Idiosyncratic risk and the equity premium*

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In an economy with aggregate risk, the equity premium is the difference between the expected return of a dollar invested in an asset bearing the same (distribution of) risk as the whole economy and the risk-free interest rate. The equity premium puzzle is the observation, first put forward by Mehra and Prescott (1985), that standard macroeconomic models with homogeneous agents and no idiosyncratic risk fail to explain the equity premia typically observed in the data.[1]

Mankiw (1986) and Constantinides and Duffie (2003) observed that the presence of counter-cyclically heteroskedastic uninsurable idiosyncratic risk increases the equity premium predicted by homogeneous agent models.[2] However, Krueger and Lustig (2010) showed that this is not the case when the representative agent has CRRA preferences, and the distribution of idiosyncratic risk follows a particular form of pro-cyclical heteroskedasticity in a two-period economy. Under these assumptions, the equity premium is not affected by the presence of idiosyncratic risk. For the same class of preferences, Storesletten et al. (2007) had shown that in an OLG economy, the effect of idiosyncratic shocks in the equity premium is significantly smaller than in the two-period case considered by Constantinides and Duffie.

This paper aims to further our understanding of the effect of idiosyncratic risk on the equity premium. We consider different classes of preferences and different co-variations between the idiosyncratic shocks' variance and the economy's aggregate income. For short-lived assets, such as those considered in Constantinides and Duffie (2003) and Krueger and Lustig (2010), we offer a complete characterization of the effect, relying on the cross-moments of different derivatives of the utility function and the aggregate income of the economy. For long-lived assets, such as those in Storesletten et al. (2007), a full characterization is elusive, but we present sufficient conditions for the reversal of the effect found by Constantinides and Duffie (2003).

All these results assume that all agents in the economy are ex-ante identical. Werning (2015) and Bilbiie (2020) argue that cyclicality of the volatility of idiosyncratic shocks and ex-ante hetero-

* This is *very* preliminary work. All comments and observations are welcome!

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[1] See also Mehra (2003).

[2] See also the empirical results in Cogley (2002).

geneity renders many conclusions of homogeneous-agent models invalid.[3] Heterogeneity makes our analysis rather un-tractable, but we present a decomposition of the multiple effects at play that isolates the effect of idiosyncratic risk on the equity premium.

1 Homogeneous Agents

Let non-degenerate random variable W represent the future wealth of an economy were all agents are identical, and let function $u : \mathbb{R}_+ \to \mathbb{R}$ be the agents' Bernoulli utility index. Suppose that the support of W is a subset of $\mathbb{R}_{++}$ and that u is $\mathbf{C}^3(\mathbb{R}_{++})$, strictly increasing and strictly concave and has non-negative third derivative.

1.1 Benchmark: only aggregate risk

In the absence of any other shocks, the portfolio problem of the agents in this economy is

$$\max_y \left\{ u_0(w_0 - q \cdot y) + \mathbb{E}[u(W + W \cdot y)] \right\},$$

where q denotes the price of the asset that pays W in the second period and y is the quantity of that asset demanded by the individual.[4] Under the assumption that all agents are identical, only a no-trade equilibrium is possible, so

$$q = \frac{\mathbb{E}[u'(W) \cdot W]}{u_0'(w_0)}.$$

If the agents could also trade a risk-less asset with payoff $\mathbb{E}(W)$, its price would equal

$$\frac{\mathbb{E}[u'(W)] \cdot \mathbb{E}(W)}{u_0'(w_0)}.$$

The *equity premium* measures how much more expensive this riskless asset would be, namely the relative price of the risk-less to the risky asset (minus one). In the absence of any other risk, thus, the equity premium is

$$\bar{p} = \frac{\mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'(W) \cdot W\right]} - 1. \tag{1}$$

1.2 Idiosyncratic risk

Let random variable S, with $\mathbb{E}(S \mid W) = 0$, be the agents' uninsurable idiosyncratic risk. The agents' consumption is $C = W + S$ and the equity premium is

$$p = \frac{\mathbb{E}\left[u'(C)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'(C) \cdot W\right]} - 1. \tag{2}$$

[3] Acharya and Dogra (2020) quantify some of the effects in Werning (2015).

[4] Here, u_0 is the instantaneous utility function for consumption in the first period, and w_0 is the individuals' wealth in that same period.

Iterating expectations, this is

$$p = \frac{\mathbb{E}\left\{\mathbb{E}\left[u'(W+S) \mid W\right]\right\} \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\mathbb{E}\left[u'(W+S) \mid W\right] \cdot W\right\}} - 1. \tag{3}$$

Remark 1. *Note from Eq.* (3) *that if the economy displays idiosyncratic risk, using Eq.* (1) *instead of Eq.* (2) *misspecifies the equity premium, as it amounts to assuming the equality*

$$\mathbb{E}\left[u'(W+S) \mid W\right] = u'\left(\mathbb{E}(W+S \mid W)\right),$$

which in general requires that the Bernoulli index be quadratic.

1.3 Idiosyncratic risk and the equity premium

Using the quadratic expansion

$$u'(w+s) \approx u'(w) + u''(w) \cdot s + \tfrac{1}{2} \cdot u'''(w) \cdot s^2, \tag{4}$$

we get that

$$\mathbb{E}\left[u'(W+S) \mid W\right] \approx u'(W) + \tfrac{1}{2} \cdot u'''(W) \cdot \mathbb{V}(S \mid W), \tag{5}$$

where $\mathbb{V}$ is the variance operator. This allows us to approximate Eq. (3) by

$$\hat{p} = \frac{\mathbb{E}\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \mathbb{V}(S \mid W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \mathbb{V}(S \mid W)\right] \cdot W\right\}} - 1. \tag{6}$$

Theorem 1 (Irrelevance, 1). *If one of the following two conditions holds, idiosyncratic risk does not affect the equity premium, in the sense that* $\hat{p} = \bar{p}$:

1. *the Bernoulli index is quadratic, or*
2. *the first derivative of the Bernoulli index is homogeneous and* $\mathbb{V}(S \mid W = w) = \sigma w^2$ *for some constant* $\sigma \geq 0$, *almost surely.*

Proof. That the first condition suffices is straightforward, as mentioned in Remark 1: under a quadratic index, $u'''(w) = 0$ for all w, and the equality follows.

To see that the second condition also suffices, suppose that $u'(w) = u'(1)w^{-\rho}$, for some $\rho > 0$, and hence that $u'''(w) = \rho(\rho+1)u(1)w^{-(\rho+2)}$. Substituting this and the functional form of the conditional variance of S, we get

$$\begin{aligned}
\hat{p} &= \frac{\mathbb{E}\left[W^{-\rho} + \rho(\rho+1)\frac{\sigma}{2} \cdot W^{-(\rho+2)} \cdot W^2\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[W^{-\rho} + \rho(\rho+1)\frac{\sigma}{2} \cdot W^{-(\rho+2)} \cdot W^2\right] \cdot W\right\}} - 1 \\
&= \frac{\mathbb{E}\left\{\left[1 + \rho(\rho+1)\frac{\sigma}{2}\right] \cdot W^{-\rho}\right\} \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[1 + \rho(\rho+1)\frac{\sigma}{2}\right] \cdot W^{-(\rho-1)}\right\}} - 1 \\
&= \frac{\mathbb{E}[W^{-\rho}) \cdot \mathbb{E}(W)}{\mathbb{E}(W^{-(\rho-1)}]} - 1 \\
&= \bar{p}.
\end{aligned} \tag{7}$$

□

Remark 2. *The second case in the theorem encompasses the whole class of CRRA Bernoulli indexes: all homogeneous functions (the case explicitly solved in Krueger and Lustig, 2010) as well as all the logarithmic indexes.*

These two cases where one obtains irrelevance, however, appear rather limiting. Quadratic preferences are the least prudent version of prudent preferences, while the assumption that the volatility of idiosyncratic shocks is pro-cyclical and follows a specific functional may be unsatisfactory, in that it requires the assumption that the relative shock S/W be independent of W.

On the other hand:

Theorem 2 (Relevance, 1). *Suppose that the Bernoulli index is exponential: $u(w) = -e^{-\alpha w}$, for some $\alpha > 0$.*

1. *If $\mathbb{V}(S \mid W = w) = {}^{\sigma}\!/_{w}$ for some $\sigma > 0$, then the equity premium is larger in the presence of idiosyncratic risk: $\hat{p} > \bar{p}$.*
2. *If $\mathbb{V}(S \mid W = w) = \sigma w$ for some $\sigma > 0$, then the equity premium is smaller in the presence of idiosyncratic risk: $\hat{p} < \bar{p}$.*

Proof. By direct computation, for this Bernoulli index

$$\hat{p} = \frac{\mathbb{E}\left\{e^{-\alpha W} \cdot \left[1 + \frac{1}{2}\alpha^2 \cdot \mathbb{V}(S \mid W)\right]\right\} \cdot \mathbb{E}(W)}{\mathbb{E}\left\{e^{-\alpha W} \cdot \left[1 + \frac{1}{2}\alpha^2 \cdot \mathbb{V}(S \mid W)\right] \cdot W\right\}} - 1$$

and

$$\bar{p} = \frac{\mathbb{E}\left(e^{-\alpha W}\right) \cdot \mathbb{E}(W)}{\mathbb{E}\left(e^{-\alpha W} \cdot W\right)} - 1.$$

Whether $\hat{p}$ is larger or smaller than $\bar{p}$ depends thus on the sign of

$$\mathbb{E}\left[e^{-\alpha W} \cdot \mathbb{V}(S \mid W)\right] \cdot \mathbb{E}\left(e^{-\alpha W} \cdot W\right) - \mathbb{E}\left(e^{-\alpha W}\right) \cdot \mathbb{E}\left[e^{-\alpha W} \cdot \mathbb{V}(S \mid W) \cdot W\right]. \tag{8}$$

For the first claim, after substituting $\mathbb{V}(S \mid W) = \sigma/W$ into Eq. (8), we need to show that

$$\mathbb{E}\left(e^{-\alpha W} \cdot W^{-1}\right) \cdot \mathbb{E}\left(e^{-\alpha W} \cdot W\right) > \mathbb{E}\left(e^{-\alpha W}\right)^2.$$

If we let V be an (ancillary) random variable distributed identically to W and independent from it, we can rewrite the latter expression as

$$\mathbb{E}\left[e^{-\alpha(W+V)} \cdot \left(\tfrac{V}{W} - 1\right)\right] > 0,$$

and the left-hand side of this inequality is proportional, by a factor of

$$\frac{\Pr(V \neq W)}{2} > 0,$$

to

$$\mathbb{E}\left[e^{-\alpha(W+V)} \cdot \left(\tfrac{V}{W} - 1\right) \mid V > W\right] + \mathbb{E}\left[e^{-\alpha(W+V)} \cdot \left(\tfrac{V}{W} - 1\right) \mid V < W\right].$$

This expression is equivalent to

$$\mathbb{E}\left[e^{-\alpha(W+V)}\cdot\left(\tfrac{V}{W}-1\right)\mid V>W\right]+\mathbb{E}\left[e^{-\alpha(V+W)}\cdot\left(\tfrac{W}{V}-1\right)\mid W<V\right],$$

which, by direct computation, is

$$\mathbb{E}\left[e^{-\alpha(W+V)}\cdot\left(\tfrac{V}{W}+\tfrac{W}{V}-2\right)\mid V>W\right]=\mathbb{E}\left[e^{-\alpha(W+V)}\cdot\tfrac{(V-W)^2}{VW}\mid V>W\right]>0,$$

as needed.

For the second claim, Eq. (8) and $\mathbb{V}(S\mid W)=\sigma W$ imply that we want to prove that

$$\mathbb{E}\left(e^{-\alpha W}\cdot W\right)^2<\mathbb{E}\left(e^{-\alpha W}\right)\cdot\mathbb{E}\left(e^{-\alpha W}\cdot W^2\right).$$

Using the same definition of V as above, this amounts to showing that

$$\mathbb{E}\left[e^{-\alpha(W+V)}\cdot W\cdot V\cdot\left(1-\tfrac{W}{V}\right)\right]<0,$$

which is true by an analogous argument. $\square$

Remark 3. *This theorem covers all CARA Bernoulli indexes.*

A comparison of the previous two theorems suggests a connection between the behavior of the coefficients of risk aversion, the behavior of the conditional variance of the idiosyncratic shocks, and the effect of the latter on the equity premium. As is well known, the relative risk aversion coefficient measures the agents' willingness to pay to insure against *multiplicative* shocks of a given variance. The second statement in Theorem 1 proves that when such willingness to pay is constant on the agents' wealth, if the variance of the multiplicative idiosyncratic shock is also constant, then the shock has no effect on the premium. The absolute risk aversion coefficient approximates the willingness to pay to insure against *additive* shocks of a given variance, and Theorem 2 suggests that when such willingness to pay is constant, the equity premium changes with the presence of additive idiosyncratic risk unless the conditional variance of such risk is constant.

Whether, conditional on aggregate wealth, the idiosyncratic risk is homoskedastic or heteroskedastic is an empirical question and we take no position about it. The irrelevance result of Krueger and Lustig (and our minimal extension in Theorem 1) requires a very particular form of heteroskedasticity, as the following theorem shows.

Theorem 3 (Relevance, 2). *Suppose that the first derivative of the Bernoulli index is homogeneous and $\mathbb{V}(S\mid W=w)=\sigma w^{\kappa}$ for constants $\sigma,\kappa\geq 0$, almost surely.*

1. *If $\kappa<2$, then the equity premium is larger in the presence of idiosyncratic risk: $\hat{p}>\bar{p}$.*
2. *If $\kappa>2$, then the equity premium is smaller in the presence of idiosyncratic risk: $\hat{p}<\bar{p}$.*

Proof. As in the proof of Theorem 1, assume that $u'(w)=u'(1)w^{-\rho}$, for some $\rho>0$. Substituting the functional form of the conditional variance of S, we get, instead of Eq. (7),

$$\hat{p}=\frac{\mathbb{E}\left\{\left[1+\rho(\rho+1)\frac{\sigma}{2}\cdot W^{\kappa-2}\right]\cdot W^{-\rho}\right\}\cdot\mathbb{E}(W)}{\mathbb{E}\left\{\left[1+\rho(\rho+1)\frac{\sigma}{2}\cdot W^{\kappa-2}\right]\cdot W^{-(\rho-1)}\right\}}-1,$$

while

$$\bar{p} = \frac{\mathbb{E}(W^{-\rho}) \cdot \mathbb{E}(W)}{\mathbb{E}[W^{-(\rho-1)}]} - 1.$$

By direct computation, for the first claim we need to show that, whenever $\kappa < 2$,

$$\mathbb{E}\left\{\left[1 + \Delta W^{\kappa-2}\right] \cdot W^{-\rho}\right\} \cdot \mathbb{E}[W^{-(\rho-1)}] > \mathbb{E}(W^{-\rho}) \cdot \mathbb{E}\left\{\left[1 + \Delta W^{\kappa-2}\right] \cdot W^{-(\rho-1)}\right\},$$

where $\Delta = \rho(\rho+1)\sigma/2 > 0$.

Let us define random variable V as in the proof of Theorem 2. By direct computation, we need to argue that

$$\mathbb{E}\left[\Delta \cdot W^{-\rho} \cdot V^{-\rho+1} \cdot \left(W^{\kappa-2} - V^{\kappa-2}\right)\right] > 0.$$

Using the same technique as in the proof of Theorem 2, the left-hand side of this expression is directly proportional, by a factor of $\Pr(W > V)$, to

$$\mathbb{E}\left[\Delta \cdot W^{-\rho} \cdot V^{-\rho} \cdot (V - W) \cdot \left(W^{\kappa-2} - V^{\kappa-2}\right) \mid W > V\right].$$

This expression is positive if, and only if, $\kappa < 2$.

The proof of the second claim is immediate. □

For CRRA indexes, the power functional form of the conditional variance is mathematically useful in Theorem 1, and we conjecture that the irrelevance cannot be guaranteed without it. More importantly, however, the assumption that the conditional variance is pro-cyclical implies that idiosyncratic risk is not as significant when an economy is in a recession as when it is in a boom, which may be untenable.

For CARA indexes, when the idiosyncratic risk is counter-cyclical (and, again, of a particular functional form), its presence increases the equity premium. If in this case the heteroskedasticity is in the direction assumed by Krueger and Lustig (2010), the equity premium is *lower* in the presence of idiosyncratic risk.

As for homoskedastic shocks:

Theorem 4 (Relevance, 3). *Suppose that $u''' > 0$ and that $\mathbb{V}(S \mid W) = \Sigma$, for some constant $\Sigma \geq 0$, almost surely. The equity premium $\hat{p}$ ranges monotonically from*

$$\bar{p} = \frac{\mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'(W) \cdot W\right]} - 1,$$

when $\Sigma = 0$, to

$$\frac{\mathbb{E}\left[u'''(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'''(W) \cdot W\right]} - 1.$$

as $\Sigma \to \infty$. This monotonicity is increasing if, and only if,

$$\frac{\mathbb{C}\text{ov}[u'(W), W]}{\mathbb{E}[u'(W)]} > \frac{\mathbb{C}\text{ov}[u'''(W), W]}{\mathbb{E}[u'''(W)]}, \tag{9}$$

where $\mathbb{C}\text{ov}$ *is the* covariance *operator.*

The proof of this theorem is technically similar to the previous arguments, so we defer it, along with the remaining proofs in the paper, to an appendix.

The denominators on both sides of Eq. (9) are positive, and risk aversion implies that the numerator on the left-hand side is negative, so the ratio on the left-hand side is negative. None of our assumptions pins down the sign of the numerator on the right-hand side, though. If the covariance between the economy's aggregate income and the third derivative of the Bernoulli index is positive, then the variance of the idiosyncratic shock *decreases* the equity premium.

COROLLARY 1. *Suppose that* $\mathbb{V}(S \mid W) = \Sigma > 0$ *almost surely.*

1. *If the first derivative of the Bernoulli index is homogeneous,* $\hat{p}$ *is increasing in* Σ.
2. *If the Bernoulli index is exponential,* $\hat{p}$ *does not depend on* Σ.

As before, the first claim applies to the class of CRRA Bernoulli indexes. It confirms that the irrelevance result in Krueger and Lustig (2010) depends on the specific type of heteroskedasticity that they assume for the idiosyncratic shock. If S is homoskedastic conditional on W, its presence implies a higher equity premium. The second claim provides another irrelevance result, which covers the class of CARA indexes, for the homoskedastic case, as expected.

2 Homogeneous Agents and Long-Lived Assets

If the risk is traded using a long-lived asset, its return must be adjusted to include the price of the asset. In a no-trade equilibrium the problem is simpler, so we concentrate on the more demanding case where the asset is traded.

2.1 Benchmark: only aggregate risk

In the absence of any other risk, the *equity premium* would be

$$\bar{p} = \frac{\mathbb{E}\left[u'(W+q)\right] \cdot \left[\mathbb{E}(W) + q\right]}{\mathbb{E}\left[u'(W+q) \cdot (W+q)\right]} - 1, \tag{10}$$

where q is the price of the asset.[5]

[5] Consider, for a specific application, the case of a stationary OLG economy where individuals live for two periods and the only asset in the economy pays W, i.i.d., every period. The problem of the young generation is

$$\max_{y} \left\{u_0(w_0 - q \cdot y) + \mathbb{E}[u((W+q) \cdot y)]\right\},$$

and its first-order condition is that

$$u_0'(w_0 - q \cdot y) \cdot q = \mathbb{E}[u'((W+q) \cdot y) \cdot (W+q)].$$

Market clearing requires that q be the solution to $q = \mathbb{E}[u'((W+q)) \cdot (W+q)]/u_0'(w_0 - q)$, while a risk-less asset with the same expected return should be priced at $\mathbb{E}[u'(W+q)] \cdot [\mathbb{E}(W) + q]/u_0'(w_0 - q)$. The relative price of the risk-less asset to the risky asset (minus 1) is Eq. (10).

2.2 Idiosyncratic risk

Under idiosyncratic risk, the premium is

$$p = \frac{\mathbb{E}\left[u'(W+q+S)\right] \cdot [\mathbb{E}(W)+q]}{\mathbb{E}\left[u'(W+q+S)\cdot(W+q)\right]} - 1. \tag{11}$$

Using the quadratic expansion

$$u'(w+q+s) \approx u'(w+q) + u''(w+q)\cdot s + \tfrac{1}{2}\cdot u'''(w+q)\cdot s^2,$$

we get the approximation

$$\hat{p} = \frac{\mathbb{E}\left[u'(W+q) + \frac{1}{2}\cdot u'''(W+q)\cdot \mathbb{V}(S \mid W)\right]\cdot[\mathbb{E}(W)+q]}{\mathbb{E}\left\{\left[u'(W+q)+\frac{1}{2}\cdot u'''(W+q)\cdot\mathbb{V}(S \mid W)\right]\cdot(W+q)\right\}} - 1 \tag{12}$$

to the equity premium.

The problem would be a trivial extension of the previous results, were it not for the dependence of q on the distribution of S. The purpose of this paper is not to develop the general comparative statics of this dependence, but to determine how that dependence affects the effect of the distribution of S on the equity premium.

2.3 Idiosyncratic risk and the equity premium

Sometimes it will be convenient to write $\mathbb{V}(S \mid W = w) = v(w, \vartheta)$, for some differentiable function $v : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}_+$, where constant ϑ is a parameter of the conditional distribution of S such that $\partial v/\partial\vartheta > 0$.

Theorem 5 (Relevance, 4). *Suppose that premium $\hat{p}$ is decreasing in asset price q, keeping ϑ fixed, and that q is increasing in ϑ.*[6] *Then, a necessary condition for $\hat{p}$ to be non-decreasing in ϑ is that*

$$\frac{\mathbb{C}\mathrm{ov}[u'(W+q), W]}{\mathbb{E}[u'(W+q)]} > \frac{\mathbb{C}\mathrm{ov}[u'''(W+q)\cdot\mathbb{V}(S \mid W), W]}{\mathbb{E}[u'''(W+q)\cdot\mathbb{V}(S \mid W)]}. \tag{13}$$

If, on the other hand, given ϑ premium $\hat{p}$ is non-decreasing in q, then Eq. (13) *suffices for $\hat{p}$ to be increasing in ϑ.*

Note that the tension between the direct effect of q and the direct effect of Σ arises when the former is negative. A complication when trying to determine the sign of the latter effect is that it involves the response of the third derivative of the Bernoulli index, since the asset is traded. Instead of attempting a full characterization, we find sufficient conditions:

Theorem 6 (Relevance, 5). *Suppose that the Bernoulli index is differentiable four times, and that $u^{[4]} < 0$.*[7] *Premium $\hat{p}$ is decreasing in asset price q if*

$$\mathbb{C}\mathrm{ov}[U''' \cdot \mathbb{V}(S \mid W), W] \leq 0 \tag{14}$$

[6] As would be the case in Aiyagari (1994), for instance.

[7] That is, assume that the Bernoulli index displays "temperance", or that the person is fourth-degree risk averse. See Ekern (1980) and Eeckhoudt et al (1995).

and

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U', W)}{\mathbb{E}(U')}, \frac{\mathbb{C}\mathrm{ov}[U''' \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U''' \cdot \mathbb{V}(S \mid W)]}\right\} \geq \max\left\{\frac{\mathbb{C}\mathrm{ov}(U'', W)}{\mathbb{E}(U'')}, \frac{\mathbb{C}\mathrm{ov}[U^{[4]} \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U^{[4]} \cdot \mathbb{V}(S \mid W)]}\right\}, \tag{15}$$

where $U^{[n]} = u^{[n]}(W + q)$.

2.3.1 CARA preferences and homoskedastic risk

Considering the case of homoskedastic idiosyncratic risk, with $\mathbb{V}(S \mid W) = \Sigma$ almost surely on W, for the rest of this subsection we a assume that q depends differentiably on Σ, with first derivative q'.

It is useful to re-write Eq. (12) as

$$\hat{p} = \frac{\mathbb{E}[u'(W + q)] \cdot [\mathbb{E}(W) + q] + \frac{1}{2} \cdot \mathbb{E}[u'''(W + q)] \cdot [\mathbb{E}(W) + q] \cdot \Sigma}{\mathbb{E}[u'(W + q) \cdot (W + q)] + \frac{1}{2} \cdot \mathbb{E}[u'''(W + q) \cdot (W + q)] \cdot \Sigma} - 1. \tag{16}$$

With multiple instances of Σ and q, a full characterization of the total differential of $\hat{p}$ with respect to Σ is possible, but cumbersome and rather uninformative. Instead, we derive independent necessary and sufficient conditions, focusing on the case when idiosyncratic risk increases the premium.

Theorems 5 and 6 immediately imply the following two results for this case:

Corollary 2 (Relevance). *Suppose that premium $\hat{p}$ is decreasing in asset price q, keeping Σ fixed, and that q is increasing in Σ. Then, a necessary condition for $\hat{p}$ to be non-decreasing in Σ is that*

$$\frac{\mathbb{C}\mathrm{ov}[u'(W + q), W]}{\mathbb{E}[u'(W + q)]} > \frac{\mathbb{C}\mathrm{ov}[u'''(W + q), W]}{\mathbb{E}[u'''(W + q)]}. \tag{17}$$

Corollary 3 (Relevance). *Suppose that the Bernoulli index is differentiable four times, and that $u^{[4]} < 0$. Premium $\hat{p}$ is decreasing in asset price q, given variance Σ, if*

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U', W)}{\mathbb{E}(U')}, \frac{\mathbb{C}\mathrm{ov}(U''', W)}{\mathbb{E}(U''')}\right\} \geq \max\left\{\frac{\mathbb{C}\mathrm{ov}(U'', W)}{\mathbb{E}(U'')}, \frac{\mathbb{C}\mathrm{ov}(U^{[4]}, W)}{\mathbb{E}(U^{[4]})}\right\}. \tag{18}$$

To be sure, note that the condition that $\mathbb{C}\mathrm{ov}[U''', W] \leq 0$, which specializes Eq. (14) to the case at hand, does not need to be assumed explicitly, as it is implied by the assumption that $u^{[4]} < 0$.

Unfortunately, these two corollaries fail to provide a full characterization of the sign of the effect of Σ on p. Still,

Theorem 7 (Relevance, 6). *If the Bernoulli index is exponential, then the equity premium $\hat{p}$ is decreasing in Σ.*

In an infinite-horizon, Markovian economy, one cannot model the price of the asset, nor the equity premium, as constants. If instead one defines, respectively, as the random variables Q and

$$P = \frac{\mathbb{E}\left[u'(W+Q+S) \mid P\right] \cdot \left[\mathbb{E}(W) + Q \mid P\right]}{\mathbb{E}\left[u'(W+Q+S) \cdot (W+Q) \mid P\right]} - 1.$$

both of which are independent of S, the results above are indicative of the direction of the effect of idiosyncratic risk on the distribution of P.

In this case, function u must be interpreted as the value function of the individual's dynamic optimization program, and Theorem 7 suggests that exponential functions[8] generate equity premia for traded long-lived assets that decreases, in first-order stochastic dominance, as the idiosyncratic shocks become homoskedastically more volatile. This result, however, is not a consequence of the direct effect of this volatility, but of its indirect effect via the price of the asset, assuming that the latter is increasing on the volatility, again in the sense of first-order stochastic dominance.

2.3.2 CRRA preferences and heteroskedastic risk

Considering now the case where $\mathbb{V}(S \mid W = w) = \sigma w^2$, with $\sigma \geq 0$.[9] As before, we a assume that q depends differentiably on σ, with first derivative q'.

THEOREM 8 (Relevance, 7). *Suppose that the Bernoulli index is homogeneous of degree $-\rho < 0$ and that*

$$q \leq \frac{\min\{\rho, 1\}}{2} \cdot \inf \mathcal{W}.$$

Then the equity premium $\hat{p}$ is decreasing in σ.

As before, this result suggests that in an infinite-horizon, Markovian economy where the agents have Epstein-Zin preferences with CRRA certainty equivalent functions, the equity premium for traded long-lived assets decreases, in first-order stochastic dominance, as σ increases.

3 Heterogeneous Agents

Our analysis so far has, by necessity, applied to no-trade equilibria of the economy. With ex-ante heterogeneity, this is no longer the case, which makes the results more evasive but also more interesting.

Going back to the case of short-lived assets, suppose now that A-dimensional random variable R denotes the returns of the existing assets. There are I classes of agents, indexed by $i = 1, \ldots, I$, and in each class there is a continuum of mass μ_i of individuals. Each individual of class i has stochastic discount factor *function* $m_i : \mathbb{R} \to \mathbb{R} \in \mathbb{C}^3$ and is endowed with an *initial* portfolio $\bar{y}_i \in \mathbb{R}^A$ of the assets. Her *initial* income, if she does not trade, is the random variable $R \cdot \bar{y}_i$, and the aggregate income is $W = R \cdot \sum_i \mu_i \bar{y}_i$. These last two random variables are exogenous.

For simplicity, we normalize $\sum_i \mu_i = 1$.

[8] For instance, those in the Epstein-Zin family with exponential certainty equivalent functions.

[9] The result can be extended to any power $1 < \beta < 3$, at the expense of some analytical complications.

3.1 Aggregate risk

After trade in assets, each individual of type i holds a portfolio y_i and her income is $W_i = R \cdot y_i$. At prices $q \in \mathbb{R}^A$, if the marginal utility of present consumption is unity, the first-order conditions of class-i individuals are that

$$q = \mathbb{E}\left[m_i(W_i)R\right]. \tag{19}$$

This implies that we can use $\sum_i \mu_i m_i(W_i)$ as pricing kernel. At equilibrium, asset prices

$$q = \mathbb{E}\left[\sum_i \mu_i m_i(W_i)R\right]$$

must ensure that $\sum_i \mu_i y_i = \sum_i \mu_i \bar{y}_i$.

The market value of the equity in the economy is

$$q \cdot \sum_i \mu_i \bar{y}_i = \mathbb{E}\left[\sum_i \mu_i m_i(W_i)R\right] \cdot \sum_i \mu_i \bar{y}_i = \mathbb{E}\left[\sum_i \mu_i m_i(W_i)W\right].$$

A risk-less asset with the same expected return would be priced at

$$\mathbb{E}\left[\sum_i \mu_i m_i(W_i)\right] \cdot E(W),$$

and hence the equity premium is

$$\bar{p} = \frac{\mathbb{E}\left[\sum_i \mu_i m_i(W_i)\right] \cdot E(W)}{\mathbb{E}\left[\sum_i \mu_i m_i(W_i) \cdot W\right]} - 1. \tag{20}$$

3.2 Idiosyncratic Risk

In order to introduce idiosyncratic risk, for each class i let S_i be a random variable with conditional expectation $\mathbb{E}(S_i \mid R) = 0$. Each individual of class i receives an *i.i.d.* realization of S_i, in addition to the return of her portfolio, so her consumption is $C_i = W_i + S_i$. In this case, Eq. (20) becomes

$$p = \frac{\mathbb{E}\left\{\mathbb{E}\left[\sum_i \mu_i m_i(W_i + S_i) \mid R\right]\right\} \cdot E(W)}{\mathbb{E}\left\{\mathbb{E}\left[\sum_i \mu_i m_i(W_i + S_i) \mid R\right] \cdot W\right\}} - 1. \tag{21}$$

The quadratic expansion

$$m(w + s) \approx m(w) + m'(w) \cdot s + \tfrac{1}{2} \cdot m''(w) \cdot s^2,$$

which is analogous to Eq. (4), yields

$$\hat{p} = \frac{\mathbb{E}\left[\sum_i \mu_i m_i(W_i)\right] \cdot \mathbb{E}(W) + \frac{1}{2}\mathbb{E}\left[\sum_i \mu_i m_i''(W_i)\mathbb{V}(S_i \mid R)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[\sum_i \mu_i m_i(W_i) \cdot W\right] + \frac{1}{2}\mathbb{E}\left[\sum_i \mu_i m_i''(W_i)\mathbb{V}(S_i \mid R) \cdot W\right]} - 1 \tag{22}$$

as an approximation to Eq. (21).

3.3 Idiosyncratic risk and the equity premium

The following result is analogous to the first claim in Theorem 1, and its argument only slightly more complicated.

THEOREM 9 (Irrelevance, 2). *Suppose that all the stochastic discount factor functions m_i are linear. Then, idiosyncratic risk has no effect on the equity premium, in the sense that $\hat{p} = \bar{p}$.*

Of course, the assumption that all m_i are linear is limiting. Unfortunately, Eq. (22) is complicated and we need to make some simplifying assumptions in order to develop the expression. To begin, we assume that all idiosyncratic shocks are homoskedastic, with constant $\mathbb{V}(S_i \mid R) = \sigma_i$. In this case, the premium simplifies to

$$\hat{p} = \frac{\sum_i \mu_i \mathbb{E}(M_i) \cdot \mathbb{E}(W) + \frac{1}{2} \sum_i \mu_i \sigma_i \mathbb{E}(M_i'') \cdot \mathbb{E}(W)}{\sum_i \mu_i \mathbb{E}(M_i \cdot W) + \frac{1}{2} \sum_i \mu_i \sigma_i \mathbb{E}(M_i'' \cdot W)} - 1,$$

where for simplicity we denote the random variables $M_i = m_i(W_i)$ and $M_i'' = m_i''(W_i)$.[10] Letting $\boldsymbol{\sigma} = (\sigma_i)_{i=1}^I$, we can re-write the right-hand side of this expression as

$$\hat{\pi}(\boldsymbol{\sigma}, \mathbf{y}) = \frac{f(\mathbf{y}) + \frac{1}{2} \sum_i \mu_i \sigma_i \mathbb{E}(M_i'') \cdot \mathbb{E}(W)}{\varphi(\mathbf{y}) + \frac{1}{2} \sum_i \mu_i \sigma_i \mathbb{E}(M_i'' \cdot W)} - 1.$$

THEOREM 10 (Relevance, 8). *The direct effect of an increase in the variance of a class's idiosyncratic shock is positive, in the sense that*

$$\frac{\partial \hat{\pi}}{\partial \sigma_i}(\boldsymbol{\sigma}, \mathbf{y}) > 0,$$

if, and only if,

$$-\frac{\mathbb{C}\text{ov}(M_i'', W)}{\mathbb{E}(M_i'' \cdot W)} > \hat{p}.$$

3.4 Further simplifying assumptions, 1

Suppose, furthermore, that $\sigma_i = \sigma$ for all i. With an obvious but minor abuse of notation, we get

$$\hat{\pi}(\sigma, \mathbf{y}) = \frac{f(\mathbf{y}) + \frac{1}{2}\sigma \cdot \sum_i \mu_i \mathbb{E}(M_i'') \cdot \mathbb{E}(W)}{\varphi(\mathbf{y}) + \frac{1}{2}\sigma \cdot \sum_i \mu_i \mathbb{E}(M_i'' \cdot W)} - 1.$$

In this case, the following is straightforward:

COROLLARY 4 (Relevance). *The direct effect of an increase in the common variance of the idiosyncratic shocks is positive, in the sense that*

$$\frac{\partial \hat{\pi}}{\partial \sigma}(\sigma, \mathbf{y}) > 0,$$

if, and only if,

$$-\frac{\sum_i \mu_i \mathbb{C}\text{ov}(M_i'', W)}{\sum_i \mu_i \mathbb{E}(M_i'' \cdot W)} > \hat{p}.$$

[10] It is important to note that these variables are not primitives of the economy, as they depend on the portfolios y_i, which are endogenous.

3.5 Further simplifying assumptions, 2

Alternative, suppose that all classes of individuals have the same stochastic discount factor function $m_i = m : \mathbb{R} \to \mathbb{R}$, while allowing for heterogeneity in the variance of the idiosyncratic shocks. Instead of Eq. (4), we can use the approximation

$$m_i(r \cdot y_i + s) = m(r \cdot \bar{y}) + m'(r \cdot \bar{y})[r \cdot (y_i - \bar{y}) + s] + \tfrac{1}{2} m''(r \cdot \bar{y})[r \cdot (y_i - \bar{y}) + s]^2.$$

This implies that $\mathbb{E}[m_i(W_i + S_i) \mid R]$ is approximately equal to

$$m(W) + m'(W)(W_i - W) + \tfrac{1}{2}[(W_i - W)^2 + \sigma_i],$$

since $E(S_i \mid R) = 0$ and $\mathbb{V}(S_i \mid R) = \sigma_i$. Aggregating, we can approximate the pricing kernel

$$\mathbb{E}\left[\textstyle\sum_i \mu_i m_i(R \cdot y_i + S_i) \mid R\right]$$

by

$$M + \tfrac{1}{2} M'' \cdot (D + \bar{\sigma}),$$

for the random variables $M = m(W)$, $M'' = m''(W)$ and $D = \sum_i \mu_i (W_i - W)^2$; and the constant $\bar{\sigma} = \sum_i \mu_i \sigma_i$. Random variable D is the cross-sectional dispersion in final income; this variable is random, as it depends on R, and endogenous, as it depends on $\mathbf{y}$. Constant $\bar{\sigma}$ is the cross-sectional average variance of idiosyncratic shocks, which is exogenous. Random variables M and M'' are exogenous too.

We can now approximate the equity premium by

$$\tilde{p} = \frac{\mathbb{E}\left[M + \frac{1}{2} M'' \cdot (D + \bar{\sigma})\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[M + \frac{1}{2} M'' \cdot (D + \bar{\sigma})\right] \cdot W\right\}} - 1. \tag{23}$$

As before, we denote the right-hand side of this expression as

$$\tilde{\pi}(\boldsymbol{\sigma}, D) = \frac{f(D) + \frac{1}{2}\bar{\sigma}\mathbb{E}(M'') \cdot \mathbb{E}(W)}{\varphi(D) + \frac{1}{2}\bar{\sigma}\mathbb{E}(M'' \cdot W)} - 1.$$

Theorem 11 (Relevance, 9). *The direct effect of an increase in the variance of a class's idiosyncratic shock is positive, in the sense that*

$$\frac{\partial \tilde{\pi}}{\partial \sigma_i}(\boldsymbol{\sigma}, D) > 0,$$

if, and only if,

$$-\frac{\mathbb{C}\mathrm{ov}(M'', W)}{\mathbb{E}(M'' \cdot W)} > \tilde{p}.$$

Proof. By direct computation, ${}^{\partial \tilde{\pi}}/_{\partial \sigma_i}(\boldsymbol{\sigma}, D) > 0$ if, and only if,

$$\mathbb{E}(M'') \cdot \mathbb{E}(W) \cdot \left[\varphi(D) + \tfrac{1}{2}\bar{\sigma}\mathbb{E}(M'' \cdot W)\right] > \mathbb{E}(M'' \cdot W) \cdot \left[f(D) + \tfrac{1}{2}\bar{\sigma}\mathbb{E}(M'') \cdot \mathbb{E}(W)\right].$$

This is equivalent to

$$\frac{\mathbb{E}(M'') \cdot \mathbb{E}(W)}{\mathbb{E}(M'' \cdot W)} > \tilde{p} + 1, \tag{24}$$

from where the result follows by definition. □

From an empirical perspective, this result is useful since the left-hand side of this expression is exogenous and the right-hand side, while endogenous, is observable. From a theoretical point of view, the result remarks that the effects of cross-sectional heterogeneity (in income) and idiosyncratic risk go in opposite directions, in the following sense: Suppose that $\sigma_i = 0$ for all i, so that there is no idiosyncratic risk initially. Using Eqs. (23) and (24), we conclude from the theorem that

$$\frac{\partial \tilde{\pi}}{\partial \sigma_i}(\mathbf{0}, D) > 0 \Leftrightarrow \frac{\mathbb{E}[m''(W)] \cdot \mathbb{E}(W)}{\mathbb{E}[m''(W) \cdot W]} - 1 > \frac{\mathbb{E}\left[m(W) + \frac{1}{2}m''(W) \cdot D\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[m(W) + \frac{1}{2}m''(W) \cdot D\right] \cdot W\right\}} - 1.$$

Notice that the condition on the right-hand side of the implication compares a primitive constant that does not depend on income heterogeneity with the equity premium that results from income heterogeneity alone. The result then says that the direct effect of introducing idiosyncratic risk is positive if, and only if, income heterogeneity by itself makes the premium sufficiently low.

Of course, the comparative statics presented in the last two theorems are only partial, and complete differentiation would require an understanding of how the individual portfolios change, at equilibrium, after a perturbation to the parameter capturing the volatility of the idiosyncratic shock(s). Such understanding is beyond the scope of this paper.

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Appendix: proofs

Proof of Theorem 4: Under the assumption of this theorem, Eq. (6) rewrites as

$$\hat{p} = \frac{\mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W) + \frac{1}{2} \cdot \mathbb{E}\left[u'''(W)\right] \cdot \mathbb{E}(W) \cdot \Sigma}{\mathbb{E}\left[u'(W) \cdot W\right] + \frac{1}{2} \cdot \mathbb{E}\left[u'''(W) \cdot W\right] \cdot \Sigma} - 1.$$

By direct computation $\lim_{\Sigma \to 0} \hat{p} = \bar{p}$ and

$$\lim_{\Sigma \to \infty} \hat{p} = \frac{\mathbb{E}\left[u'''(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'''(W) \cdot W\right]} - 1.$$

Also, note that, since $\mathbb{E}[u'''(W)] > 0$, $\hat{p}$ is increasing in Σ if, and only if,

$$\frac{\mathbb{E}\left[u'(W)\right]}{\mathbb{E}\left[u'(W) \cdot W\right]} < \frac{\mathbb{E}\left[u'''(W)\right]}{\mathbb{E}\left[u'''(W) \cdot W\right]}.$$

By monotonicity and (strict) prudence, both the numerators and the denominators on both sides of Eq. (9) are positive numbers, and we can rewrite the expression as

$$\frac{\mathbb{E}\left[u'(W) \cdot W\right]}{\mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W)} - 1 > \frac{\mathbb{E}\left[u'''(W) \cdot W\right]}{\mathbb{E}\left[u'''(W)\right] \cdot \mathbb{E}(W)} - 1.$$

which is equivalent to Eq. (9). □

Proof of Corollary 1: For the first statement, by Theorem 4 it suffices to show that Eq. (9) holds strictly. Assume, as in the proof of the second part of Theorem 1, that $u'(w) = u'(1)w^{-\rho}$, where $\rho > 0$ since u is strictly concave. By direct computation, it suffices to show that

$$\mathbb{E}(W^{-\rho+1}) \cdot \mathbb{E}(W^{-\rho-2}) - \mathbb{E}(W^{-\rho-1}) \cdot \mathbb{E}(W^{-\rho}) > 0.$$

Letting V be an (ancillary) random variable distributed identically to W and independent from it, the latter expression is equivalent to

$$\mathbb{E}\left(V^{-\rho+1} \cdot W^{-\rho-2} - V^{-\rho} \cdot W^{-\rho-1}\right) > 0.$$

Now, this expectation equals

$$\mathbb{E}\left[(V \cdot W)^{-\rho} \cdot \left(\tfrac{V}{W} - 1\right) \cdot \left(\tfrac{1}{W} - \tfrac{1}{V}\right)\right] + \mathbb{E}\left[(V \cdot W)^{-\rho} \cdot \left(\tfrac{1}{W} - \tfrac{1}{V}\right)\right].$$

Note that the first summand of the last expression is strictly positive, since both random variables are non-degenerate and take only strictly positive values, so the integrand is positive. The second summand is null, since they are identically distributed. It follows that the sum is strictly positive, as needed.

To prove the second claim, suppose that $u(w) = -e^{-\alpha w}$, where $\alpha > 0$. Note that

$$\frac{\mathbb{E}\left[u'(W)\cdot W\right]}{\mathbb{E}\left[u'(W)\right]} = \frac{\mathbb{E}(e^{-\alpha W}\cdot W)}{\mathbb{E}(e^{-\alpha W})} = \frac{\mathbb{E}\left[u'''(W)\cdot W\right]}{\mathbb{E}\left[u'''(W)\right]}.$$

It follows from Theorem 4 that $\hat{p} = \bar{p}$, regardless of the distribution of the idiosyncratic risk. □

Proof of Theorem 5: Obviously,

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}\vartheta} = \frac{\partial\hat{p}}{\partial q}\cdot\frac{\mathrm{d}q}{\mathrm{d}\vartheta} + \frac{\partial\hat{p}}{\partial\vartheta}.$$

For the first result, note that the first summand on the right-hand side of the last expression is negative, so a necessary condition for the sum to be positive is that the second summand be positive. For the second result, under the assumptions the first summand is non-negative, so the sum is positive if so is the second summand. In both cases, all one needs to observe is that $\partial\hat{p}/\partial\vartheta > 0$. The proof that this inequality is equivalent to Eq. (13) is similar to the proof of Theorem 4, so we omit it. □

Proof of Theorem 6: We can write Eq. (12) as

$$\hat{p} = \frac{f(q) + \Sigma\cdot g(q)}{\varphi(q) + \Sigma\cdot\gamma(q)},$$

where

$$f(q) = \mathbb{E}[u'(W+q)]\cdot[E(W)+q],$$
$$g(q) = \tfrac{1}{2}\mathbb{E}[u'''(W+q)\cdot\mathbb{V}(S\mid W)]\cdot[E(W)+q],$$
$$\varphi(q) = \mathbb{E}[u'(W+q)\cdot(W+q)]$$

and

$$\gamma(q) = \tfrac{1}{2}\mathbb{E}[u'''(W+q)\cdot\mathbb{V}(S\mid W)\cdot(W+q)].$$

With this formulation, $\hat{p}$ is decreasing in q if, and only if,

$$[f'(q) + \Sigma\cdot g'(q)]\cdot[\varphi(q) + \Sigma\cdot\gamma(q)] < [\varphi'(q) + \Sigma\cdot\gamma'(q)]\cdot[f(q) + \Sigma\cdot g(q)],$$

which holds true if

$$f'(q)\cdot\varphi(q) < \varphi'(q)\cdot f(q) \tag{25}$$
$$f'(q)\cdot\gamma(q) \le \varphi'(q)\cdot g(q) \tag{26}$$
$$g'(q)\cdot\varphi(q) \le \gamma'(q)\cdot f(q) \tag{27}$$
$$g'(q)\cdot\gamma(q) \le \gamma'(q)\cdot g(q). \tag{28}$$

Upon substitution, Eq. (25) is equivalent to

$$\left\{\mathbb{E}(U'')[\mathbb{E}(W)+q] + \mathbb{E}(U')\right\}\cdot\mathbb{E}[U'\cdot(W+q)] < \left\{\mathbb{E}[U''\cdot(W+q)] + \mathbb{E}(U')\right\}\cdot E(U')\cdot[\mathbb{E}(W)+q],$$

which is, by direct computation,

$$\left\{\mathbb{E}(U'')\cdot\mathbb{E}[U'\cdot(W+q)] - \mathbb{E}(U')\cdot\mathbb{E}[U''\cdot(W+q)]\right\}\cdot[\mathbb{E}(W)+q] + \mathbb{E}(U')\cdot\mathbb{C}\mathrm{ov}(U',W) < 0. \tag{$*$}$$

Since $u' > 0$ and $u'' < 0$, we have that $\mathbb{E}(U') > 0$ and $\mathbb{C}\mathrm{ov}(U',W) < 0$, so it suffices that

$$\mathbb{E}(U'')\cdot\mathbb{E}[U'\cdot(W+q)] \le \mathbb{E}(U')\cdot\mathbb{E}[U''\cdot(W+q)],$$

for inequality $(*)$ to hold, as $\mathbb{E}(W)+q>0$. As in the proof of Theorem 4, this is equivalent to

$$\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')} \geq \frac{\mathbb{C}\mathrm{ov}(U'',W)}{\mathbb{E}(U'')},$$

which is one of the inequalities that are part of Eq. (15).

Similarly, Eq. (26) is equivalent to the requirement that the sum of

$$\big\{\mathbb{E}(U'')\cdot\mathbb{E}[U'''\cdot\mathbb{V}(S\mid W)\cdot(W+q)]-\mathbb{E}[U'''\cdot\mathbb{V}(S\mid W)]\cdot\mathbb{E}[U''\cdot(W+q)]\big\}\cdot[\mathbb{E}(W)+q] \qquad (**)$$

and

$$\mathbb{E}(U')\cdot\mathbb{C}\mathrm{ov}[U'''\cdot\mathbb{V}(S\mid W),W] \qquad (***)$$

be non-positive.

Since $u'>0$ and $\mathbb{C}\mathrm{ov}[U'''\cdot\mathbb{V}(S\mid W),W]\leq 0$, we have that the expression in $(***)$ is non-positive. On the other hand, since $\mathbb{E}(W)+q>0$, for inequality $(**)$ to hold it suffices that

$$\mathbb{E}(U'')\cdot\mathbb{E}[U'''\cdot\mathbb{V}(S\mid W)\cdot(W+q)]\leq\mathbb{E}[U'''\cdot\mathbb{V}(S\mid W)]\cdot\mathbb{E}[U''\cdot(W+q)],$$

which is equivalent to

$$\frac{\mathbb{C}\mathrm{ov}[U'''\cdot\mathbb{V}(S\mid W),W]}{\mathbb{E}[U'''\cdot\mathbb{V}(S\mid W)]} \geq \frac{\mathbb{C}\mathrm{ov}(U'',W)}{\mathbb{E}(U'')}.$$

For Eqs. (25) and (26) to hold true, it thus suffices that

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')},\frac{\mathbb{C}\mathrm{ov}[U'''\cdot\mathbb{V}(S\mid W),W]}{\mathbb{E}[U'''\cdot\mathbb{V}(S\mid W)]}\right\} \geq \frac{\mathbb{C}\mathrm{ov}(U'',W)}{\mathbb{E}(U'')}.$$

By a virtually identical analysis, using that $u'''>0$ and $u^{[4]}<0$, one can prove that

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')},\frac{\mathbb{C}\mathrm{ov}[U'''\cdot\mathbb{V}(S\mid W),W]}{\mathbb{E}[U'''\cdot\mathbb{V}(S\mid W)]}\right\} \geq \frac{\mathbb{C}\mathrm{ov}[U^{[4]}\cdot\mathbb{V}(S\mid W),W]}{\mathbb{E}[U^{[4]}\cdot\mathbb{V}(S\mid W)]}$$

suffices for Eqs. (27) and (28) $\square$

Proof of Theorem 7: Let $u(w)=-e^{-\alpha w}$, for some $\alpha>0$. Then,

$$u^{[n]}(w)=(-1)^{n+1}\alpha^n e^{-\alpha w}=(-\alpha)^n u(w),$$

which implies that $u^{[4]}<0$. Also,

$$\mathbb{C}\mathrm{ov}[u^{[n]}(W+q),w]=(-\alpha)^n\mathbb{C}\mathrm{ov}[u(W+q),w]$$

and

$$\mathbb{E}[u^{[n]}(W+q)]=(-\alpha)^n\mathbb{E}[u(W+q)],$$

which imply that

$$\frac{\mathbb{C}\mathrm{ov}[u^{[n]}(W+q),w]}{\mathbb{E}[u^{[n]}(W+q)]}=\frac{\mathbb{C}\mathrm{ov}[u(W+q),w]}{\mathbb{E}[u(W+q)]}$$

for all orders of differentiation. Corollary 3 implies that, with Σ fixed, $\hat{p}$ is decreasing in q. If we can argue that the price of the asset is increasing in Σ, then Corollary 2 implies that condition (13), which does not hold true, is necessary for $\hat{p}$ to be non-decreasing in Σ.

To see that, indeed, $q' > 0$, note that

$$q = \mathbb{E}\left[u'(W + q + S) \cdot (W + q)\right]$$

re-writes, in the case of exponential preferences, as

$$q = e^{-\alpha q} \cdot \left\{\mathbb{E}[u'(W + S) \cdot W] + \mathbb{E}[u'(W + S)] \cdot q\right\},$$

so

$$e^{\alpha q} = \frac{\mathbb{E}[u'(W + S) \cdot W]}{q} + \mathbb{E}[u'(W + S)].$$

This expression is transcendental, so we can only obtain q' by implicit differentiation:

$$\left\{e^{\alpha q} + \frac{\mathbb{E}[u'(W + S) \cdot W]}{q^2}\right\} \cdot q' = \frac{\partial}{\partial \Sigma}\left\{\frac{\mathbb{E}[u'(W + S) \cdot W]}{q} + \mathbb{E}[u'(W + S)]\right\}.$$

Since exponential preferences are strictly increasing and strictly prudent, we know that

$$\mathbb{E}[u'(W + S) \cdot W] = \mathbb{E}\left\{\mathbb{E}[u'(W + S) \mid W] \cdot W\right\}$$

and

$$\mathbb{E}[u'(W + S)] = \mathbb{E}\left\{\mathbb{E}[u'(W + S) \mid W]\right\}$$

are both increasing in Σ, which implies that $q' > 0$, as needed. $\square$

Proof of Theorem 8: The strategy for the proof is the same as in Theorem 7: we will argue that, under the assumption of the theorem, all the conditions that make Eq. (13) necessary for $\hat{p}$ to be non-decreasing in σ are satisfied, but not Eq. (13) itself. Using the functional forms $u'(w) = u'(1)w^{-\rho}$ and $\mathbb{V}(S \mid W) = \sigma W^2$, we need to prove the following:

(a) that q is increasing in σ, namely that implicitly differentiating

$$q = u'(1) \cdot \mathbb{E}[(W + q + S)^{-\rho} \cdot (W + q)] \tag{29}$$

with respect to σ yields $q' > 0$;

(b) Equation (14);

(c) that

$$\frac{\mathbb{C}\text{ov}(U', W)}{\mathbb{E}(U')} \geq \frac{\mathbb{C}\text{ov}(U'', W)}{\mathbb{E}(U'')} \quad \text{and} \quad \frac{\mathbb{C}\text{ov}[U''' \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U''' \cdot \mathbb{V}(S \mid W)]} \geq \frac{\mathbb{C}\text{ov}[U^{[4]} \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U^{[4]} \cdot \mathbb{V}(S \mid W)]},$$

two of the inequalities in Eq. (15), which in the case are

$$\frac{\mathbb{E}[(W + q)^{-\rho} \cdot W]}{\mathbb{E}[(W + q)^{-\rho}]} \geq \frac{\mathbb{E}[(W + q)^{-(\rho+1)} \cdot W]}{\mathbb{E}[(W + q)^{-(\rho+1)}]} \tag{30}$$

and

$$\frac{\mathbb{E}[(W + q)^{-(\rho+2)} \cdot W^3]}{\mathbb{E}[(W + q)^{-(\rho+2)} \cdot W^2]} \geq \frac{\mathbb{E}[(W + q)^{-(\rho+3)} \cdot W^3]}{\mathbb{E}[(W + q)^{-(\rho+3)} \cdot W^2]}; \tag{31}$$

(d) that

$$\frac{\mathbb{C}\text{ov}[U''' \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U''' \cdot \mathbb{V}(S \mid W)]} \geq \frac{\mathbb{C}\text{ov}(U'', W)}{\mathbb{E}(U'')},$$

another one of the inequalities in Eq. (15), which is

$$\frac{\mathbb{E}[(W + q)^{-(\rho+2)} \cdot W^3]}{\mathbb{E}[(W + q)^{-(\rho+2)} \cdot W^2]} \geq \frac{\mathbb{E}[(W + q)^{-(\rho+1)} \cdot W]}{\mathbb{E}[(W + q)^{-(\rho+1)}]}; \tag{32}$$

(e) that

$$\frac{\mathbb{C}\text{ov}(U', W)}{\mathbb{E}(U')} \geq \frac{\mathbb{C}\text{ov}[U^{[4]} \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U^{[4]} \cdot \mathbb{V}(S \mid W)]}$$

the final inequality in Eq. (15), which is

$$\frac{\mathbb{E}[(W+q)^{-\rho} \cdot W]}{\mathbb{E}[(W+q)^{-\rho}]} \geq \frac{\mathbb{E}[(W+q)^{-(\rho+3)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+3)} \cdot W^2]} \tag{33}$$

under the functional forms; and

(f) that Eq. (13) fails, namely that

$$\frac{\mathbb{E}[(W+q)^{-\rho} \cdot W]}{\mathbb{E}[(W+q)^{-\rho}]} \leq \frac{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^2]}. \tag{34}$$

Of course, we proceed one by one:

(a) From Eq. (29), by the implicit function theorem q' equals the product of

$$\frac{u'(1)}{\mathbb{E}\left[1 + \rho(W+q+S)^{-(\rho+1)} \cdot (W+q) - (W+q+S)^{-\rho}\right]} \tag{$*$}$$

and

$$\mathbb{E}\left\{\frac{\partial}{\partial \sigma}\mathbb{E}[(W+q+S)^{-\rho} \mid W] \cdot (W+q)\right\}, \tag{$**$}$$

so long as the denominator on the former expression is non-zero. We actually want that denominator to be strictly positive, which is the case, since $\rho > 0$, $W + q > 0$ with probability one by assumption, and

$$\rho(w+q) > 0 \Leftrightarrow 1 + \frac{\rho(w+q)}{(w+q+s)^{\rho+1}} > \frac{1}{(w+q+s)^{\rho}}.$$

Since $u'(1) > 0$, it follows that the term in Eq. ($*$) is strictly positive.

That the term in Eq. ($**$) is also positive is immediate, since $(w+q+s)^{-\rho}$ is strictly convex in s, and an increase in σ is a mean-preserving spread of S given W.

(b) For Eq. (14), it suffices for us to argue that $u'''(W+q) \cdot \mathbb{V}(S \mid W)$ and W are anti-comonotone with probability one. Letting the function

$$\eta(w) = u'''(w+q) \cdot \mathbb{V}(S \mid W = w) = \sigma\rho(\rho+1)u'(1)w^2,$$

we have that $\eta'(w) \leq 0$ so long as $w \geq 2q/\rho$. Since $q \leq {}^{\rho}\!/_{2} \inf \mathcal{W}$, by assumption, we have that this inequality holds with probability 1, as needed.

(c) Define now the function

$$h(n) = \frac{\mathbb{E}[(W+q)^{-n} \cdot W^m]}{\mathbb{E}[(W+q)^{-n} \cdot W^{m-1}]}$$

over $n > 0$, given any $m \geq 0$. For Eqs. (30) and (31), it suffices to observe that q is non-increasing in n.

By direct computation, $h'(n) \leq 0$ if, and only if,

$$\mathbb{E}[(W+q)^{-n} \cdot W^m \cdot \ln(W+q)] \cdot \mathbb{E}[(W+q)^{-n} \cdot W^{m-1}]$$

is at least as large as

$$\mathbb{E}[(W+q)^{-n} \cdot W^{m-1} \cdot \ln(W+q)] \cdot \mathbb{E}[(W+q)^{-n} \cdot W^{m}].$$

Letting random variable V be i.i.d. with W, this is the requirement that

$$\mathbb{E}\left\{(W-V) \cdot (VW)^{m-1} \cdot [(V+q)(W+q)]^{-n} \cdot \ln(W+q)\right\} \geq 0.$$

This expectation is proportional, by a factor of $\Pr(V \neq W)/2$, to the sum of

$$\mathbb{E}\left\{(W-V) \cdot (VW)^{m-1} \cdot [(V+q)(W+q)]^{-n} \cdot \ln(W+q) \mid V > W\right\}$$

and

$$\mathbb{E}\left\{(W-V) \cdot (VW)^{m-1} \cdot [(V+q)(W+q)]^{-n} \cdot \ln(W+q) \mid V < W\right\}.$$

Since V and W follow the same distribution, the latter is

$$\mathbb{E}\left\{(V-W) \cdot (VW)^{m-1} \cdot [(V+q)(W+q)]^{-n} \cdot \ln(V+q) \mid V > W\right\},$$

so the sum equals

$$\mathbb{E}\left\{(W-V) \cdot (VW)^{m-1} \cdot [(V+q)(W+q)]^{-n} \cdot [\ln(W+q) - \ln(V+q)] \mid V > W\right\},$$

which is, indeed, non-negative.

(d) Note that Eq. (32) is equivalent to the requirement that

$$\mathbb{E}[(W+q)^{-(\rho+2)}W^3] \cdot \mathbb{E}[(W+q)^{-(\rho+1)}] \geq \mathbb{E}[(W+q)^{-(\rho+2)}W^2] \cdot \mathbb{E}[(W+q)^{-(\rho+1)}W].$$

With V defined as above, this is

$$\mathbb{E}\left\{V \cdot [(V+q)(W+q)]^{-(\rho+1)} \cdot \left(\frac{V^2}{V+q} - \frac{W^2}{W+q}\right)\right\} \geq 0,$$

or

$$\mathbb{E}\left\{(V-W) \cdot [(V+q)(W+q)]^{-(\rho+1)} \cdot \left(\frac{V^2}{V+q} - \frac{W^2}{W+q}\right) \;\middle|\; V > W\right\} \geq 0.$$

In order to guarantee this, we need to argue that

$$v > w \Rightarrow \frac{v^2}{v+q} \geq \frac{w^2}{w+q},$$

or, equivalently, that the ratio $w^2/(w+q)$ is non-decreasing for $w \in \mathcal{W}$. By direct computation, this is true since $\mathcal{W} \subseteq \mathbb{R}_{++}$ and $q \geq 0$.

(e) Using the same technique, Eq. (33) is equivalent to the requirement that

$$\mathbb{E}\left[(V-W) \cdot W^2 \cdot (V+q)^{-\rho} \cdot (W+q)^{-(\rho+3)}\right] \geq 0,$$

or, equivalently, that

$$\mathbb{E}\left\{(V-W) \cdot [(V+q)(W+q)]^{-\rho} \cdot \left(\frac{W^2}{(W+q)^3} - \frac{V^2}{(V+q)^3}\right) \;\middle|\; V > W\right\} \geq 0.$$

For this, it suffices that the ratio $w^2/(w+q)^3$ be non-increasing at all $w \in \mathcal{W}$. This is guaranteed, indeed, by the assumption that $q \leq {}^1\!/\!_2 \inf \mathcal{W}$.

(f) Finally, note again that Eq. (34) is equivalent to

$$\mathbb{E}[(V-W)\cdot V^2\cdot(V+q)^{-(\rho+2)}(W+q)^{-\rho}]\geq 0,$$

or

$$\mathbb{E}\left\{(V-W)\cdot[(V+q)(W+q)]^{-\rho}\cdot\left(\frac{V^2}{(V+q)^2}-\frac{W^2}{(W+q)^2}\right)\;\middle|\; V>W\right\}\geq 0.$$

For this inequality to hold true, it suffices that $w/(w+q)$ be non-increasing at all $w\in\mathcal{W}$, which is true since $\mathcal{W}\subseteq\mathbb{R}_{++}$ and $q\geq 0$. □

Proof of Theorem 9: If $m_i''=0$ for all i, Eq. (22) reduces to

$$\hat{p}=\frac{\mathbb{E}\left[\sum_i \mu_i m_i(W_i)\right]\cdot\mathbb{E}(W)}{\mathbb{E}\left[\sum_i \mu_i m_i(W_i)\cdot W\right]}-1,$$

which depends on the idiosyncratic shocks only through the agents' portfolios $\mathbf{y}=(y_i)_{i=1}^I$.

Note, however, that the first-order condition (19) reduces to

$$q=\mathbb{E}\left[m_i(\mathbb{E}(W_i+S_i\mid R))\cdot R\right],$$

since m_i is linear. Hence, to complete the argument it suffices to note that S_i vanishes on conditional expectation, so the equality

$$q=\mathbb{E}\left[m_i(W_i)\cdot R\right]$$

determines y_i, regardless of the presence of S_i. □

Proof of Theorem 10: By direct computation, $\partial\hat{\pi}/\partial\sigma_i(\boldsymbol{\sigma},\mathbf{y})>0$ if, and only if,

$$\mathbb{E}(M_i'')\cdot\mathbb{E}(W)\cdot\left[\varphi(\mathbf{y})+\tfrac{1}{2}\textstyle\sum_j\mu_j\sigma_j\mathbb{E}(M_j''\cdot W)\right]>\mathbb{E}(M_i''\cdot W)\cdot\left[f(\mathbf{y})+\tfrac{1}{2}\textstyle\sum_j\mu_j\sigma_j\mathbb{E}(M_j'')\cdot\mathbb{E}(W)\right],$$

which is equivalent to

$$\frac{\mathbb{E}(M_i'')\cdot\mathbb{E}(W)}{\mathbb{E}(M_i''\cdot W)}>\hat{p}+1.$$ □