $q \neq \mathbf{0}$.[24]

The latter gives *a* solution for $D_I x_n(p, p \cdot \omega_n)$, but does not imply its uniqueness. However, since $rank(D_{\omega_n} Z_n(p, \omega_n)) = \ell - 1$, it follows that Δ_n is indeed one-dimensional, from where $D_I x_n(p, p \cdot \omega_n)$ is uniquely recovered.

Example 4. The original equilibrium is shown in Fig. 3: given endowments $\omega = (\omega_1, \omega_2)$, prices p clear markets. Now consider Fig. 4. Keeping prices at p, if one shocks the endowment of consumer 1 by $\mathrm{d}\omega_1$, so that ω_1' is the new endowment, his demand changes to x_1', but his excess demand remains constant: $x_1 - \omega_1 = x_1' - \omega_1'$. This defines the linear space Δ_1. If nothing changes for consumer 2, his excess demand also remains the same. Let $\omega' = (\omega_1', \omega_2)$. At ω', with prices p, the aggregate excess demand is the same as before: zero. Hence, as shown in Fig. 5, prices p are also equilibrium at endowments ω'.[25]

The problem then reduces to identifying individual demands from income effects. Under the assumptions that for no individual or commodity are income effects constant as income changes, and that for every individual there are two commodities for which the (pseudo-)elasticities of the derivatives of their income effects with respect to income differ, it is shown that individual demands can be uniquely recovered.

[24] To see this, notice that if $p^\top q = 0$, then, $\mathbf{0} = D_{\omega_n} Z_n(p, \omega_n)q = [D_I x_n(p, p \cdot \omega_n)p^\top - \mathbb{I}]q = -q$.

[25] This is just an illustration. Of course, this analysis should be differential.

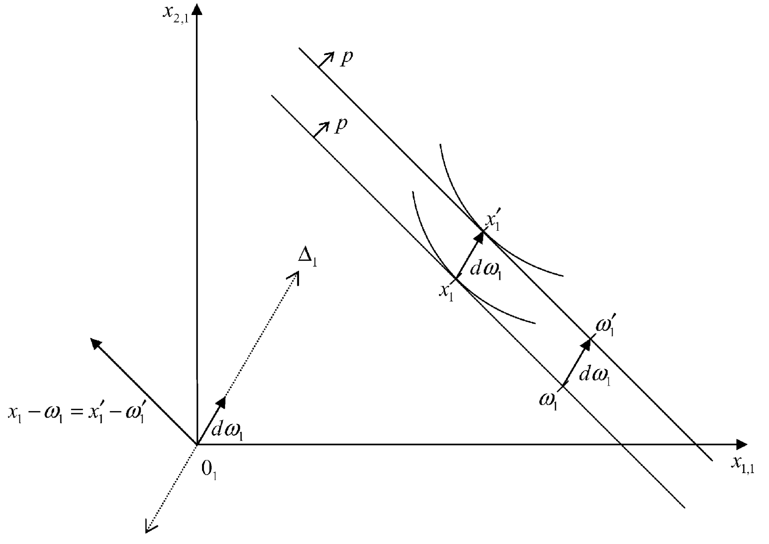

Fig. 4. Endowment shock.

The conclusion is, then, that the equilibrium manifold, in general, contains all the information implicit in the utility maximization problem; there is no need for the observation of individual decisions.

As this conclusion stands in sharp contrast with the wisdom that the profession derived from the Sonnenschein–Mantel–Debreu literature, it is worthwhile to determine what

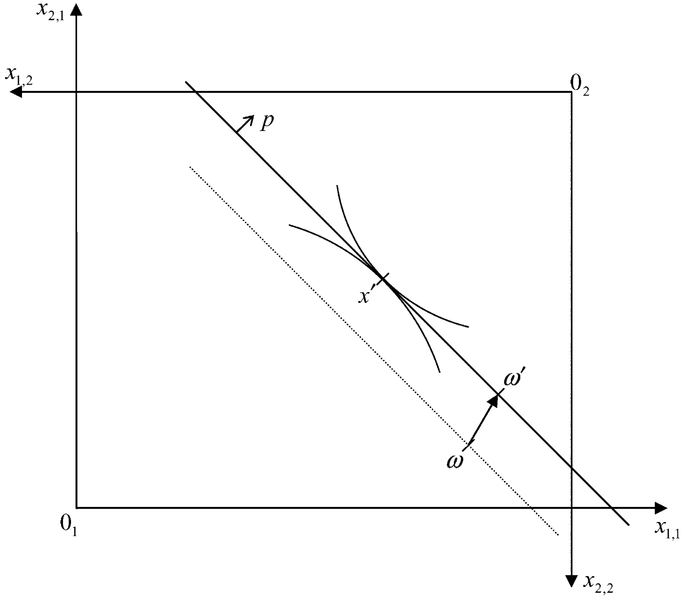

Fig. 5. New equilibrium.

drives the difference. The intuition obtained from Brown and Matzkin (1996) was that information on individual constraints imposes restrictions on aggregate variables, without observation of individual choices. Chiappori et al. (2002) confirm this intuition, using results from Chiappori and Ekeland (1999b): given a smooth function mapping aggregate endowments into prices, if the assumption that there are at least as many consumers as there are commodities is acceptable, then one can never reject the hypothesis that for some individual utility functions and perfectly egalitarian distribution of aggregate income, the function maps the aggregate endowment into equilibrium prices. Data on individual income appears then to be necessary for refutability of the equilibrium hypothesis.

3.3. Uncertainty

As before, there are N individuals and ℓ commodities, $\ell \geq 3$. However, there are now two periods, present and future; the future state of the world is uncertain and can be any one amongst S finitely many possibilities. No consumption takes place in the present period, but financial markets for J many assets are open, $J \geqslant 3$. In the future period, assets yield their returns and individuals obtain endowments of commodities, both of which may depend on the realized state of the world. Also, subject to the value of their wealths, individuals demand and consume commodities. At each state of the world, commodity 1 acts as a numeraire.

Formally, the return of asset j in state s is $r_{j,s}$ units of commodity 1. We denote $r_j = (r_{j,1}, \ldots, r_{j,s})^\top$, while $R_s = (r_{1,s}, \ldots, r_{J,s})$. It is assumed that there are no redundant assets, so that the vectors $r_1, \ldots, r_J$ are linearly independent (implicitly requiring that $J \leqslant S$). It is also assumed that the first asset return is positive ($r_1 > 0$) and that at every state of the world at least one asset has non-null yield ($R_s \neq 0$, for some s).

In the present period, individual n has no endowment of commodities, but is endowed with a vector $\phi_n \in \mathbb{R}^J$ of assets. In the future period, individuals receive no assets; and endowments of commodities in the future depend on the state of the world, i.e., in s, endowment of individual n is $\omega_{n,s} \in \mathbb{R}^\ell_{++}$.

Individual n has preferences over future consumption, considering all possible states of nature: $U_n : \mathbb{R}^{\ell S}_+ \to \mathbb{R}$. These preferences are assumed to be additively separable: for each s, there exists $v_{n,s} : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that $U_n((x_1, \ldots, x_S)) = \sum_{s=1}^S v_{n,s}(x_s)$. Each instantaneous utility, $v_{n,s}$ is assumed to be smooth, strictly monotone, strongly concave and possessing interior contour sets.

As before, $\mathcal{S}^{\ell-1}_{++} = \{p \in \mathbb{R}^\ell_{++} : p_1 = 1\}$. Spot-market prices for commodities in state s are $p_s \in \mathcal{S}^{\ell-1}_{++}$, which implies that commodity 1 is numeraire at all states. Asset 1 is also a numeraire; so, prices of assets q, are restricted to lie in $\mathcal{S}^{J-1} = \{q \in \mathbb{R}^J : q_1 = 1\}$.

Given prices $(p_s)_{s=1}^S$ and q and endowments ϕ_n and $(\omega_{n,s})_{s=1}^S$, the problem that individual n faces is: $\max_{(x_s)_{s=1}^S, \theta} U_n((x_1, \ldots, x_S))$ subject to the constraints that $q \cdot \theta \leq q \cdot \phi_n$ and that, for every s, $p_s \cdot x_s \leq p_s \cdot \omega_{n,s} + R_s\theta$, where $(x_s)_{s=1}^S$ represents consumption of commodities and θ represents a portfolio of assets. This problem gives a unique interior demand. Let $(x_n, \theta_n) : \mathcal{S}^{\ell-1}_{++} \times \mathcal{S}^{J-1} \times \mathbb{R}^{\ell S}_{++} \times \mathbb{R}^J \to \mathbb{R}^{\ell S}_{++} \times \mathbb{R}^J$ denote the maximizer of this problem.

Associated to the above maximization problem, there are two more series of maximization problems:

1. $\max_x v_{n,s}(x)$ subject to $p_s \cdot x \leq p_s \cdot \omega_{n,s} + R_s\theta$, for each s, and each $\theta \in \mathbb{R}^J$ such that $p_s \cdot \omega_{n,s} + R_s\theta > 0$. Let $\tilde{x}_{n,s}(p_s, \omega_{n,s}, \theta)$ denote the maximizer and $V_{n,s}(p_s, \omega_{n,s}, \theta)$ the value of this problem.
2. $\max_{\theta\in\mathbb{R}^J} \sum_{s=1}^{S} V_{n,s}(p_s, \omega_{n,s}, \theta)$ subject to $q \cdot \theta \leq q \cdot \phi_n$.

It is obvious that (x^*, θ^*) solves the original maximization problem if and only if at each s, x_s^* solves the problem in 1, given θ^*, and θ^* solves the maximization problem in 2.

Kübler et al. (2002) also impose the following condition.

Condition 2. For each individual n, preferences are such that:

- regularity on assets: at every point in the domain, there exist assets j and j', other than the numeraire, such that the following matrix is non-singular:

$$\begin{bmatrix} \dfrac{\partial^2\theta_{j,n}}{\partial\phi_{1,n}^2}(\cdot) & \dfrac{\partial^2\theta_{j',n}}{\partial\phi_{1,n}^2}(\cdot) \\ \dfrac{\partial^3\theta_{j,n}}{\partial\phi_{1,n}^3}(\cdot) & \dfrac{\partial^3\theta_{j',n}}{\partial\phi_{1,n}^3}(\cdot) \end{bmatrix}$$

- regularity on commodities: at each state s and at every point in the domain, there exist commodities l and l', other than the numeraire, such that the following matrix is non-singular:

$$\begin{bmatrix} \dfrac{\partial^2\tilde{x}_{l,n,s}}{\partial\omega_{1,n,s}^2}(\cdot) & \dfrac{\partial^2\tilde{x}_{l',n,s}}{\partial\omega_{1,n,s}^2}(\cdot) \\ \dfrac{\partial^3\tilde{x}_{l,n,s}}{\partial\omega_{1,n,s}^3}(\cdot) & \dfrac{\partial^3\tilde{x}_{l',n,s}}{\partial\omega_{1,n,s}^3}(\cdot) \end{bmatrix}$$

- at each state s and every point in the domain, the vectors $\tilde{x}_{n,s}(\cdot)$ and $D_{\omega_{1,n,s}}\tilde{x}_{n,s}(\cdot)$ are linearly independent.
- at every point in the domain and for every state s,

$$(D_q\theta_n(\cdot) + D_{\phi_{1,n}}\theta_n(\cdot)(\theta_n(\cdot) - \phi_n)^\top)R_s^\top \neq 0$$

The first two parts of the above condition are analogous to the regularity condition of the deterministic case; income effects are significant enough in the sense that at least a pair of assets or commodities, in each state of nature, can be distinguished. The third part, namely, demands for commodities and their income effects are linearly independent, rules out some preferences (e.g. homothetic) but holds generically when income effects are not constant (Kübler et al., 2002). The fourth part implies that substitution effects of the demand for assets affect the conditional demands for commodities in all states via the returns of assets.

Let $(x, \theta) : (\mathcal{S}_{++}^{\ell-1})^S \times \mathcal{S}^{J-1} \times \mathbb{R}_{++}^{\ell SN} \times \mathbb{R}^{JN} \to \mathbb{R}_+^{\ell S} \times \mathbb{R}^J$ denote the aggregate demand: $(x, \theta)(p, q, \{\omega_n\}, \{\phi_n\}) = \sum_{n=1}^{N}(x_n, \theta_n)(p, q, \omega_n, \phi_n)$.

Also define $D_s = \{(p, \{\omega_n\}, \{\theta_n\}) \in \mathcal{S}^{\ell-1}_{++} \times \mathbb{R}^{\ell N}_{++} \times \mathbb{R}^{JN} : \forall n,\ p \cdot \omega_n + R_s\theta_n > 0\}$, for each s, and let $\tilde{x}_s : D_s \to \mathbb{R}^{\ell}_{+}$ denote the conditional aggregate demand in state s and be defined by $\tilde{x}_s(p, \{\omega_n\}, \{\theta_n\}) = \sum_{n=1}^{N} \tilde{x}_{n,s}(p_s, \omega_n, \theta_n)$.

Theorem 15 (Kübler et al., 2002). *The aggregate demand for assets θ, identifies the individual demand functions for assets, $\{\theta_n\}$.*

The argument that proves this result is similar to the one used in the deterministic case in exploiting the income effects and regularity on assets. Similarly, using the condition of regularity on commodities, the following theorem follows from the deterministic results.

Theorem 16 (Kübler et al., 2002). *For each state of nature s the conditional aggregate demand, $\tilde{x}_s$, identifies the individual demand functions for assets, $\{\tilde{x}_{n,s}\}$.*

Kübler et al. go beyond these two results. Given the assumption of concavity of all utility functions, they argue that each (unconditional) demand (x_n, θ_n) identifies the conditional demand $\tilde{x}_{n,s}$ for each state of the world, and hence conclude that (x, θ) identifies $\tilde{x}$.

Corollary 1. *The aggregate demand (x, θ) identifies the individual demand functions for assets $\{\theta_n\}$, and the conditional demands for commodities $\{\tilde{x}_{n,s}\}$, for all states s.*

This whole process can now be exploited for identification of preferences. Under Condition 2, Proposition 1 in Geanakoplos and Polemarchakis (1990) implies that knowledge of the individual demand functions for assets θ_n, and the conditional demands for commodities $\tilde{x}_{n,s}$, is sufficient for identification of the utility function U_n up to cardinal equivalence. Hence, the following corollary.

Corollary 2. *The aggregate demand (x, θ) identifies the individual utility functions $\{U_n\}$, up to cardinal equivalence.*

As Kübler et al. point out, although the deterministic results imply that each $\tilde{x}_s$ identifies $\{v_{n,s}\}$, these results only hold up to ordinal equivalence. Since non-affine transformations cannot be distinguished, this does not identify $\{U_n\}$ (for which equivalence has to be cardinal). Indeed, it is necessary to know θ, for the identification of preferences up to cardinal equivalence.

The analysis of Kübler et al. goes even further. Define effective endowments as:

$$\tilde{\omega}_n = \begin{pmatrix} \omega_{n,0} \\ \omega_{n,1} + (1, 0, \cdots, 0)^{\top} R_1 \cdot \theta_n \\ \vdots \\ \omega_{n,S} + (1, 0, \cdots, 0)^{\top} R_S \cdot \theta_n \end{pmatrix}$$

which accounts for the return of the endowment of assets. Define the equilibrium manifold as

$$E = \{(p, q, \{\omega_n\}, \{\phi_n\}) \in (\mathcal{S}_{++}^{\ell-1})^S \times \mathcal{S}^{J-1} \times \mathbb{R}_{++}^{\ell SN} \times \mathbb{R}^{JN} : (x, \theta)(p, q, \{\omega_n\}, \{\phi_n\}) = \sum_{n=1}^{N} (\tilde{\omega}_n, \phi_n)\}$$

and for each state s, and $\{\theta_n\}$, define the conditional equilibrium manifold as

$$E_s(\{\theta_n\}) = \{(p_s, \{\omega_{n,s}\}) \in \mathcal{S}_{++}^{\ell-1} \times \mathbb{R}_{++}^{\ell N} : z_s(p_s, \{\omega_{n,s}\}, \{\theta_n\}) = \sum_{n=1}^{N} \tilde{\omega}_{n,s}\}$$

Kübler et al. claim that under the regularity assumption, manifold E identifies the profile of demands for assets $\{\theta_n\}$, at least locally. Once that result is obtained, it is easy to see that E also identifies all conditional manifolds $E_s(\cdot)$, and hence, by the deterministic arguments, all the profiles of conditional demands $\{\tilde{x}_{n,s}\}$, for all s.

Theorem 17 (Kübler et al., 2002). *Let $(p, q, \{\omega_n\}, \{\phi_n\}) \in E$ be given. For each individual n, E identifies U_n, up to a cardinal equivalence, on an open neighborhood of $x_n(p, q, \omega_n, \phi_n)$.*

4. Empirical implications of game-theoretic equilibrium concepts

We now turn to analyze the testable restrictions of aggregate behavior in a different setting, namely, the game-theoretic model. There is no analog to the Sonnenschein–Mantel–Debreu results here. The literature has been quite silent on this problem until a number of recent papers appeared (Zhou, 1997, 2002; Sprumont, 2000, 2001; Bossert and Sprumont, 2002, 2003; Ray and Zhou, 2001; Carvajal, 2003a; Ray and Snyder, 2003). As in the general equilibrium framework, this literature has built upon the existing revealed preference theory as developed for individual-level data.

Almost half a century back, social-choice theorists looked at the notion of revealed preference a la Samuelson (1938) in the context of individual choices. The question asked was: given a series of observations of choices and choice sets, when can an individual's choices be rationalized by a preference ordering? From the work of Chernoff (1954), Arrow (1959), Richter (1966), Sen (1971) and others, it is now well known that a choice function (possibly set-valued) is rationalizable by a complete and transitive preference relation if and only if it satisfies the so-called α and β conditions. The same question can naturally be asked in the context of collective choices and of interactive decisions. Here one would observe outcomes of game forms and determine if each player's choices could be rationalized with a preference ordering, given some concept of equilibrium play.

In this section, we describe the approaches of this emerging literature and some of its results. We describe the basic problem in Section 4.1. Rationalization of Nash equilibrium in normal forms and of subgame-perfect equilibrium in extensive forms are discussed in

Sections 4.2 and 4.3, respectively. Section 4.4 discusses the difference between Nash and subgame-perfect rationalization while Section 4.5 addresses other solution concepts.

4.1. The general problem

The general problem under consideration in this literature takes the following form. Let Γ be a collection of related game forms that several individuals play.[26] Suppose that we can observe an (or, possibly, a set of) outcome(s), $O(G)$ for every game form G in Γ. What conditions must the observed outcomes satisfy so that $O(G)$ coincides with the equilibrium outcome (set) of a game associated with G, for every G?

Notice that in this framework, as in individual choice theory, only outcomes are observed, not the preferences of the players. From the observations, one has to construct preferences of individuals so that the observed outcomes can be rationalized by the equilibrium notion with these constructed preferences. Also, note that no restrictions are imposed on the preferences (except that they must be orderings). Any restriction would indeed imply further conditions for rationalizability of solutions.

Since in strategic situations players can move simultaneously or sequentially, and in each case there are a variety of theoretical equilibrium concepts, we have a rich class of models to investigate. Clearly, different structures of strategic interactions and different solution concepts will have different implications on the observed outcomes.

A data set is a realization of the outcomes $O(G)$ for every game form G. There is an obvious distinction in the nature of the data sets observed in the simultaneous and the sequential game forms. In extensive game forms, we assume that we observe outcomes and not strategies (complete plans of actions), whereas in normal game forms, strategies (equivalently, actions) are assumed to be observed. Thus, a data set in the extensive form context has missing observations compared to the corresponding normal form data set. To see this consider the data set from the game tree (and all reduced forms) as in Fig. 6a. The tree has two choice nodes; player 1 moves in the first node and has two choices, namely, L and R. Player 2 moves in the second (after player 1 moves L) and also has two choices, namely, l and r. Here, the game form G has three possible (non-trivial) reduced game forms, denoted respectively by G_1, G_2 and G_3 in Fig. 6a.

The corresponding normal game form obviously has a 2×2 structure as shown in Fig. 6b. There are four possible (non-trivial) reduced normal forms; three of them correspond to G_1, G_2 and G_3. Clearly, if we observe the outcomes in the tree, we do not observe the outcome in G_4, player 2's choice of action when player 1 chooses to play R.

4.2. Nash behavior in normal game forms

Sprumont (2000) examines the testable restrictions of normal form games. Sprumont considers finite sets of actions, A_i, one for each player, i; the product set, A, is called the set of joint actions. A joint choice function, f, assigns to every possible subset B of A a

[26] Some of the symbols in this section have been used earlier in this paper to denote different notions. We are allowing such duplications as we wish to keep standard notations used in the literature. We hope our readers will not be confused.

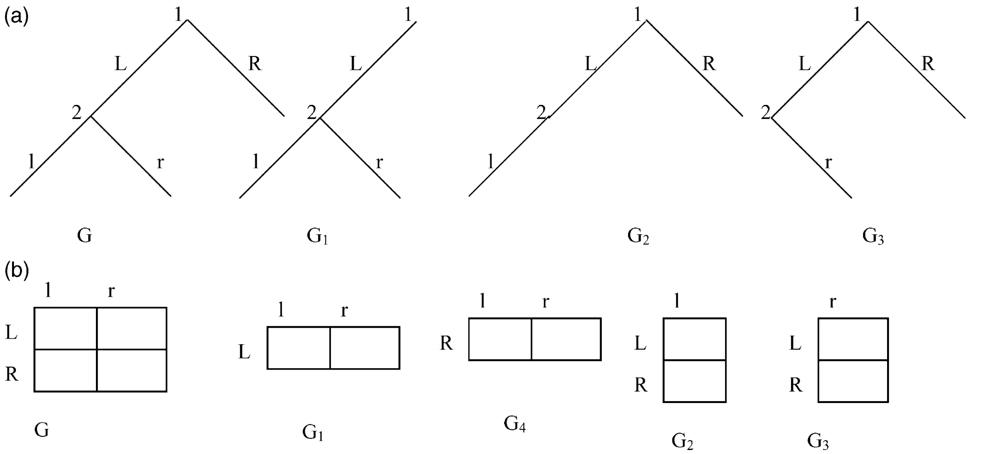

Fig. 6. Data sets in extensive and normal form.

non-empty set. A data set is a realization of a joint choice function. The joint choice function f may not be singleton-valued.

A data set is *Nash rationalizable* if there exist preference orderings on A such that for every B, $f(B)$ coincides with the set of Nash equilibrium for the game defined by the set of actions B with those preferences. Nash behavior imposes nontrivial restrictions on the data sets. To observe this, consider the following data set as shown in Fig. 7 for the 2×2 normal game form described in Fig. 6b. It is easy to see that this outcome function is not Nash-rationalizable. Since player 2 (strictly) prefers (R, r) to (R, l) in G_4, and 1 prefers (R, r) to (L, r) in G_3, (R, r) must be a Nash equilibrium when all actions are available. Yet, (R, r) is not selected by f in G.

Sprumont provides necessary and sufficient conditions for a data set to be Nash rationalizable. A *line* is defined as a subset of A that takes the form $\{(a_i, a_{-i}) \mid a_i \in A_i\}$, where i and a_{-i} are arbitrary but fixed. Let $\tilde{A}$ denote the set of all non-empty subsets of A which are Cartesian products; a typical element of $\tilde{A}$ is written as $B = \prod_i B_i$. A choice function f is defined over elements in $\tilde{A}$.

For any $B, B' \in \tilde{A}$, denote by $B \vee B'$ the smallest set in $\tilde{A}$ that contains $B \cup B'$.

Condition 3 (persistence under expansion). For all $B, B' \in \tilde{A}$, $f(B) \cup f(B') \subset f(B \vee B')$.

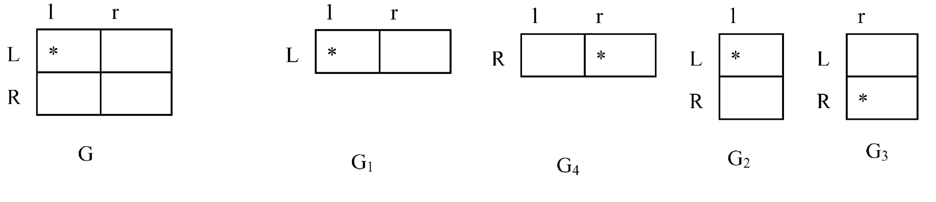

Fig. 7. Data that are not Nash rationalizable.

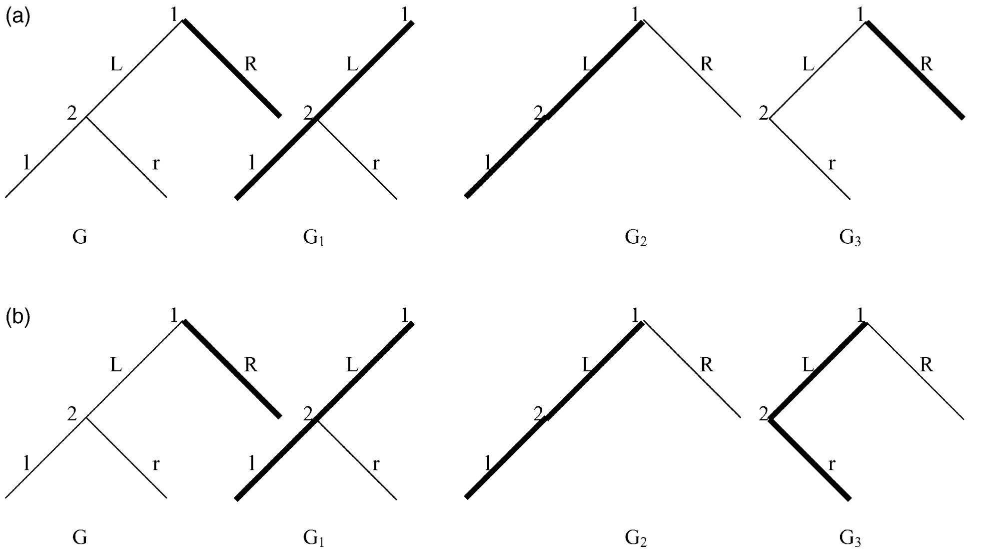

Fig. 8. Data not consistent with Subgame-perfection.

As the data set in Fig. 7 suggests, this condition is a necessary condition for f to be Nash-rationalizable. The condition by itself is, however, not sufficient. Sprumont provides the following condition which is required as well.

Condition 4 (persistence under contraction). For all $B, B' \in \tilde{A}$ such that $B' \subset B$, $f(B) \cap B' \subset f(B')$. If, moreover, B is included in a line of A and $f(B) \cap B' \neq \phi$, then $f(B') \subset f(B)$.

Theorem 18 (Sprumont, 2000). *An n-agent joint choice function is Nash-rationalizable if and only if it satisfies Persistence under Expansion and Persistence under Contraction.*

4.3. Subgame-perfect behavior in extensive game forms

As a complement to the work of Sprumont (2000), Ray and Zhou (2001) consider situations in which the players move sequentially with perfect information. They fix an extensive game form (tree) G with complete information. A reduced game form, G', is obtained from G by deleting branches of G. For Ray and Zhou, the data are the outcomes of all possible reduced game forms. They look for necessary and sufficient conditions for a data set to be rationalizable as the unique subgame-perfect equilibrium in every reduced game form.

Consider for example the following two distinct data sets, as described in Fig. 8a and b, on the same game tree as in Fig. 6a.

Clearly, neither of these data sets can be rationalized as subgame-perfect equilibrium. The choice of player 1 to play R in the game form G is not subgame perfect as player 1 prefers the outcome (L, l) to R as revealed in the game form G_2.

An extensive game form G is a finite rooted tree with set of nodes, X, with a distinct initial node x_0, and a *precedence function* $p : X/x_0 \to X$. If $p(y) = x$, then x is called an *immediate predecessor* of y. Also y is called an *immediate successor* of x, or $y \in s(x)$. Let $S(x)$ denote the set of all successors of x. A node z is called a terminal node, or an outcome, if there exists no $x \in X$ such that $p(x) = z$. The set of all terminal nodes is Z. A path ρ is a finite sequence of nodes: $(x_k : k = 0, \ldots, m)$ where $x_k = p(x_{k+1})$ for each k and x_m is a terminal node. A path leading to a terminal node x_m, $\rho(x_m)$, can be uniquely identified.

A reduced extensive game form G' of an extensive game form G is an extensive game form consisting of (I) terminal nodes $Z' \subseteq Z$ and (iI) all the non-terminal nodes that belong to $\rho(z')$ for any $z' \in Z$. For each reduced extensive game form G' and a non-terminal node $x \in X'/Z'$, the subgame form beginning at x, G'_x, is the reduced extensive game form consisting of (I) terminal nodes $Z'(x) = Z' \cap S(x)$ and (iI) all the non-terminal nodes that belong to $\rho(z')$ for any $z' \in Z' \cap S(x)$.[27]

Given an outcome function O, one can construct incomplete preference orderings for players over the terminal nodes. Consider the paths that lead to two different terminal nodes u and v. Take the player i who has to play at the node where these two paths diverge. Player i's preference over u and v can be determined by his choice in the reduced game form G' which has only two terminal nodes, u and v. This incomplete order, P_i, for player i, is known as the *revealed base relation*. Formally, for any $u, v \in Z$, let x be the node at which the paths to u and v diverge. If $x \in X_i$, then uP_iv if and only if $u = O(G')$, where G' is the reduced game form which has only two terminal nodes, u and v.

Ray and Zhou (2001) proposed three independent conditions that together turn out to be necessary and sufficient for subgame-perfect rationalization.

Condition 5 (acyclicity of the revealed base relation). For each player i, the revealed base relation, P_i, is acyclic.

Under this condition, P_i actually can be extended, via Zorn's lemma, to a strict preference ordering on Z which is both complete and transitive.

If a terminal node u is the unique (subgame-perfect Nash equilibrium) outcome for a reduced game G' and u is a successor of x, then u should also be the (subgame-perfect Nash equilibrium) outcome for any subgame G'_x of G'.

Condition 6 (internal consistency). For any G', if x is a predecessor of $O(G')$, then $O(G'_x) = O(G')$.

Finally, if u is the unique (subgame-perfect Nash equilibrium) outcome of a game, then at each node x on the path leading to u, the player who moves at x should prefer u to any other terminal node that could have been reached from x had he moved otherwise.

[27] The subgame form G'_x is the reduced game form consisting of the path from x_0 to x and the subgame below the node x.

Condition 7 (subgame consistency). Consider any G', $x \in X \setminus Z$ and $u = O(G'_x)$. For any $y \in s(x)$ such that y is not on the path to u and $v = O(G'_y)$, then $O(G') = u$, where G' is the reduced-game form which has only two terminal nodes, u and v.

Clearly, the data sets in Fig. 8a and b violate this condition and therefore cannot be rationalized as subgame-perfect equilibrium.

Theorem 19 (Ray and Zhou, 2001). *An outcome function is subgame-perfect rationalizable if and only if acyclicity of the revealed base relation, internal consistency, and subgame consistency are all satisfied*.

4.4. From Nash to subgame perfection

One would naturally be interested in the differences between the Nash and subgame-perfect behavior in extensive games. As observed earlier, one cannot use Sprumont's conditions for Nash rationalization in extensive game forms by testing the conditions in the corresponding normal game forms.

It is indeed possible to observe data on extensive game forms that are not rationalizable by subgame-perfect equilibrium, yet can still be rationalized as Nash behavior. Consider once again the data sets in Fig. 8a and b. As described earlier, neither of these data sets satisfies the subgame consistency condition of Ray and Zhou and therefore cannot be rationalized as a subgame-perfect equilibrium. The data in Fig. 8a, however, can be rationalized by a Nash equilibrium. The choice of player 1 to play R in the game form G can be justified as a Nash behavior on his part that assumes that player 2 would play r (although actually, player 2 prefers to play l when given the choice).[28] The data in Fig. 8b can not be rationalized even by Nash equilibrium as there is no choice of player 2 that would justify player 1's choice of playing R in the game form G.

Also, notice that, under the (revealed) preferences that rationalize the outcomes in Fig. 8a, the game G has multiple Nash equilibria. There is a Nash equilibrium (indeed, subgame perfect) outcome (L, l) in the game, which however, is not observed, as we assume that only one outcome is observed in each reduced game form.

Ray and Snyder (2003) work on the difference between Nash and subgame-perfect behavior in data sets. They first provide a necessary and sufficient condition for partial Nash rationalization; i.e., they rationalize the data in each reduced game as one of the possibly multiple Nash equilibria. They provide a necessary and sufficient condition, called *extensive form consistency*, which compares the outcomes of a set of reduced extensive form games, varying the set of feasible strategies for one player while the other players' strategies are fixed. Extensive form consistency also implies that the revealed base relation is acyclic.

In the data in Fig. 8b, there are two strategies consistent with the given outcome in the game form G, namely, (R, l) and (R, r); if we fix player 2's strategy at either l or r, we see from the outcomes of the reduced games G_2 and G_3, that player 1 prefers to play L. Extensive form consistency is not satisfied here and R cannot be rationalized as Nash

[28] This is precisely the case of "incredible threat" often used to show the difference between Nash and subgame-perfect equilibrium.

behavior in game G. In the data set in Fig. 8a, the condition is satisfied and the outcome in game G can be rationalized using the strategy profile (R, r).

Ray and Snyder then provide a condition, *subgame-perfect consistency*, which uses observations of reduced game outcomes that are proper subgames below a node. This condition ensures that the strategies played are not only Nash but are also consistent with subgame-perfect behavior.

The data set in Fig. 8a does not satisfy this condition because player 2 is active in G_1, which is a proper subgame of G, and is observed to move l; under this circumstance, we know from G_2, player 1 prefers L to R. Thus, the outcome R in G violates subgame-perfect consistency.

Subgame-perfect consistency together with extensive form consistency are necessary and sufficient for subgame-perfect equilibrium rationalization. These two conditions together are equivalent to Ray and Zhou's conditions. The advantage of the Ray and Snyder conditions are that they can be used to test for Nash alone and also to distinguish between Nash and subgame-perfect behavior.

4.5. Other solution concepts

In the framework of normal game forms, Sprumont (2000) also discusses Pareto efficiency as a solution concept. Interestingly, the outcome in Fig. 7 is Pareto-rationalizable. If both agents (strictly) prefer (L, l) to (R, r) and (R, r) to the remaining joint actions, the Pareto optima for each pair of feasible sets do coincide with the reported choices. The falsifiable implications of the Nash and Pareto hypotheses are therefore different. Sprumont considers two-agent joint choice functions which select a single joint action from each pair of feasible sets and proves that if any such function is Nash-rationalizable, it is also Pareto-rationalizable. Also, every choice function is partially Pareto-rationalizable, even if strict preferences are required.

Bossert and Sprumont (2003) add to this literature by exploring testable restrictions for (Pareto) efficiency and individual rationality. To understand individual rationality, they use the *status quo* model as introduced by Zhou (1997) and used by Rubinstein and Zhou (1997), and Masatlioglu and Ok (2003). They find a condition, namely *END-congruence*, that is necessary for arbitrary domains. This property is also sufficient in two cases: (I) when feasible sets under consideration coincide with the universal set; and (iI) for arbitrary domains, when the cardinality of the universal set is smaller than $2(n + 1)$.

In this research agenda it is generally required that the outcomes of all possible reduced trees be observable. There are many simple and interesting game situations, such as dividing-a-dollar, in which the players have infinitely many strategies but practically only finitely many observations can be made. Hence, one would like to know how to extend the above results to such cases with only limited observations. Carvajal (2003a) attempts to model such a situation and finds testable conditions.

The violation of all these necessary and sufficient conditions can also provide information that one can use to form beliefs about the consistency of individual decision making within games, and thereby rationalize outcomes with alternative hypotheses of behavior such as multiple preference orderings. It would be interesting to find general results analogous to Kalai et al. (2002) in this context.

5. Conclusion

A broad set of research questions is involved in the study of the empirical implications of markets and games. In our survey, we have explored only a small set of papers that provides the background for much more work. We have limited our survey to focus on testability and identification in competitive equilibrium, using a particular framework of analysis. We have also described how this framework of analysis extends to game-theoretic models.

Our choice of topics very much follows and extends two research lines that appear in the work of Brown and Matzkin: the testable restrictions of the equilibrium manifold, and further the identification of economic fundamentals from the equilibrium manifold; and the implications of the revealed preference theory of individual behavior for collective choice.

A common goal of the work we discuss is to describe a way to derive the positive implications of models of collective behavior in a systematic way without making parametric specifications or ad hoc assumptions. The aim is to try to derive restrictions of the most basic form of the model itself. However, it is of course impossible to completely avoid maintained hypotheses in testing. For example, in describing the testable implications of the pure exchange model of general equilibrium, we assume that utility functions do not vary over time, and that the agents are myopic. Whether these assumptions are less problematic than the assumption of a particular functional form for utility remains to be determined.

As in the work in general equilibrium, the game theoretic literature uses an extension of revealed preference theory to a collective choice situation; we get restrictions on collective choice behavior, derived from restrictions on individual choice. What is however different in the game theory literature discussed here is that there is no algorithm for deriving the comparative statics results; they are derived by intuitions and proofs, not by “methods”.

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