Insurance Contracts and Financial Markets

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December 21, 2019

Abstract

We study the interaction between insurance and financial markets. Individuals who differ only in risk have access to insurance contracts offered by a monopolist firm, and can also save through a competitive market. We show that an equilibrium always exists in that economy and identify an externality imposed on the insurer's decision by the endogeneity of prices in the financial market. We argue that, because of such externality and in contrast to the case of pure contract theory, equilibrium may exhibit pooling and always exhibits under-insurance even for the riskiest agents in the economy.

Key words: Insurance, Screening, Pooling, Finance.
JEL classification: D82, D86, D52, G1

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1. Introduction

At least since [?], [?] and [?] studied the problem of screening, insurance contracts have been a major theme of research in economic theory. But a limiting feature of the resulting literature is that it abstracts from any other relation or trade occurring in the economy–it is, by assumption, partial equilibrium analysis.

The motivation for this paper is that there are settings in which such limitation may render the analysis invalid. The considerations that explain the behavior of, say, the insurance market for domestic appliances would likely be insufficient to understand the effects of a reform to the social security regime of the same economy. In particular, when the insurance industry is considered in isolation, its relation with other financial markets is overlooked: on one hand, the analysis will ignore that the distribution of wealth in the society is endogenous, as the individuals react to different insurance coverages by purchasing different financial portfolios; on the other, it will ignore that the prices of financial assets affect the individuals' willingness to pay for insurance.

Here, we couple a version of the usual contract theory program to a general equilibrium model of a financial economy. A monopolist is considered who is able to insure individuals in an economy that presents idiosyncratic risk. This firm charges a premium in the present, in exchange for state-contingent promises of coverage in the future. At the same time, a financial market that is complete with respect to aggregate shocks is available to the agents. Individuals are assumed to be plenty and to be divided in types according only to their risk distributions. They simultaneously choose whether to buy an insurance contract, and which, and a portfolio of financial assets. At equilibrium, these choices must be individually rational, and asset prices must guarantee that the financial markets clear. The monopolist understands this, and optimally chooses a menu of insurance contracts that is self-enforcing, taking into account how this menu impacts the equilibrium of the financial market and, therefore, the insurees' willingness to pay for insurance.

Our main result is that the equilibrium menu of contracts can pool agents of different riskiness together, which is in stark contrast to the basic results of contract theory. Moreover, equilibrium always displays under-insurance of *all* agents, even the riskiest ones, which again differs from the usual lessons obtained in partial equilibrium.

We then identify the driving force behind the pooling result. First, we argue that what hinders separation is not the incentive constraints. That fact differentiates our result from the literature that deals with the failure of the Spence-Mirrlees condition in mechanism design. We conclude, instead, that the source of pooling is an externality induced on the decision of the firm through general equilibrium effects. Such externality operates through the individual rationality (or participation) constraint, rather than through incentives. It makes separation costly for the monopolist by making it more expensive to ensure the participation of individuals in a potential separating menu.[1] The externality is generated by the endogeneity of all agents' reservation utilities to the whole menu of contracts, which is in itself a consequence of the fact that individual decisions are linked together by the financial market. We call this externality the *market effect*.

In existing insurance theory models, at the optimal decision of the firm the riskiest individual is "undistorted" relative to the Pareto Efficient outcome.[2] The intuition behind this result is that there is no need to distort the riskiest agents, since their incentives are always to under-report their own risk and never to over-state it.[3] Since the market effect is an individual rationality phenomenon, our equilibria display distortions for every type of insuree, for the same reason that explains the existence of pooling equilibria.

Our insight is not specific to insurance contracts and financial markets. Except for some works in public finance,[4] mechanism design problems are solved in a partial equilibrium approach. Some of the literature, like [?], has recognized that trade in related markets may induce failures of the Spence-Mirrlees property that underlies the partial equilibrium approach. Such effect would operate through the incentive compatibility constraints and is different from ours. In general, our point is that, to the extent that general equilibrium effects are relevant, participation constraints may play as important a role as the one of incentives, through the endogeneity of the outside options.

[1] To be sure, our result is not a consequence of countervailing incentives and type-dependent reservation utilities, as in [?] or [?]. Rather, in our model the outside options do depend on private information, on the specific contract each agent is offered and, what is new, on the whole set of contracts posted by the firm.

[2] Namely, his marginal utility equalizes the marginal cost of providing insurance, which usually implies full insurance at the top of the risk distribution.

[3] That implies that there is no distortion "at the top", because the source of any inefficiency in the canonical model stems from incentive compatibility.

[4] See, for instance, [?] and [?].

2. The environtment

While our results hold true in much more general environments, we present them in a simple setting that will result familiar to the reader: a two-period version of the canonical monopolistic insurance problem posited by [?]. Thus, we assume that the economy evolves over the present and a future period, and consists of only one commodity per period.

2.1. The population and their risks

The economy is populated by a continuum of individuals whose mass we normalize to one. In the future, independently, each individual will find herself in one of two idiosyncratic states: she may have an accident, or not; we denote these idiosyncratic states, respectively, by A and N. If an individual does not suffer the accident, her wealth is $\bar{\omega} > 0$; if she does, she sustains a loss of $0 < \lambda < \bar{\omega}$, which results in a wealth of $\bar{\omega} - \lambda$.

There are two types of individuals, L and H. The only difference between the two types is the probability with which they have the accident in the future, with individuals of type H being more likely to have it: we denote by π^i the probability that a person of type i experiences the accident in the future, and assume that $\pi^H > \pi^L$. The fraction of type-i individuals in the population is μ^i.

If an individual of type i has a consumption plan, x, consisting of x_0 units in the present, x_A units upon having the accident in the future, and x_N units otherwise, her *ex-ante* utility is

$$U^i(x) = x_0 + \pi^i u(x_A) + (1 - \pi^i) u(x_N). \tag{1}$$

The cardinal utility index, $u : \mathbb{R}_+ \to \mathbb{R}$, is increasing, strictly risk averse and differentiable three times on $\mathbb{R}_{++}$, and satisfies the standard Inada conditions.

Having assumed that preferences are quasilinear in present consumption, we do not need to specify individual endowments for that period.

2.2. The insurance firm

An insurance contract specifies a premium charged in the present and the amount that is paid to the insuree, in the future, if she has the accident. Denote these terms by p and a, respectively. In the absence of other trade, upon signing an insurance contract the agent has a present wealth of $-p$; in the future, her wealth is $\bar{\omega}$ if she does not have the accident, and $\bar{\omega} - \lambda + a$ if she does. It will be useful to define $w = \bar{\omega} - \lambda + a$ and to write the insurance contracts as pairs (p, w). We assume that $a \in [0, \lambda]$, so that $w \in [\bar{\omega} - \lambda, \bar{\omega}]$, and refer to w as the *coverage* of the contract.

There is a monopolistic insurance company in the economy, whose goal is to maximize the profit it makes in the present. In that goal, the firm is subject only to the constraint that it must be able to honor the future commitments it makes. In the first period, this company posts a menu of contracts, M, from which each agent selects at most one. We consider only menus of the form $M = \{(p^H, w^H), (p^L, w^L)\}$, where there is one contract that is recommended for each type.[5] This implies no loss of generality.

Given the choices of the individuals from the menu it has offered, the company collects the resulting premia as revenue in the present. In order to be able to honor these contracts, the firm must constitute enough savings to be able to cover its *ex-post* obligations. For the moment, we assume that the firm has access to some exogenous savings technology that allows it to secure future revenue, with no risk, at a unitary cost of κ. By the law of large numbers, the aggregate obligation of the firm is riskless: assuming that, indeed, contract (p^i, w^i) attracts the individuals of type i, and only them, the firm must guarantee future funds for $\mu^i \pi^i [w^i - (\bar{\omega} - \lambda)]$ in order to cover the obligations acquired with this type of insuree.

Both the revenue collected and the portfolio required are determined by the menu chosen by the firm. Denoting these variables as R and Y, respectively, the monopolist's profits are

$$G(M, \kappa) = R(M) - \kappa \cdot Y(M), \tag{2}$$

where $R(M) = \mu^L p^L + \mu^H p^H$ and

$$Y(M) = \mu^L \pi^L [w^L - (\bar{\omega} - \lambda)] + \mu^H \pi^H [w^H - (\bar{\omega} - \lambda)]. \tag{3}$$

Importantly, for Eq. (**??**) to be a valid objective function for the firm, it must consider only menus where the insurees self-select to the contract recommended for their respective types.

[5] If the two contracts are the same, the company is trying to pool the two types; if one of them is $(0, \bar{\omega} - \lambda)$, the company is trying to exclude that type from the insurance market.

2.3. *The financial market*

We assume that besides the insurance contracts there exists a market where all individuals can competitively trade a riskless bond. This instrument allows them to transfer wealth across time periods, and is therefore an imperfect substitute for insurance. The price of the bond, which will be determined endogenously, is denoted by q; the holdings of an individual of type i, by y^i.

For the moment, we assume that the insurance company does not use this "local" financial market to constitute its savings: its cost of saving is κ, which is exogenous and may differ from q.[6] In Section **??** we dispense with this assumption.

3. Equilibrium

3.1. *Individual choice*

If an individual of type i has taken an insurance contract (p, w) and the price of the bond is q, she chooses her optimal level of savings by solving the program

$$\max_y \left\{-p - qy + \pi^i u(w + y) + (1 - \pi^i)u(\bar{\omega} + y) : y \geq -w\right\}. \tag{ISP}$$

Let $V^i(p, w, q)$ represent the indirect utility from this problem. The individual's choice of an insurance contract is, then, given by the problem

$$\max_{(p,w)} \left\{V^i(p, w, q) : (p, w) \in M \cup \{(0, \bar{\omega} - \lambda)\}\right\}. \tag{IIP}$$

For simplicity of notation, we will write $\bar{M} = M \cup \{(0, \bar{\omega} - \lambda)\}$. This set contains the menu of contracts offered by the monopolist, as well as the no-insurance option that is always available to the individual. With this notation, we can re-write Program (**??**) as stating that the agent chooses a $(p, w) \in \bar{M}$ such that,

$$V^i(p, w, q) \geq V^i(\tilde{p}, \tilde{w}, q) \text{ for all } (\tilde{p}, \tilde{w}) \in \bar{M}. \tag{IC and IR}$$

Of course, the individual chooses her insurance contract and her savings simultaneously. The nested manner in which we wrote the problem is equivalent to the simultaneous choice and will

[6] And the firm does not arbitrage this difference away.

be useful below for the purposes of presentation. To be sure, note that in the choice of a contract the individual takes into account the way in which that choice will affect her level of savings. Also, note that Program (**??**) summarizes the two canonical considerations of contract theory: the incentive compatibility (IC) constraint,[7] and the individual rationality (IR) constraint.[8]

3.2. *Financial market equilibrium*

Nesting the individual choice problem the way we did will be useful next.

Definition 1. *Given a menu* M*, an equilibrium in the financial market, FME, is a triple* $\{y^H, y^L, q\}$*, consisting of a level of savings for each type and a price for the bond, such that:*

1. *for each* i*,* y^i *solves Problem* (**??**)*, given* q *and* (p^i, w^i)*;*
2. *for each* i*, contract* (p^i, w^i) *solves Problem* (**??**)*, given* q *and* M*; and*
3. *the bond market clears:* $\mu^L y^L + \mu^H y^H = 0$.

This concept serves as a prediction of the general equilibrium implications of the firm's choice of a menu. We assume that the firm understands that the menu impacts the savings decisions of the individuals, and that these decisions affect the contract that they will choose. Assuming that each insuree chooses the contract intended for them, the first and third conditions determine the equilibrium demands and price in the market for the riskless bond. Using the indirect utility functions at equilibrium prices, the second condition says that all individuals choose to participate in the insurance market and self-select to their contracts, as the firm intended. We henceforth assume that the firm is aware of this mechanism and makes predictions that are correct according to it.

3.3. *Equilibrium*

In terms of the technique that we will employ in solving the model, what we do amounts to analyzing the problem of the insurance firm, as in contract theory, after having re-written it in terms of the

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7 That $V^i(p^i, w^i, q) \geq V^i(p^j, w^j, q)$, for $j \neq i$.

8 That $V^i(p^i, w^i, q) \geq V^i(0, \bar{\omega} - \lambda, q)$.

indirect utility functions at the equilibrium prices induced by the menu of contracts. That is, if we denote by $E(M)$ the set of FME for menu M:

Definition 2. *An* equilibrium *is a tuple* $\{M, y^H, y^L, q\}$, *consisting of a menu, individual savings and a price for the bond, such that:*

1. *given* M, *individual savings and the asset price constitute an FME:* $\{y^H, y^L, q\} \in E(M)$, *and*
2. *menu* M *solves the firm's maximization problem:*

$$\max_{M} \{G(M, \kappa) : E(M) \neq \varnothing\}. \tag{FMP}$$

Again, the first condition states that the insurance firm understands that the menu of contracts affects the financial market, and that its predictions are consistent with that effect. The second says that, given that understanding and the cost of transferring funds to the future, the firm chooses the optimal menu. In this condition, problem (**??**) guarantees that the firm is ruling out menus for which, at equilibrium prices, either an IR or an IC constraint would fail.[9]

3.4. Existence of equilibrium

Note that an equilibrium exists if, and only if, the firm's maximization problem has a solution. In order to show that that is the case we first use the quasi-linear structure of the economy to write the non-emptiness constraint in Program (**??**) in a more tractable way. Then, our argument shows that the firm does not have unlimited market power, in the sense that it cannot construct menus giving itself unbounded profits and which are still acceptable to the agents. This is so because the firm's capacity of manipulating outside options is naturally constrained by agents' ability to use the market as a substitute for expensive insurance.

Proposition 1. *An equilibrium exists.*

[9] A subtle problem is the possibility that set $E(M)$ be non-singleton: in the presence of multiple financial equilibria, we would need to stipulate a mechanism by which the firm selects one of them. Since we are considering preferences that are quasilinear in present consumption, the literature on general equilibrium guarantees that the set of arrays that satisfy the first and third conditions of the definition of FME is a singleton. Adding the second condition of that definition gives us that $E(M)$ is either empty or a singleton, so we do not need to worry about multiplicity of FME.

Proof. Denote by $y^i(w, q)$ the solution of Program (**??**), noting that it does not depend on p, by quasi-linearity. Denote also by $Q(w^L, w^H)$, the set of all q such that

$$\mu^L y^L(w^L, q) + \mu^H y^H(w^H, q) = 0.$$

It is immediate, from the first-order conditions of (**??**), that $q \in Q(w^L, w^H)$ if, and only if,

$$q = \pi^L u'(w^L + y^L) + (1 - \pi^L) u'(\bar{\omega} + y^L)$$

for some $y = (y^L, y^H)$ that solves

$$\max_{\tilde{y}^L, \tilde{y}^H} \left\{ \sum_i \mu^i \left[\pi^i u(w^i + \tilde{y}^i) + (1 - \pi^i) u(\bar{\omega} + \tilde{y}^i) \right] : w^i + y^i \geq 0, \; \sum_i \mu^i \tilde{y}^i = 0 \right\}. \tag{SMP}$$

By continuity this program has a solution, which is unique by strict concavity. It follows that set $Q(w^L, w^H)$ is a singleton, so we can denote its unique element by $q(w^L, w^H)$.

With this notation, we can re-write Program (**??**) as

$$\max_M \left\{ G(M, \kappa) : \forall (p, w) \in \bar{M}, \; V^i(p^i, w^i, q(w^L, w^H)) \geq V^i(p, w, q(w^L, w^H)) \right\}, \tag{FMP'}$$

and it suffices to show that this program has a solution. Since its objective function is continuous, we just need to show that its domain is compact. By Berge's Theorem of the Maximum, continuity of the indirect utility functions guarantees that the domain is closed. Coverages are bounded, by assumption. As for premia, it suffices to consider the individual rationality constraints only. For each individual, the surplus gained by taking the intended insurance contract, when the coverages offered are $w = (w^L, w^H)$ and insurance is free, is at most

$$\max_i \left\{ \max_w \left\{ V^i(0, w^i, q(w)) - V^i(0, \bar{\omega} - \lambda, q(w)) : w \in [\bar{\omega} - \lambda, \bar{\omega}]^2 \right\} \right\}$$

which exists, by continuity and compactness. This value is an upper bound for the premia that the insurance company can charge, if it is to satisfy that $V^i(p^i, w^i, q(w)) \geq V^i(0, \bar{\omega} - \lambda, q(w))$. $\square$

The functions defined in the previous proof are well understood in general equilibrium theory. Some of their comparative statics are introduced in the following lemma and proved in the appendix.

Lemma 1. *Both functions* y^i *and function* q *are continuously differentiable and strictly decreasing.*

4. The predictions of contract theory

The existence proof provides us, in Program (??), with a characterization of the problem of the firm that resembles the problem of mechanism design in partial equilibrium environments. The challenge is that (??) is written in terms of indirect utility functions evaluated at market-clearing prices. The difficulty that arises, then, is that any perturbation in the menu affects the whole set of constraints.[10]

4.1. Benchmark: the economy without financial markets

In order to have a background against which to compare our results, we first characterize equilibrium in the absence of financial markets. In such setting, the value of contract (p, w) for an agent of type i would be

$$\tilde{U}^i(p, w) = -p + \pi^i u(w) + (1 - \pi^i) u(\bar{\omega}),$$

while the insurer's problem would be

$$\max_{M} \left\{ G(M, \kappa) : \forall (p, w) \in \bar{M},\ \tilde{U}^i(p^i, w^i) \geq \tilde{U}^i(p, w) \right\}.$$

Equilibrium would amount to the choice of the insurer's optimal menu.

The following proposition makes it clear that our two-period version of the insurance problem is but an instance of the canonical monopolistic insurance model.

Lemma 2. *In the absence of financial markets, the optimal menu* M *is such that:*

1. $U^L(p^L, w^L) = U^L(0, \bar{\omega} - \lambda)$*;*
2. $U^H(p^H, w^H) = U^H(p^L, w^L)$*; and*
3. $u'(w^H) = \kappa$ *and* $u'(w^L) > \kappa$*.*

The first two parts of the lemma say that the safest type of agent is left indifferent to her outside option, while the riskiest type is indifferent between the two contracts in the menu. The third

[10] That complete dependence of the constraint set on all the entries of the menu is precisely the consequence of relaxing the assumption of an isolated insurance market. The only independence is induced by quasi-linearity: the price of savings does not depend on the premia. This simplicity gives us tractability but does not amount to making the problem one of partial equilibrium!

condition is more interesting for us: the type-H contract is undistorted in comparison to the Pareto optimal allocation, by receiving insurance so that $u'(w^H) = \kappa$, while the contract for type L individuals is distorted. It follows that separation and the absence of distortions "at the top" are properties of equilibrium in the absence of a financial market.

4.2. Pooling

Our first result, and perhaps the most striking one in the paper, is that separation is *not* a robust property of the equilibrium contracts in our setting with endogenous savings.[11]

Proposition 2. *In the presence of a financial market, there exist economies where the two types are offered the same insurance contract at equilibrium.*

In order to argue this proposition it suffices to show examples where a pooling menu arises at equilibrium. For that, we assume that the individuals have CARA cardinal utility indexes with absolute risk aversion $\alpha > 0$. That is, for the rest of this subsection we let $u(x) = -e^{-\alpha x}$.

We proceed by first solving for FME in closed form, which provides us with the functions needed for the monopolist's problem. First, suppose that an i-type individual accepts the contract (p^i, w^i). Then, his optimization problem at savings price q is:

$$\max_{y \in \mathbb{R}} \left\{ -p^i - yq - F^i(w^i) e^{-\alpha y} \right\},$$

where $F^i(x) = \pi^i e^{-\alpha x} + (1 - \pi^i) e^{-\alpha \bar{\omega}}$. The first-order conditions of this problem, which are necessary and sufficient, can be rearranged and rewritten in logarithmic form so as to get

$$y^i = \frac{1}{\alpha} \ln \frac{\alpha}{q} F^i(w^i).$$

Using market clearing, individual savings vanish and we can find the equilibrium price as a function of the two insurance coverages. Defining $f(x, y) = F^H(x)^{\mu^H} F^L(y)^{\mu^L}$, we have $q = \alpha f(w^H, w^L)$.

As a consequence, it is possible to characterize savings for an agent of type i when accepting coverage w, if the profile of coverages in society is (w^H, w^L), as

$$y^i(w, q(w^H, w^L)) = \frac{1}{\alpha} \ln \frac{F^i(w)}{f(w^H, w^L)}.$$

[11] We distinguish pooling from market exclusion: exclusion happens when any of the individuals is left receiving zero insurance; pooling, when both individuals receive the same non-zero contract.

And, finally, indirect utilities at the FME are

$$V^i(p, w, q(w^H, w^L)) = -p - f(w^H, w^L)\left[\ln \frac{F^i(w)}{f(w^H, w^L)} - 1\right].$$

Noticing that terms that are constant between types cancel out, we can replace the indirect utility functions by the simpler functions

$$v^i(p, w, q(w^H, w^L)) = -p - f(w^H, w^L) \ln F^i(w).$$

This value function is the composition of the indirect utility function and the equilibrium price function. The expression is informative in that it separates precisely the two effects our model is interested in. That is: on one hand, there is the direct effect of coverage on utility, measured by $-\ln F^i(w)$;[12] on the other, changes in coverage for any type have an indirect effect via market changes, and the $f(w^H, w^L)$ term captures that.

We can now solve the firm's problem numerically. Because the problem lacks concavity, we cannot rely directly on standard local optimization algorithms and need to resort to both *grid-optmization* and *multistart* methods that provide, respectively, approximations of the global optimum and robustness checks. Luckily, in our case local optimization provides the correct results, and the equilibrium is unique.

Table **??** shows the parameter specification for which the optimal menus were calculated. The risk aversion parameter, α, is not present in the table because we let it vary. We want to see whether the optimal menu of contracts presents pooling or separation.

Table 1: Parameter Specification

$\bar{\omega}$	λ	μ^L	π^H	π^L	κ
1	0.8	0.3	0.6	0.3	1

Figure **??** shows the values of w^H and w^L at equlibrium, for different values of α. The graph shows that for very small risk aversion insurance is absent in this economy. This is compatible with

[12] Simple algebra shows that this function is increasing and concave.

Lemma ??, for the case without financial markets.[13] As α increases, coverage is first improved for the riskier type but, when individuals get more risk averse, pooling appears, which is the novelty of our model and proves Proposition ??.

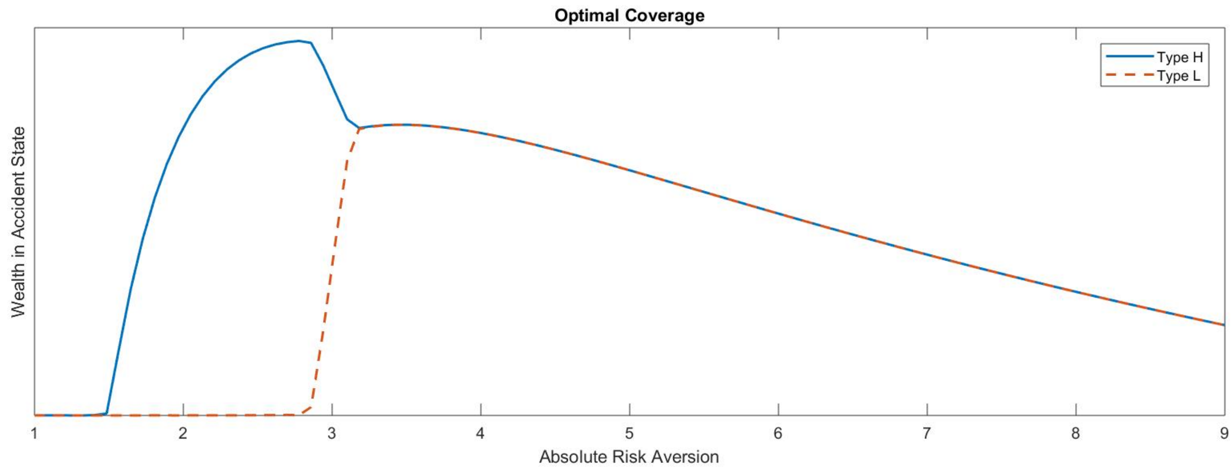

Figure 1: Equilibrium values of of w^H and w^L

4.3. Distortion also at the top

The reader can easily check that the equilibrium values of w^H in Figure ?? are lower than the values that would be given by the efficiency condition that $u'(w^H) = \kappa$. It is immediate that, unlike in the model without financial markets, in this example economy there are distortions from the first best even for the riskier type. Our next result is that this distortion is robust in the presence of financial markets.

Proposition 3. *In the economy with financial markets, let $\{M, y^H, y^L, q\}$ be an equilibrium. If the allocation is interior, in the sense that $\bar{\omega} > w^H \geq w^L > \bar{\omega} - \lambda$, then*

$$u'(w^L + y^L) > u'(w^H + y^H) > \kappa.$$

In words, the allocation fails first-best efficiency for the riskiest type (too).[14]

[13] At very low risk aversion, marginal utilities for CARA are below κ which, in our case, is taken to be 1.

[14] A stronger result is true, in fact: $w^H > w^L$.

Proof. The individual rationality constraint for individuals of type H is redundant, so only their incentive compatibility constraint binds in the firm's maximization problem, along with the individual rationality constraint of the type L contract. Using interiority, the first-order conditions with respect to p^H and w^H are that

$$\mu^H = -\ell^H \tfrac{\partial V^H}{\partial p^H}(p^H, w^H, q) \tag{4}$$

and

$$\kappa \mu^H \pi^H = \ell^H \tfrac{\partial V^H}{\partial w^H}(p^H, w^H, q) + \tfrac{\partial q}{\partial w^H}(w) \cdot D, \tag{5}$$

where ℓ^H and ℓ^L are, respectively, the multipliers of the incentive compatibility constraint of type H and the individual rationality constraint of type L, and

$$D = \ell^L \left[\tfrac{\partial V^L}{\partial q}(p^L, w^L, q) - \tfrac{\partial V^L}{\partial q}(0, \bar{\omega} - \lambda, q) \right] + \ell^H \left[\tfrac{\partial V^H}{\partial q}(p^H, w^H, q) - \tfrac{\partial V^H}{\partial q}(p^L, w^L, q) \right].$$

If either $D = 0$ or ${}^{\partial q}/_{\partial w^H} = 0$, Eq. (**??**) is the same as in the case without financial markets, and we recover the property of no distortion at the top. Lemma **??** tells us that ${}^{\partial q}/_{\partial w^H} < 0$. As for D, using Roy's identity,

$$D = \ell^L[y^L(0, \bar{\omega} - \lambda, q) - y^L(p^L, w^L, q)] + \ell^H[y^H(p^L, w^L, q) - y^H(p^H, w^H, q)].$$

Since $w^H \geq w^L > \bar{\omega} - \lambda$, by Lemma **??** we have that $D > 0$, so it follows that

$$\kappa \mu^H \pi^H < \ell^H \tfrac{\partial V^H}{\partial w^H}(p^H, w^H, q) = \ell^H \pi^H u'(w^H + y^H).$$

Using Eq. (**??**) and the envelope theorem we get $\ell^H = \mu^H$, which gives the result upon substitution. □

5. The mechanism

In classical (partial equilibrium) contract theory, pooling equilibria appear only when the Spence-Mirrlees[15] condition is violated: if a canonical contract-theoretical model satisfies single-crossing, it is always possible for the mechanism designer to dominate each pooling contract by a separating contract that is arbitrarily close.

[15] In our model, with premium and coverage, the Spence-Mirrlees condition simply reads: $\pi^H > \pi^L$. We use the expressions Spence-Mirrlees condition an single-crossing property interchangeably.

Lemma **??** shows that our version of the insurance problem is no exception to this rule, in the absence of a financial market. This is because of a specific feature of classical contract theoeretical models: reservation utilities are exogenous.[16] Then, for any pooling contract that is individually rational, it suffices to exhibit an incentive compatible menu that is more profitable and that makes each individual better off to guarantee that such menu is feasible and dominates the original pooling.

Although this logic seems straightforward, is is just not true when outside options are endogenous. In that case, building a new menu will *per se* alter reservation utilities, and it is not immediate that individual rationality constraints continue to hold.

5.1. Single crossing of indirect utilities

We will say that *incentive separation* is possible if, for any individually-rational pooling menu, there is an incentive-compatible separating menu that dominates it and is such that all individuals prefer their contracts under this new menu. In the language of our model: incentive separation is possible if, for any pooling menu $M = \{(p, w)\}$ such that $\bar{\omega} - \lambda < w < \bar{\omega}$, there exists an incentive-compatible menu $\{(p^H, w^H), (p^L, w^L)\}$ such that, for both i,

$$V^i(p^i, w^i, q(w^L, w^H)) \geq V^i(p, w, q(w^L, w^H)).$$

Importantly, notice that incentive separation does not rule out the possibility of there being pooling equilibria, as separation would also require that the deviation menu $\{(p^H, w^H), (p^L, w^L)\}$ be individually rational. In classical settings, where the reservations utilities are exogenous, incentive separation suffices for separation, naturally. Thus, in those settings any condition that guarantees incentive separation suffices for separation, and that is precisely the role of the single-crossing property.

The following concept extends the logic of the Spence-Mirrlees condition.

Definition 3. *We say that the economy satisfies the* ordering property *if for any point* (w, w)*, with* $w \in (\bar{\omega} - \lambda, \bar{\omega})$*,*

$$\frac{\partial V^H}{\partial w}(p, w, q) + \frac{\partial V^H}{\partial q}(p, w, q)\frac{\partial q}{\partial w^H}(w, w) > \frac{\partial V^L}{\partial w}(p, w, q) + \frac{\partial V^L}{\partial q}(p, w, q)\frac{\partial q}{\partial w^H}(w, w)$$

[16] In any insurance problem reservation utilities are type-dependent, and in our model they are *also* endogenous. Type-dependence and endogeneity are important but distinct phenomena. Although a large literature has been devoted to the former (see [?], [?]), not much has been said about the latter.

where $q = q(w, w)$.

That any economy obeying this property satisfies incentive separability goes as follows. Assume that the economy is at a given pooling menu that presents coverage w. Suppose that the firm proposes a new menu which is separating, gives more coverage to the riskier individuals and keeps the low-risk type accepting w. The price of savings is changed, but if at this new price marginal utilities are still ordered, then these coverages can be incentive compatible for appropriate choices of premia. The ordering property is simply the expression for ordering marginal utilities at new prices when the new contract is arbitrarily close to w.

Our next result is that our model with endogenous reservation utilities still displays the ordering property, and hence satisfies incentive separability.

Proposition 4. *Even in the presence of financial markets, the economy satisfies the ordering property.*

Proof. This is a matter of computing derivatives. Consider a contract (p, w). By using the envelope theorem we get, for any q, that

$$\tfrac{\partial V^i}{\partial w}(p, w, q) = \pi^i u'(w + y^i(w, q)),$$

while, by Roy's identity,

$$\tfrac{\partial V^i}{\partial q}(p, w, q) = -y^i(w, q).$$

Since $\partial q / \partial w^H < 0$, by Lemma **??**, it suffices to show that $\pi^H u'(w + y^H(w, q)) > \pi^L u'(w + y^L(w, q))$ for any $w \in [\bar{\omega} - \lambda, \bar{\omega}]$ given a price q; and that $y^H(w, q) > y^L(w, q)$.

Let W be the random variable paying w in the accident state and $\bar{\omega}$ otherwise. The first-order conditions of the two types' (**??**) programs imply that:

$$E_H[u'(W + y^H(x, q))] = E_L[u'(W + y^L(x, q))] \tag{6}$$

Let us assume, by way of contradiction, that $\pi^H u'(w + y^H(w, q)) \leq \pi^L u'(w + y^L(w, q))$. Then, as $\pi^H > \pi^L$ and because u is concave, it must be true that $y^H(w, q) > y^L(w, q)$. For (**??**) to be possible given the contradiction assumption, we would need

$$(1 - \pi^H) u'(\bar{\omega} + y^H(x, q)) > (1 - \pi^L) u'(\bar{\omega} + y^L(x, q)).$$

But again, because $\pi^H > \pi^L$ and by concavity, if $y^H(w, q) > y^L(w, q)$, such inequality cannot be satisfied.

On the other hand, just notice that $u'(\bar{\omega} + y) < u'(w + y)$, for $w \in [\bar{\omega} - \lambda, \bar{\omega})$ and any $y \in \mathbb{R}$. Then, because $\pi^H > \pi^L$, $E_H[u'(W + y)] > E_L[u'(W + y)]$. Therefore, for the first-order conditions to hold at equilibrium, $y^H(w, q) > y^L(w, q)$ is a necessary condition. □

The proof of this proposition relies on showing that marginal utilities can be ordered at any prices. That is a consequence of the fact that, even though savings approximate the marginal utilities in the accident state between types, they are not enough to make high-risks more satiated than low-risks.[17] Otherwise, the low-risk type agents would be willing to pay more than the high-risk ones for one unit of the riskless bond, contradicting equilibrium in the savings market. As a consequence of the proposition, it follows pooling equilibria result from the individual rationality constraints.

5.2. The market effect

We now identify the externality that prevents separation from being a necessary condition for equilibrium in our model. This externality arises from the firm's understanding that the choices of menu affects the equilibrium price of savings, which impacts the agents' willingness to pay for insurance.

Recall that menu M is feasible if $E(M) \neq \varnothing$, and say that a feasible M is a "best pooling menu" if it is a solution for the firm's problem with the added constraint that menu be pooling. Also, given a pooling menu $M = \{(p, w)\}$, denote by $x^i = w + y^i(w, q(w, w))$ the amount consumed by an i-type in state A; and by X^i the random variable that takes the value x^i in state A and $\bar{\omega} + y^i(w, q(w, w))$ in state N.

Lemma 3. *Assume that M is a best pooling menu with coverage $\tilde{w}$, and let $q = q(\tilde{w}, \tilde{w})$. There is* no *local feasible deviation $\tilde{M}$ from M such that $G(\tilde{M}, \kappa) > G(M, \kappa)$ if, and only if,*

$$u'(x^H) - k + \frac{u''(x^H)\left[k(\pi^L\mu^L + \pi^H\mu^H) - \pi^L u'(x^L)\right]}{\mu^H\pi^H u''(x^H) + \mu_L\pi^L u''(x^L)\frac{E_H[u''(X^H)]}{E_L[u''(X^L)]}} \leq 0. \tag{7}$$

[17] That is, the possibility of inversion of the order of marginal utilities, as in [?], does not occur in our model, and is not the reason why separation may fail.

This lemma provides (locally) necessary and sufficient condition for there to be a profitable and feasible deviation from the optimal pooling menu,[18] which is useful for understanding the following heuristics. First, note that Eq. (**??**) can be rewritten as

$$\mu^{H}\pi^{H}\left[u'(w+y^{H}(w,q))-\kappa\right]+\left[y^{L}(\bar{\omega}-\lambda,q)-y^{L}(w,q)\right]\frac{\partial q}{\partial w^{H}}(w,w)\leq 0.$$

The first term measures the distance from efficient insurance for the high-risk individual at the best pooling contract. This term is always positive, so if price impacts disappear such inequality cannot be satisfied and pooling cannot be optimal. This observation recovers the qualitative result of the benchmark framework with no financial markets.

The second term is the key. By Lemma **??**, this term is always negative, so the inequality in Eq. (**??**) may be satisfied for some economies. We submit that this term measures the general equilibrium effect on prices imposes an additional layer of costs for separation. In order to see heuristically the source of the term, assume that the economy is at a pooling menu M, with coverage $w>\bar{\omega}-\lambda$, such that the low-risk agents' individual rationality constraints are binding:[19]

$$V^{L}(p,w,q)-V^{L}(0,\bar{\omega}-\lambda,q)=0.$$

Suppose that the monopolist considers a new menu providing more coverage to the high-risk agents and keeping the low-risks at the original contract. If this new menu is sufficiently close, the individual rationality constraint for individuals of type L is changed to

$$\frac{\partial V^{L}(p,w,q)}{\partial q}-\frac{\partial V^{L}(0,\bar{\omega}-\lambda,q)}{\partial q}=\left[y^{L}(\bar{\omega}-\lambda,q)-y^{L}(w,q)\right]\frac{\partial q}{\partial w^{H}}<0,$$

where we used the envelope theorem to compute the differentials of the indirect utility functions, and where the inequality comes from Lemma **??** and the fact that $w>\bar{\omega}-\lambda$. The rationale above shows that price variations lead to a violation of an individual rationality constraint.

In words, when the firm increases insurance for riskier individuals, they save less on the risk-free asset, which results in a reduction in the price of saving. Such change in prices affects the decision of the low-risk individuals between accepting their contract or not, and price variations benefit disproportionately the outcome obtained through the outside option vis-à-vis the utility of accepting the

[18] Its expression is convoluted, but it is solely written in terms of first and second derivatives of the utility function, and of the best pooling coverage, w.

[19] It is proved in the appendix that this is indeed the case in the best pooling menu. See Lemma **??**.

contract.[20] As a consequence, the price variation alters the attractiveness of the menu, making it less appealing for the safer agents.

Naturally, if the firm is trying to separate agents, it must take into account with this additional cost. This is the *market effect* in our model. It tends to prevent separation by affecting the willingness of individuals to continue to accept a contract after the price of savings changes. Together, Proposition **??** and Lemma **??** identify the source of pooling as arising from individual rationality constraints, rather than from incentive compatibility.

6. Internal saving by the insurance company

So far, we have assumed that the insurance company constitutes its savings in a market that is external to the economy, so that its demand for savings has no effect over the price of the bond. We now proceed to lift this assumption, which was made for simplicity of presentation.

If the price paid by the firm is the one determined in the same savings market where the individuals trade, the definition of the firm's profits, Eq. (**??**), must be redefined as

$$G(M) = R(M) - q(M) \cdot Y(M), \tag{8}$$

and the market-clearing condition of Definition **??** must be re-written to require that

$$\mu^L y^L + \mu^H y^H + Y(M) = 0.$$

These changes are not innocuous. Following the intuition developed in Section **??**, suppose that the firm is considering whether a separating deviation from a given pooling contract will improve its profits. Our argument so far has been that the latter need not be the case, since better insured high-risk individuals will demand less savings, the price of the bond will decrease, and low-risk individuals will save more and will be willing to pay less for the initial insurance contract. Implicit in the argument is the fact that the lower demand for savings from the riskier individuals will *not* be compensated by another agent's increase in demand. But note that when the insurance company offers better coverage

[20] This is so because the direct effect of a price change is to make savings cheaper. For any individual, her asset holding are higher when she is not covered — that is, in the outside option — than when she is. Consequently, she gains more through the devaluation of savings when she does not accept any contract.

for any type, it must increase its savings in order to honor the extra coverage it is promising. At least in principle, it could occur that if the firm were to demand the riskless bond in the local market, its extra demand for the bond could offset the decrease in demand by the riskier type, and the overall demand and the resulting price could be higher, in which case the low-risk individuals would be willing to pay more, not less, for their initial contract.

To determine whether this could be possible, we re-compute the equilibrium coverages of the example presented as proof of Proposition **??**, under the new conditions. Here, we maintain the assumption that the level of savings demanded by the insurance firm is still given by Eq. (**??**). For the same parameters as before, namely those in Table **??**, we obtain the equilibrium coverages presented in Figure **??**. The numerical results are different, but the qualitative conclusion is the same as before: *even when the insurance firm demands its savings in the internal financial market, there exist economies where the two types are offered the same insurance contract at equilibrium.*

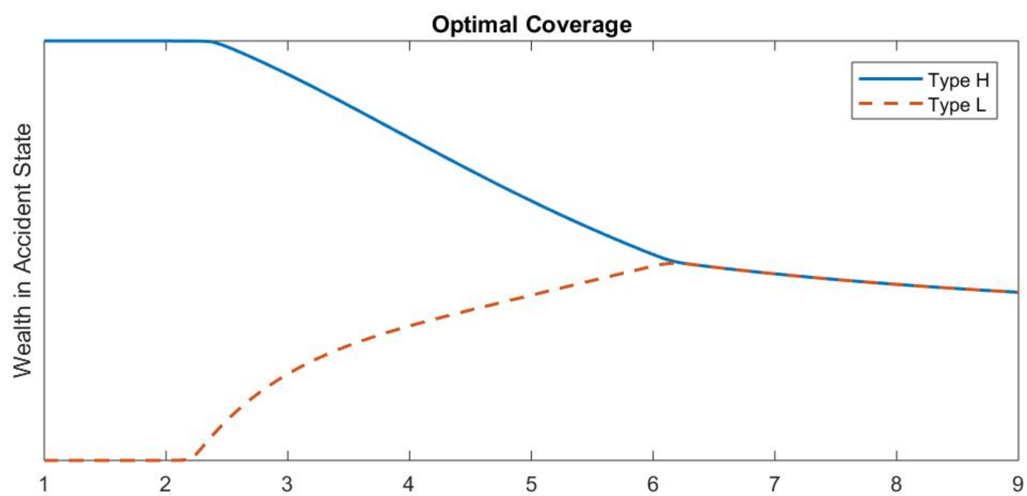

Figure 2: Equilibrium coverages with internal funding

The refinement of the intuition for this result is subtle and depends on three features of the example. First, the individuals' CARA preferences in the example display risk-aversion and prudence.[21] Secondly, we assume that, by the law of large numbers, the firm faces no uncertainty. And, finally, we are assuming that the firm only saves the amount that is needed to remain solvent in the second

[21] That is, $u''' \geq 0$.

period.

Under these three features, the mechanism behind our results continues to operate. To begin, recall that our insurees are prudent and are risk averse, while the firm faces *no* risk. Suppose that the firm contemplates and improvement of the coverage in the high-risk contract by dw^H. Because the firms faces no risk, it will increase its bond holdings by $\pi^H dw^H$, per insuree. The high-risk individuals, on the other hand, react by decreasing their savings by

$$\frac{dy^H}{dw^H} = -\pi^H \frac{u''(w^H + y^H)}{E[u''(W^H + y^H)]},$$

where W^H is the random variable that pays w^H in the accident state and $\bar{\omega}$ otherwise. Now, for prices to still drop after the firm offers this increased coverage, we must have that

$$1 < \frac{u''(w^H + y^H)}{E[u''(W^H + y^H)]}.$$

Since u'' is increasing (prudence), the result follows because u'' is negative and $w^H < \bar{\omega}$.

This guarantees that when the firm improves the coverage offered to the high-risk agents, their demand for savings decreases by more than the extra savings that the firm itself will need to constitute. Since the firm does not over-save in order to manipulate the price, the aggregate demand for savings decreases, the price of the bond falls, and the high-risk individuals reduce their willingness to pay for insurance. If this effect is significant enough, the price concession that the firm would need to give to retain the insurance demand by the low-risk insurees may prevent it from improving the contract offered to the high-risk types, hence choosing to remain in a pooling (and inefficient) situation.

7. Related literature

This paper focuses on an insurance problem with adverse selection, where the agents live in a competitive financial economy. It studies, then, the relation between contract economics and general equilibrium. A wide literature has been concerned with this issue since [?]: see [?] and [?] or, in a different perspective, [?]. These papers generally study economies that are competitive and subject to informational problems that generate the possibility of adverse selection. In these economies, they model contracts as exclusive relations between a given firm and a consumer. In them, contracts, which are usually lotteries on future consumption, are either offered by firms that are competitive or are exogenously given. The latter is the case, for instance, in [?], where individuals buy insurance by choosing among a number of pools to trade in. Contracts are sold in a competitive market in which both firms and agents take prices as given and, in equilibrium, prices reflect the profile of types of agents buying them. Furthermore, the two fundamental questions for this literature are whether equilibrium exists and is efficient.

This is in contrast with our model, in which the insurer is a monopolist company exercising power over not only the price of contracts, but also over other prices. The key here is that we assume a market structure that is divided in two. On one hand, there is the contract market, which is strategic, and in the other a financial market that is competitive. None of the papers cited go in this direction.

On the other hand, [?] follows a similar path to ours, by identifying the externality that is associated to adverse selection in the competitive economy. In their model, competitive firms sell promises of future consumption in separated markets, one for each type of agent, who vary only in risk. Risk is private information but, in equilibrium, insurees self-select to the market designed to their own type. Exclusivity of contracts and the fact that markets are competitive and naturally separated precludes risk-sharing between types of agents, and generates inefficiency, and it turns out that the level of net trades an agent can make in a market is linked to trades in the other one through incentive compatibility.

We, too, identify an externality that the existence of competitive markets imposes on the screening problem. But, in contrast, our externality is reflected directly in the problem of the firm, rather than on another individual's problem. This is so because we assume a firm that internalizes the possible deviations by agents. If we see this work as trying to understand contract theory in the realm of general equilibrium, it inverts what is usually done in the mentioned literature: we take competitive markets as the background, and look at the problem in a mechanism design framework.

Lastly, our paper relates to the literature on pooling in contract theory. It provides a new justification for pooling that is not solely due to either of the two most usual ones: the absence of Spence-Mirrlees conditions as in, for instance, [?]; or countervailing incentives as in [?].[22] In our model, we

[22] Countervailing incentives happen when a given agent may have incentives either to understate or overstate their private information depending on the actual realization of such private variable. In many cases that is associated with type-dependent outside options, usually when trading with the firm demands the agent to forego an opportunity outside the contractual relation that varies with his type. In pure contract theoretical models of insurance, foregone opportunities

show that incentive constraints do not pose a problem for separation: a condition that plays the role of the single-crossing property is preserved by the endogenous counterparts of utility functions in our framework. That excludes the former as an explanation for pooling in our model.

As for the latter, countervailing incentives in the sense of [?] and [?] are indeed present in our work. Because in our model agents can use the financial market to alter their own levels of wealth, countervailing incentives naturally arise and, moreover, they do so endogenously. Therefore, we are adding an extra layer of mixed incentives to the countervailing phenomenon. In our model, outside options are not only type-dependent, but also endogenous because two different contracts generate two levels of savings that affect distinctly the relation between the insurance outcome and the outside option. However, the key factor behind pooling in the economies we study is not that individuals endogenously react to offered contracts — that also happens, for instance, in [?]. Rather, the driving force for pooling here is that the effect of such reaction leads to changes that affect all the constraints in the economy, including other agents' participation constraints. Namely, individual deviations, when occurring for all individuals of a given type, alter the prices in the competitive financial market and, therefore, have a widespread effect in the whole feasibility set, and not only in their own constraints.

Other properties of our equilibrium and the extension of our results to competition between insurance firms are left as subjects of future research.

Appendix: Lemmata

Proof of Lemma **??**. By Inada, both Programs (**??**) and (**??**) have interior solution. The first-order conditions of the former problem are that

$$q = \pi^i u'(w^i + y^i) + (1 - \pi^i) u'(\bar{\omega} + y^i). \tag{$*$}$$

By strict concavity and the implicit function theorem, function y^i is continuously differentiable and strictly decreasing.

The first-order conditions that characterize function $q(w)$, from the solution to Program (**??**), are that Eq. (**??**)

are always present, but type-dependence of the outside options is easily controlled as incentive compatibility implies that the difference between the inside outcome and the outside option is monotonous in types.

hold for both types and $\mu^L y^L + \mu^H y^H = 0$. Differentiating this system one more time, with respect to w^L, gives

$$\begin{pmatrix} -1 & D^2U^L & 0 \\ -1 & 0 & D^2U^H \\ 0 & \mu^L & \mu^H \end{pmatrix} \begin{pmatrix} dq \\ dy^L \\ dy^H \end{pmatrix} = \begin{pmatrix} -\pi^L u''(w^L + y^L)dw^L \\ 0 \\ 0 \end{pmatrix},$$

where

$$D^2U^i = \pi^i u''(w^i + y^i) + (1 - \pi^i)u''(\bar{\omega} + y^i).$$

By strict concavity and the implicit function theorem, q is differentiable with respect to w^L and

$$\frac{\partial q}{\partial w^L}(w) = \frac{\mu^L \pi^L u''(w^L + y^L) D^2u^L}{\mu^L D^2u^L + \mu^H D^2u^H} < 0.$$

The same argument applies to w^H. □

Proof of Lemma **??**. Individual rationality for the riskier type, H, is redundant. Next, assume that the incentive compatibility constraint for individuals of type Lis not binding. Then, the argument of the following proposition closely parallels the characterization of an optimal contract in the canonical model of contract theory, and its details thus be omitted: our assumption that $u'(\bar{\omega} - \lambda) > \kappa > u'(\bar{\omega})$ guarantees that efficient insurance is achievable;[23] then, the three claims in the proposition follow simply by taking first-order conditions of the monopolist's problem.

Of course, to complete the argument we need to show that, indeed, $U^L(p^L, w^L) \geq U^L(p^H, w^H)$ at the solution of the relaxed problem above. Explicitly, this requires that $P_H - P_L \geq \pi_L[u(w_H) - u(w_L)]$. By the second part of the proposition, $P_H - P_L = \pi_H[u(w_H) - u(w_L)]$, which implies, since $\pi_H > \pi_L$, that we just need to show that $w_H \geq w_L$. The third part of the proposition implies that this is the case, since u is concave. □

Lemma 4. *For any* q *and* $\bar{\omega} - \lambda \leq w \leq \bar{\omega}$, *let* $\Delta V^i(p, w) = V^i(p, w, q) - V^i(0, \bar{\omega} - \lambda, q)$. *Then,* $\Delta V^H(p, w) > \Delta V^L(p, w)$.

Proof. By the envelope theorem, we can write $\Delta V^i(p, w) = -p + \int_{\bar{\omega}-\lambda}^{w} \pi^i u'(x + y^i(x, q))dx$. But then, by Claim 1, we have that, $\pi^H u'(x + y^H(x, q)) > \pi^L u'(x + y^L(x, q))$. Using this information,

$$\Delta V^H(p, w) = -p + \int_{\bar{\omega}-\lambda}^{w} \pi^H u'(x + y^H(x, q))dx > -p + \int_{\bar{\omega}-\lambda}^{w} \pi^L u'(x + y^L(x, q))dx = \Delta V^L(p, w).$$

□

[23] In the absence of the assumption on marginal utilities, this proposition only changes in that market exclusion can be optimal. That is, when $u'(\bar{\omega} - \lambda) < \kappa$, it makes no sense to provide insurance for any of the agents, then both are excluded from the market. Besides, if $u'(\bar{\omega}) > \kappa$, we would have $w^H = \bar{\omega}$.

Lemma 5. *If a best pooling menu* $M = \{(p, w)\}$ *consists of an interior contract, it satisfies*

$$\pi^L u'(w + y^L(w, q)) + \left[y^L(\bar{\omega} - \lambda, q) - y^L(w, q)\right] Dq = \kappa(\pi^L \mu^L + \pi^H \mu^H).$$

where $q = q(w, w)$ *and* $Dq = \partial q/\partial w^L + \partial q/\partial w^H$. *Moreover,* $p = \Delta V^L(0, w)$.[24]

Proof. Solving for the best pooling menu consists of facing the problem of the firm when the latter is searching for optima in the space of pooling contracts. In that space, incentive constraints hold trivially. Additionally, by Lemma **??**, in such space the individual rationality constraint for the riskiest type is redundant. Therefore, (**??**) can be rewritten as

$$\max_{p,w} \left\{ p - \nu \sum_i \mu^i \pi^i w : -p + V^L(0, w, q(w, w)) \geq V^L(0, \bar{\omega} - \lambda, q(w, w)) \right\}.$$

Because we know that, for fixed coverage, q is a constant, it is always in the interest of the firm to set equality in the individual rationality constraint of the L-types, which establishes the second claim in the lemma.

The first claim then follows from taking the necessary first order conditions of the problem above, assuming interiority.[25] □

Lemma 6. *Let* $\mathbf{A}$ *be a matrix with dimensions* $K \times L$, *with* $(\mathbf{A}_k)_{k \leq K}$ *its rows, and* $\mathbf{b} \in \mathbb{R}^L$. *Suppose that, for the matrix* $\bar{\mathbf{A}} = (\mathbf{A}_k)_{2 \leq k \leq K}$, *there is* $z \in \mathbb{R}^L$ *such that* $\bar{\mathbf{A}} z \geq \bar{\mathbf{b}}$, *where* $\bar{\mathbf{b}} = (\bar{\mathbf{b}}_k)_{2 \leq k \leq K}$. *Then, one and only one of the following two statements is true:*

(A) *There exists a vector* $z \in \mathbb{R}^L$ *such that* $\mathbf{A} z \geq \mathbf{b}$ *with* $\mathbf{A}_1 z > \mathbf{b}_1$*; or*

(B) *There exists a vector* $\gamma \in \mathbb{R}^K_+$ *such that* $\gamma_1 > 0$, *and with the property that* $\sum_k \gamma_k \mathbf{A}_k = 0$ *and* $\sum_k \gamma_k \mathbf{b}_k \geq 0$.

Proof. This is just a minor variation of Farkas's Lemma. See Theorem 22.2 in [**?**]. □

Proof of Lemma **??**. We first add some notation to avoid cumbersome expressions. Define $u_i' = u'(x^i)$, $\Delta y^i = y^i(\bar{\omega} - \lambda, q) - y^i(w, q)$, $\Delta V^i = V^i(p, w, q) - V^i(0, \bar{\omega} - \lambda, q)$ and $\partial_i q = {}^{\partial q}/_{\partial w^i}(w, w)$.

As in Proposition **??**, if a deviation is close enough to the original contract, it must satisfy in differential form all the constraints. If it is profitable for the firm, then it must also satisfy

$$\mu^L(dp^L - \pi^L \kappa dw^L) + \mu^H(dp^H - \pi^H \kappa dw^H) > 0.$$

[24] That is, $\Delta V^L(p, w) = 0$.

[25] Without interiority, the same condition holds with the left hand side being less or equal to the right hand side.

Now, notice that incentive compatibility constraints are binding at the original menu, $\mathcal{M}$. Therefore, for them to hold in the deviation menu, for each type,

$$-dp^i + \frac{\partial V^i(0,w,q)}{\partial w}dw^i \geq -dp^j + \frac{\partial V^i(0,w,q)}{\partial w}dw^j,$$

given that the derivatives of V^i relative to prices appear on both sides of the inequality and, hence, cancel out. By the envelope theorem, the previous inequality can be written as

$$-dp^i + \pi^i u_i'(x^i)dw^i \geq -dp^j + \pi^i u_i'(x^i)dw^j.$$

For individual rationality, we must have, for each i,[26]

$$-dp^i + \pi^i u'(x^i)dw^i + \Delta y^i(\partial_i q dw^i + \partial_j q dw^j) \geq -\Delta V^i.$$

It is possible to arrange this system of five linear inequalities in matrix form, as $\mathbf{A}z \geq \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} \mu^H & \mu^L & -\mu^H\pi^H\kappa & -\mu^L\pi^L\kappa \\ -1 & 1 & \pi^H u_H' & -\pi^H u_H' \\ 1 & -1 & -\pi^L u_L' & \pi^L u_L' \\ -1 & 0 & \partial_H q\Delta y^H + \pi^H u_H' & \partial_L q \Delta y^H \\ 0 & -1 & \partial_H q \Delta y^L & \partial_L q\Delta y^L + \pi^L u_L' \end{pmatrix},$$

$z = (dp^H, dp^L, dw^H, dw^L)'$ and $\mathbf{b} = (0, 0, -\Delta V^H, 0)'$.

We want to find a necessary and sufficient condition for the existence of γ in statement (B) of Lemma **??**. First of all, notice that the hypothesis on the statement of Lemma **??** is satisfied: $\bar{\mathbf{A}}0 \geq \bar{\mathbf{b}}$. Now, assume there is a γ satisfying (B), that is, the pooling contract allows no local deviation. We shall show that this is so whenever (p, w) satisfies Eq. (**??**), and vice-versa.

By the second part of property (B) we know that $\gamma\mathbf{b} = \gamma_4(-\Delta V^H) \geq 0$. By Lemma **??**, $\Delta V^H > \Delta V^L = 0$, where the last equality is a consequence of Lemma **??**, since we are assuming (p, w) to be a best pooling contract that is interior. Therefore, because $\gamma_4 \geq 0$, it must be the case that $\gamma_4 = 0$.

Write as $\mathbf{A}^\ell$, for $\ell = 1, \ldots, 4$, the columns of matrix $\mathbf{A}$. By statement (B) in Lemma **??**, $\gamma\mathbf{A}^\ell = 0$ for all ℓ. By taking $\gamma(\mathbf{A}^1 + \mathbf{A}^2) = 0$ we get $\gamma_1 = \gamma_5$. Substituting that in $\gamma\mathbf{A}^1$, we also get $\gamma_2 = \gamma_1\mu^H + \gamma_3$. Taking $\gamma(\mathbf{A}^3 + \mathbf{A}^4) = 0$ and using $\gamma_1 = \gamma_5$ we get that

$$\gamma_1 \cdot \left[\pi^L u_L' + (\partial_L q + \partial_H q)\Delta y^L - \kappa(\pi^L\mu^L + \pi^H\mu^H)\right] = 0.$$

[26] We could rightfully ignore individual rationality for the H-types. We opt not to do so just to maintain everything clear.

Because we are assuming (p, w) to be a best interior pooling contract, the term in braces is zero by Lemma **??**. Therefore, this equation is satisfied automatically. So, the last equation to be checked is either $\gamma \mathbf{A}^3 = 0$ or $\gamma \mathbf{A}^4 = 0$. Choosing the first one gives us

$$\gamma_1 \left[\mu^H \pi^H (u'_H - \kappa) + \partial_H q \Delta y^L\right] = \gamma_3 (\pi_l u'_L - \pi^H u'_H).$$

As we proved in Lemma **??**, $\pi^L u'_L - \pi^H u'_H < 0$, so that for γ_1 and γ_3 to be greater or equal to zero, it must be true that

$$\mu^H \pi^H (u'_H - \kappa) + \partial_H q \Delta y^L \leq 0. \tag{9}$$

In order to conclude the proof of sufficiency, it suffices to show that the inequality in Eq. (**??**) is equivalent to the one in Eq. (**??**). That is done by using the necessary condition a best pooling contract must satisfy by Lemma **??** to substitute for Δy^L in the above inequality.

Note that by rearranging the expression in Lemma 3,

$$\Delta y_L = \frac{k(\pi^L \mu^L + \pi^H \mu^H) - \pi^L u'(x_L)}{\partial_H q + \partial_L q}.$$

Plugging this into Eq. (**??**), we have that

$$\mu^H \pi^H (u'(x_H) - k) + \left[k(\pi^L \mu^L + \pi^H \mu^H) - \pi^L u'(x_L)\right] \frac{\partial_H q}{\partial_H q + \partial_L q} \leq 0 \tag{10}$$

Now, we just need to calculate $\partial_i q$. Recall first order condition for type i is that $\mathrm{E}_i\, u'(X_i) = q$. After a change in w_i, this equation changes so that

$$dq = \pi^i u''(x^i) dw^i + \mathrm{E}_i[u''(X^i)] dy^i$$

Similarly, with the first order condition for type j, we get that $dq = \mathrm{E}_j[u''(X^j)] dy^j$ and, by market clearing, $\mu^H dy^H + \mu^L dy^L = 0$. We can solve these last three equations, we get that

$$\partial_i q = \pi^i u''(x^i) \frac{\mu^i\, \mathrm{E}_j[u''(X^j)]}{\mu^i\, \mathrm{E}_j[u''(X^j)] + \mu^j\, \mathrm{E}_i[u''(X^i)]}.$$

Then,

$$\frac{\partial_H q}{\partial_H q + \partial_L q} = \mu^H \pi^H \frac{u''(x^H)}{\mu^H \pi^H u''(x^H) + \mu^L \pi^L u''(x^L) \frac{\mathrm{E}_H[u''(X^H)]}{\mathrm{E}_L[u''(X^L)]}}.$$

Plugging this result into Eq. (**??**), and dividing by $\pi^H \mu^H > 0$, we get Eq. (**??**).

Notice, however, that necessity is also proved. If (p, w) is a best pooling contract that allows no local deviations that are profitable for the firm, then a γ satisfying the conditions in (B) must exist. Therefore, Eq. (**??**) must hold. □