Journal of Mathematical Economics 77 (2018) 87–100

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Risk externalities: When financial imperfections are not the problem, but part of the solution[☆]

Maria Arvaniti [a], Andrés Carvajal [b,c,*]

[a] Umeå University, Sweden
[b] University of California, Davis, United States
[c] FGV, EPGE Escola Brasileira de Economia e Finanças c, Brazil

ARTICLE INFO

Article history:
Received 4 April 2017
Received in revised form 2 January 2018
Accepted 23 March 2018
Available online 1 May 2018

Keywords:
Externalities
Financial markets
Constrained inefficiency

ABSTRACT

We model an economy with complete financial markets where one agent's actions impose an externality on other agents by altering the probability distribution of their risks, and show that limiting the ability of that agent to diversify his risks creates incentives for him to internalize the welfare effects of his decisions, leading to a welfare improvement. Hence, in the presence of risk externalities, disturbing the functioning of perfect financial markets can be socially beneficial. An important implication is, for instance, that allowing oil companies to diversify their exploration risks may result in an inefficiently high risk of environmental catastrophes.

On April 20, 2010, a concrete seal on the Macondo well, where oil rig *Deepwater Horizon* was positioned in the Gulf of Mexico, burst as a consequence of a surge of natural gas. The spill that occurred during the next 87 days is the largest ever in the history of petroleum production, and constitutes the worst environmental catastrophe in US history. In 2011, a report commissioned by the US government concluded that the explosion of the well could have been prevented, and that its causes could be traced directly to decisions made by the rig's owner, operator and contractors: during the installation of the seal, nitrogen gas was used to accelerate its "curing", a technique that is known to weaken the pressure that this type of concrete seal withstands.

In this paper, we argue that limiting the ability of an agent to participate in financial trade to diversify his risks may induce a Pareto improvement in the allocation of resources in the economy. In the context of the Gulf of Mexico catastrophe, this is to say that had the companies in charge of Deepwater Horizon been restricted in their diversification of risks, the resulting change in their operational decisions could have been beneficial to all agents, including themselves. That this is the case is not obvious: the literature on financial economics shows that competitive financial markets deliver Pareto efficiency when, and only when, all the agents can potentially diversify all the risks they face. But this result is established under the assumption that no agent in the market causes an externality on others.

If an agent who causes an externality via the probability distribution of risks is limited in his access to financial markets, he certainly will respond by altering those probabilities in a way that is optimal to him. It seems easy to conclude that such policy can then be used to effect a welfare improvement — in a partial equilibrium model, this is immediately the case. The obvious effect that this partial equilibrium argument ignores is the reaction of asset prices to the perturbation of the agent's financial portfolio. More subtle, but equally important, is the fact that, after such price change, other agents in the economy will change their portfolios too. The general equilibrium trade-off that we study, and which makes the conclusion less immediate and more interesting, is that, in the absence of this kind of externalities, disturbing the functioning of *perfect* financial markets would be socially harmful. Our argument shows that the cost of a correctly chosen asset reallocation is of second order to gains of having a different probability distribution, generically.

To be sure, any departure from economic efficiency in a market can also be studied under a mechanism design approach, and there is very influential work that does it — from the canonical Clarke (1971), Groves (1973), Groves and Ledyard (1977) and D'Aspremont and Gérard-Varet (1979); to current contributions such as Bierbrauer and Hellwig (2016) and Kuzmics and Steg (2017). But the nature of the problems we wish to study fits equally well in a general equilibrium context. In particular, we think of economies with a large number of individuals who are exposed to a catastrophic risk, and our aim is to contribute to the literature on

☆ We are very grateful to different seminar audiences and to Luis Braido, Eduardo Engel, Herakles Polemarchakis and Yiannis Vailakis for their comments. We also thank the Co-Editor in charge, Alessandro Citanna, and two referees for their extremely careful and useful feedback.

* Corresponding author at: University of California, Davis, United States.
E-mail addresses: maria.arvaniti@econ.umu.se (M. Arvaniti), acarvajal@ucdavis.edu (A. Carvajal).

https://doi.org/10.1016/j.jmateco.2018.03.008
0304-4068/

public economics where the results on market inefficiency usually capture a different type of externality and consider different policy tools. We believe that our results add a new insight to that literature and are complementary to the kind of policy recommendation that would arise from mechanism design.

We study an economy where, initially, all agents participate in the exchange of a complete set of financial assets. The only market failure in this economy is the externality that one agent imposes on the others via the probability distribution of their risks. We show that, in a generic sense, everybody can be made better-off if the agent that causes the externality is not allowed to choose his portfolio of financial assets in an optimal manner (while everybody else continues to trade without constraints). Besides the restriction of that agent's financial decisions, lump-sum transfers are used to make sure that *all* agents are, indeed, made better-off. We first allow for a lump-sum transfer to the agent who imposes the externality. Being lump-sum, this transfer does *not* affect that agent's incentives, only his welfare. Still, it may be that for institutional reasons such transfer is not possible. Under the assumption that there is an external source of funding for the rest of the agents, which we call "relief aid", we show that the Pareto improvement is still generically possible, even when the agent that imposes the externality is excluded from any lump-sum aid. We also check the robustness of our results to a more sophisticated behaviour by the agent who causes the externality, where he now recognizes that his decisions with respect to probabilities affect the willingness to pay of other agents for the existing financial assets and, therefore, the equilibrium level of assets prices. Again, we find that the equilibrium allocation is constrained inefficient in this case. To conclude the analysis, we extend the results to economies with uninsurable idiosyncratic risk.

While the spirit of our results, and the techniques we use, are similar to those in the literature on incomplete financial markets, it is worthwhile to notice that in our economy there is only one commodity per state, so that it is *not* via relative commodity prices that the Pareto improvement is induced by our financial policies.

The paper is organized as follows. A short literature review is presented in Section 1. The following three sections describe the general kind of economies with externalities for which our analysis holds and define competitive equilibrium and Pareto efficiency for this kind of economies. Section 5 introduces notions of weak and strong constrained inefficiency for those economies and states the main theorem of our paper. Section 6 then gives the proof for that result: the genericity of strong constrained inefficiency. In Section 7 we study the case of a more sophisticated behaviour by agent 0 while in Section 8 we extend the results to economies with uninsurable idiosyncratic risk. A technical appendix completes the paper.

1. Existing literature

Geanakoplos and Polemarchakis (1986) argue that in a numéraire asset model with incomplete markets, generically, every equilibrium is constrained inefficient, provided that there are more than one commodities and there is an upper bound in the number of individuals: reallocations of existing assets support superior allocations.[1] Following Geanakoplos and Polemarchakis (1986) and Citanna et al. (1998) show that equilibria are generically constrained inefficient even without an upper bound on the number of households. They also show that perfectly anticipated lump-sum transfers in a limited number of goods are typically effective.

As Polemarchakis and Siconolfi (1997) point out, incomplete markets are just a special case of an asset market with restricted participation, and Cass, Cass et al. (2001) extend the literature by accommodating a wide range of portfolio constraints, including any smooth, quasi-concave inequality constraint on households' portfolio holdings. Gori et al. (2012) focus on price-dependent borrowing restrictions, and show that equilibria associated with a sufficiently high number of strictly binding participation constraints in the financial markets can be Pareto improved upon by a local change in these constraints. Our results will point in the opposite direction, that in the presence of externalities a social improvement can be attained by introducing portfolio constraints on some agents' trades.

For the case of abstract economies with externalities and multiple commodities, Geanakoplos and Polemarchakis (2008) show that the competitive equilibrium allocation is, generically, constrained inefficient: there exists an anonymous taxation policy that leaves all agents better off. While the concept of equilibrium used in that paper, and the spirit of the results are the same as ours, neither the setting nor the type of externality are the same, and the mechanism through which the Pareto improvement is induced differs too, our argument does not rely on the relative prices of commodities.

Of course, any situation in which an externality induces a departure from economic efficiency can be studied from the perspective of the literature on mechanism design. We do *not* take that approach, which we consider complementary to our results, and instead apply the ideas of the literature on incomplete financial markets. Relatedly, Braido (2005) presents a general equilibrium model as a two-stage game where agents act as producers, as consumers and as financial intermediaries with intermediation costs. Each individual is allowed to design a financial structure that consists of specifying securities pay-offs in each state and transaction constraints that restrict the participation of some agents in some markets. He offers an example with two agents where (only) one of them is risk averse and faces a production risk where the probability of the "good" state is increasing in the costly, unobservable effort he exerts. In this setup, Braido shows that an incomplete financial structure, in the form of trading constraints, is Pareto superior to complete markets. Our framework is more general: all agents are risk-averse and face aggregate and/or idiosyncratic risk. In our setup, the agent who exerts effort creates an externality. Moreover, by imposing trading constraints, we are limiting the insurance opportunities of the whole society and, since the rest of the agents are risk averse too, it is therefore more demanding to show that a Pareto improving policy exists.

2. An externality via probabilities

Consider a two-period economy populated by $I+1$ individuals, who are denoted by $i = 0, \ldots, I$. In period 0, agents receive an endowment of a single consumption good. In period 1, there is uncertainty regarding endowments: there are (only) two states of the world, which we denote by $s = 1, 2$, and agents receive an endowment $e_{i,s}$ of the consumption good in state s.[2]

Rather than being exogenous, the probabilities of the two states depend on some action of agent 0. In period 0, he has the choice of choosing a costly *action*, α, which makes state 1 more likely. The cost of this action is that it subtracts from the agent's consumption at date 0. For individuals $i \geqslant 1$, their preferences over consumption plans are given by a function $v_i : \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}$, with

$$v_i(c_0, c_1, c_2) = c_0 + \pi(\alpha) \cdot u_{i,1}(c_1) + [1 - \pi(\alpha)] \cdot u_{i,2}(c_2), \qquad (1)$$

where each $u_{i,s} : \mathbb{R}_+ \to \mathbb{R}$ is $\mathbb{C}^3$ on $\mathbb{R}_{++}$ and satisfies the usual assumptions that $u'_{i,s} > 0$, $u''_{i,s} < 0$, $\lim_{c \downarrow 0} u'_{i,s}(c) = \infty$ and $\lim_{c \to \infty} u'_{i,s}(c) = 0$.

[1] See, also, Stiglitz (1982), and Greenwald and Stiglitz (1986).

[2] The results below will hold true if there are multiple commodities in the second period. In fact, results of the type we are studying are easier to argue in that case, only at the cost of heavier notation.

For agent $i = 0$, who causes the risk externality, the utility function is $v_0 : [\underline{\alpha}, \infty) \times \mathbb{R} \times \mathbb{R}^2_+ \to \mathbb{R}$, with the same functional form, and the same assumptions on the instantaneous utility over consumption. Additionally, the probability of state 1 is given by $\pi : [\underline{\alpha}, \infty) \to (0, 1)$ which satisfies $\pi' > 0$ and $\pi'' < 0$. Moreover we assume that, over the relevant domain of future consumption levels and the agent's action, agent 0 has strictly concave preferences, i.e., that matrix

$$\begin{pmatrix} \pi(\alpha) \cdot u''_{0,1}(c_{0,1}) & 0 & \pi'(\alpha) \cdot u'_{0,1}(c_{0,1}) \\ 0 & [1-\pi(\alpha)] \cdot u''_{0,2}(c_{0,2}) & -\pi'(\alpha) \cdot u'_{0,2}(c_{0,2}) \\ \pi'(\alpha) \cdot u'_{0,1}(c_{0,1}) & -\pi'(\alpha) \cdot u'_{0,2}(c_{0,2}) & \pi''(\alpha) \cdot [u_{0,1}(c_{0,1}) - u_{0,2}(c_{0,2})] \end{pmatrix} \quad (2)$$

is negative definite.

The assumption that all individuals have preferences that are quasilinear in present consumption simplifies the mathematical arguments, but, in principle, implies a loss of generality. The gained simplicity is that, since utility is transferable in this case, we can express our arguments in terms of aggregate *social welfare* functions. Without that simplicity, the results continue to hold true, but our arguments become ordinal and would require us to do genericity analysis on the space of preferences of the individuals, and not only on their endowments. Importantly, imposing quasi-linearity also guarantees that the type of policy intervention we consider be rather weak, which makes our results stronger and more interesting.[3]

For all agents, we allow for negativity of their present consumption, $c_{i,0}$, so it is not necessary for us to specify date-0 endowments.

Note that we assume that $\alpha \in [\underline{\alpha}, \infty)$, for a lower bound on the action $\underline{\alpha}$. Now, in order to have interior solutions, we impose the following condition. For each state s, let $(\hat{c}_{i,s})_{i=0}^{I}$ solve the following maximization problem:

$$\max_{(c_i)_{i=0}^{I}} \left\{ \sum_{i=0}^{I} u_{i,s}(c_i) : \sum_{i=0}^{I} c_{i,s} = \sum_{i=0}^{I} e_{i,s} \right\}. \quad (3)$$

The following assumptions will be technically useful later on:

Assumption 1 (*Interiority*)**.** At the consumption plan $\hat{c}_0$, agent 0 prefers state 1 to state 2, in the sense that $u_{0,1}(\hat{c}_{0,1}) > u_{0,2}(\hat{c}_{0,2})$, and there exists a level of the action $\hat{\alpha} \in (\underline{\alpha}, \infty)$ such that

$$\pi'(\hat{\alpha}) = \frac{1}{u_{0,1}(\hat{c}_{0,1}) - u_{0,2}(\hat{c}_{0,2})}.$$

Assumption 2 (*Heterogeneity*)**.** At consumption allocation $\hat{c}$, agents other than 0 have heterogeneous preferences: for all $i, j \geqslant 1$, $u''_{i,s}(\hat{c}_{i,s}) \neq u''_{j,s}(\hat{c}_{j,s})$, if $i \neq j$, for both $s = 1, 2$.

3. Competitive equilibrium

Financial markets are assumed to be complete: there is an elementary (Arrow) security for each state, with asset s paying one unit of the consumption good in state s. Holdings of these securities are denoted by $\vartheta_{i,s}$.

3.1. The problem of agents $i \geqslant 1$

All agents other than 0 have only one decision to make in period 0: they have to choose their holdings of securities, and therefore their consumption in that period and in both states in period 1. We assume that they take the prices of the securities and the probabilities of the states as given.

[3] See Footnote 6 below.

Letting q_1 and q_2 denote the prices of the securities, the problem of individual i is, simply,

$$\max_{\vartheta_{i,1}, \vartheta_{i,2}} \left\{ -q_1 \cdot \vartheta_{i,1} - q_2 \cdot \vartheta_{i,2} + \pi(\alpha) \cdot u_{i,1}(e_{i,1} + \vartheta_{i,1}) + [1 - \pi(\alpha)] \cdot u_{i,2}(e_{i,2} + \vartheta_{i,2}) \right\}. \quad (4)$$

The first-order conditions of this problem are standard:

$$q_1 = \pi(\alpha) \cdot u'_{i,1}(e_{i,1} + \vartheta_{i,1}) \text{ and } q_2 = [1 - \pi(\alpha)] \cdot u'_{i,2}(e_{i,2} + \vartheta_{i,2}). \quad (5)$$

These conditions are necessary and sufficient to characterize the solutions of Program (4) since, under the Inada conditions and with positive prices, they are interior.

3.2. The problem of agent $i = 0$

Agent 0 has an extra decision to make in period 0: apart from choosing his holdings of securities, he must choose his action. His problem is, then:

$$\max_{\alpha, \vartheta_{0,1}, \vartheta_{0,2}} \left\{ -\alpha - q_1 \cdot \vartheta_{0,1} - q_2 \cdot \vartheta_{0,2} + \pi(\alpha) \cdot u_{0,1}(e_{0,1} + \vartheta_{0,1}) + [1 - \pi(\alpha)] \cdot u_{0,2}(e_{0,2} + \vartheta_{0,2}) \right\}. \quad (6)$$

Assuming that this agent, too, takes asset prices as given, the following are the first-order conditions:

$$1 = \pi'(\alpha) \cdot [u_{0,1}(e_{0,1} + \vartheta_{0,1}) - u_{0,2}(e_{0,2} + \vartheta_{0,2})], \quad (7)$$

while

$$q_1 = \pi(\alpha) \cdot u'_{0,1}(e_{0,1} + \vartheta_{0,1}) \text{ and } q_2 = [1 - \pi(\alpha)] \cdot u'_{0,2}(e_{0,2} + \vartheta_{0,2}). \quad (8)$$

As before, concavity of agent 0's preferences implies that these conditions are both necessary and sufficient, while the Inada conditions and Assumption 1 guarantee that the solutions are interior. Note that the first-order condition with respect to the action, Eq. (7), implies that when agent 0 prefers state 1 to state 2 sufficiently, he is willing to choose a high action to raise the likelihood of state 1. In his choice of action, however, he does not internalize the effect of a more likely state 1 on the well-being of the society as a whole. Since we have made no assumptions on aggregate endowments and social welfare in each state, all agents other than agent 0 could be worse-off or better-off in state 1. It is this feature that reflects the non-alignment of interest across agents.

3.3. Equilibrium

Competitive equilibrium is defined by individual rationality and market clearing requirements. We write competitive equilibria as a tuple $\{\bar{\vartheta}, \bar{\alpha}, \bar{q}\}$, where $\bar{\vartheta} = [(\bar{\vartheta}_{i,s})_{s=1,2}]_{i=0}^{I}$ is the allocation of the two assets and $\bar{q} = (\bar{q}_1, \bar{q}_2)$ is the vector of asset prices, such that:

1. action $\bar{\alpha}$ and portfolio $(\bar{\vartheta}_{0,s})_{s=1,2}$ solve Program (6) when the prices are $q = \bar{q}$;
2. for each $i \geqslant 1$, portfolio $(\bar{\vartheta}_{i,s})_{s=1,2}$ solves Program (4) when the prices are $q = \bar{q}$ and the probability of state 1 is $\pi(\bar{\alpha})$; and
3. both of the securities markets clear: $\sum_{i=0}^{I} \bar{\vartheta}_{i,s} = 0$, for $s = 1, 2$.

This definition of equilibrium is standard in the literature, and Proposition 1 in Ghosal and Polemarchakis (1997) can be invoked to argue that equilibrium is guaranteed to exist.

For agents $i \geqslant 1$, individual rationality is characterized by the first-order conditions, Eq. (5). Importantly, these agents take as

$$\begin{pmatrix} \pi u''_{0,1} & 0 & \pi' u'_{0,1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \pi u''_{0,1} \\ 0 & (1-\pi)u''_{0,1} & -\pi' u'_{0,2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \pi' u'_{0,1} & -\pi' u'_{0,2} & \pi''(u_{0,1}-u_{0,2}) & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \pi' u'_{0,1} \\ 0 & 0 & \pi' u'_{1,1} & \pi u''_{1,1} & 0 & \dots & 0 & 0 & \pi u''_{1,1} & 0 & 0 \\ 0 & 0 & -\pi' u'_{1,2} & 0 & (1-\pi)u''_{1,2} & \dots & 0 & 0 & 0 & (1-\pi)u''_{1,2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \pi' u'_{I,1} & 0 & 0 & \dots & \pi u''_{I,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\pi' u'_{I,2} & 0 & 0 & \dots & 0 & (1-\pi)u''_{I,2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & \dots & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & \dots & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Box I.

given not only prices, but also the probabilities of the two states. In the case of agent 0, individual rationality is characterized by Eqs. (7) and (8).[4] Note, then, that for agent 0, who causes an externality via his choice of the action, our assumption is that he takes prices as given, and considers the effects of the action on his own well-being only.

For later usage, let us define the function

$$\mathcal{F}(q, \vartheta, \alpha, e) = \begin{pmatrix} \pi(\alpha) \cdot u'_{0,1}(e_{0,1} + \vartheta_{0,1}) - q_1 \\ [1 - \pi(\alpha)] \cdot u'_{0,2}(e_{0,2} + \vartheta_{0,2}) - q_2 \\ \left[u_{0,1}(e_{0,1} + \vartheta_{0,1}) - u_{0,2}(e_{0,2} + \vartheta_{0,2})\right] \cdot \pi'(\alpha) - 1 \\ \pi(\alpha) \cdot u'_{1,1}(e_{1,1} + \vartheta_{1,1}) - q_1 \\ [1 - \pi(\alpha)] \cdot u'_{1,2}(e_{1,2} + \vartheta_{1,2}) - q_2 \\ \vdots \\ \pi(\alpha) \cdot u'_{I,1}(e_{I,1} + \vartheta_{I,1}) - q_1 \\ [1 - \pi(\alpha)] \cdot u'_{I,2}(e_{I,2} + \vartheta_{I,2}) - q_2 \\ \sum_{i=0}^{I} \vartheta_{i,1} \\ \sum_{i=0}^{I} \vartheta_{i,2} \end{pmatrix}.$$

Note that the roots of this function characterize the competitive equilibrium of the economy.

The following Lemma holds under Assumption 1.

Lemma 1 (*Determinacy and Trade*)**.** *Except on a closed and negligible set of individual endowments, whenever $\mathcal{F}(q, \vartheta, \alpha, e) = 0$, we have that the partial Jacobian of $\mathcal{F}$, matrix $D_{q,\vartheta,\alpha}\mathcal{F}$, is non-singular and $\vartheta_{0,1} \neq 0$.*

Proof. With the arguments in the order

$$(\vartheta_{0,1}, \vartheta_{0,2}, \alpha, \vartheta_{1,1}, \vartheta_{1,2}, \dots \vartheta_{I,1}, \vartheta_{I,2}, e_{1,1}, e_{1,2}, e_{0,1}),$$

the Jacobian matrix $D\mathcal{F}$ writes as in Box I.

By strict concavity of preferences, the submatrix that results from deleting the last two rows and the last three columns is block-triangular, and each of the matrices in the block-diagonal is negative definite, so this matrix is non-singular. Now, note that the result of subtracting the column corresponding to $e_{1,1}$ from the one corresponding to $\vartheta_{1,1}$ is null everywhere, except at the previous-to-last entry, where it is 1. The same is true for the last entry, if one subtracts the column corresponding to $e_{1,2}$ from the one corresponding to $\vartheta_{1,2}$. This means that the matrix (even without the last column) has full row rank.

The latter shows that function $\mathcal{F}$ is transverse to 0, which we denote by $\mathcal{F} \pitchfork 0$. By the transversality theorem, then, generically on endowments, function $\mathcal{F}(\cdot, e) \pitchfork 0$ too, which proves the first statement.

For the second statement, consider the mapping

$$\begin{pmatrix} \mathcal{F}(q, \vartheta, \alpha, e) \\ \vartheta_{0,1} \end{pmatrix},$$

and suppose that it takes the value 0. The Jacobian of this mapping is the matrix $D\mathcal{F}$ above, with the row

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

added at the bottom of it.

We know from above that without this last row and the last column, the matrix has full row rank. Note that the difference between the sum of the columns corresponding to $\vartheta_{0,1}$ and $e_{1,1}$ and the sum of the columns corresponding to $e_{0,1}$ and $\vartheta_{1,1}$ is null everywhere, except at its last entry, where it is 1. This proves that the whole matrix has full row rank, and hence that the mapping is transverse to 0.

Again by the transversality theorem, the mapping is transverse to 0, generically on e, when only (q, ϑ, α) are allowed to vary. This requires the Jacobian to have full row rank when the mapping becomes 0, but this is impossible as such matrix now has one more row than it has columns. The implication is, then, that $\vartheta_{0,1} \neq 0$ when $\mathcal{F}(q, \vartheta, \alpha, e) = 0$, generically on e. □

4. Pareto efficiency

4.1. Definition

The definition of Pareto efficient allocation is as usual: a feasible allocation of agent 0's action and the consumption of the unique commodity both at date 0 and in the two future states across all agents is efficient, if it is impossible to find an alternative, feasible allocation of these same variables that makes at least one agent better-off without making any other agent worse-off. Given that all individuals have quasilinear preferences, Pareto efficiency

[4] Our argument remains valid even if we weaken the assumption of concavity of v_0 to hold only locally around $\hat{c}_0$ and $\hat{\alpha}$, with matrix

$$\begin{pmatrix} \pi(\hat{\alpha}) \cdot u''_{0,1}(\hat{c}_{0,1}) & 0 & \pi'(\hat{\alpha}) \cdot u'_{0,1}(\hat{c}_{0,1}) \\ 0 & [1 - \pi(\hat{\alpha})] \cdot u''_{0,2}(\hat{c}_{0,2}) & -\pi'(\hat{\alpha}) \cdot u'_{0,2}(\hat{c}_{0,2}) \\ \pi'(\hat{\alpha}) \cdot u'_{0,1}(\hat{c}_{0,1}) & -\pi'(\hat{\alpha}) \cdot u'_{0,2}(\hat{c}_{0,2}) & \pi''(\hat{\alpha}) \cdot [u_{0,1}(\hat{c}_{0,1}) - u_{0,2}(\hat{c}_{0,2})] \end{pmatrix}$$

being negative definite. In this case, our argument proves that our claims of constrained inefficiency hold at any root of function $\mathcal{F}$ defined below.

amounts to the choice of an allocation of consumption and an action so as to solve the program

$$\max_{\alpha,(c_{i,1},c_{i,2})_{i=0}^{I}} \left\{ -\alpha + \sum_{i=0}^{I} \{\pi(\alpha)\cdot u_{i,1}(c_{i,1}) + [1-\pi(\alpha)]\cdot u_{i,2}(c_{i,2})\} \mid \sum_{i=0}^{I} c_{i,s} = \sum_{i=0}^{I} e_{i,s}, s = 1,2 \right\}. \tag{9}$$

Here, the date-0 consumption levels are left undetermined, but any allocation that exhausts the remaining aggregate endowment, net of α, will be Pareto efficient.

The first-order conditions that characterize Pareto efficiency are, therefore, that

$$\pi'(\alpha)\cdot \sum_{i=0}^{I}[u_{i,1}(c_{i,1}) - u_{i,2}(c_{i,2})] = 1; \tag{10}$$

and that, for each pair of agents $i, j = 0, \dots, I$,

$$u'_{i,s}(c_{i,s}) = u'_{j,s}(c_{j,s}) \tag{11}$$

for each state $s = 1, 2$; as well as the feasibility condition.

4.2. Inefficiency of competitive equilibrium

Comparison of the first-order conditions of the individual, for the competitive choices of consumption, Eqs. (5) and (8), with the first-order condition defining efficiency of consumption plans, Eq. (11), shows that the allocations of consumption prescribed by the competitive equilibrium are efficient in the sense that the marginal rates of substitutions would be equalized across agents in each state.

On the other hand, from the first-order conditions for the choice of action, Eqs. (7) and (10), it is immediate that action implied by the competitive equilibrium solution is generically not Pareto optimal: while agent 0 takes into account only the effects on his own welfare, the social planner considers the effects on social welfare when choosing the optimal action.

5. Constrained inefficiency of competitive equilibrium

5.1. Two definitions

We have shown that competitive equilibria need not yield the Pareto efficient action. This result says that if a planner could choose the action exerted by agent 0, he would choose a different level, and then would reallocate date-0 consumption to make sure that all agents, including 0 himself, are made better-off.

As is usual in the General Equilibrium literature, the latter observation does not mean that a social planner who faces constraints in terms of the policies he can apply would indeed be able to effect a welfare improving policy. Here, we consider the case in which the planner is constrained in the sense that he can distort the asset holdings of agent 0, but the choice of the action remains the agent's decision. All agents other than 0 continue to choose their portfolios without constraints.

Besides the determination of the portfolio of agent 0, we also allow the planner to effect lump-sum transfers of revenue across all agents. If such a policy exists that leaves everybody better-off, we shall say that the competitive equilibrium is *constrained inefficient*. Whether a transfer to agent 0 is required determines how strong the definition is.

Formally, we say that an allocation (α, c), where $c = [(c_{i,s})_{s=0,1,2}]_{i=0}^{I}$,[5] is *weakly constrained inefficient* if there exist an alternative level of the action, $\tilde{\alpha}$; asset prices, $\tilde{q}$; a profile of asset holdings, $\tilde{\vartheta}$; and a profile of date-0 lump-sum transfers, $(\tau_i)_{i=0}^{I}$, such that:

1. $\tilde{\alpha}$, the action of agent 0, solves
$$\max_{\alpha} \{-\alpha + \pi(\alpha)\cdot u_{0,1}(e_{0,1} + \tilde{\vartheta}_{0,1}) + [1-\pi(\alpha)]\cdot u_{0,2}(e_{0,2} + \tilde{\vartheta}_{0,2})\}; \tag{12}$$
2. for each $i \geqslant 1$, portfolio $(\tilde{\vartheta}_{i,s})_{s=1,2}$ solves Program (4) when the prices are $q = \tilde{q}$ and the probability of state 1 is $\pi(\tilde{\alpha})$;
3. both of the securities markets clear: $\sum_{i=0}^{I}\tilde{\vartheta}_{i,s} = 0$, for $s = 1, 2$;
4. the profile of lump-sum transfers is balanced: $\sum_{i=0}^{I}\tau_i = 0$;
5. agent 0 is better-off, in that
$$-\tilde{\alpha} - \tilde{q}_1\cdot\tilde{\vartheta}_{0,1} - \tilde{q}_2\cdot\tilde{\vartheta}_{0,2} + \tau_0 + \pi(\tilde{\alpha})\cdot u_{0,1}(e_{0,1} + \tilde{\vartheta}_{0,1}) + [1-\pi(\tilde{\alpha})]\cdot u_{0,2}(e_{0,2} + \tilde{\vartheta}_{0,2})$$
is higher than
$$-\alpha + c_{0,0} + \pi(\alpha)\cdot u_{0,1}(c_{0,1}) + [1-\pi(\alpha)]\cdot u_{0,2}(c_{0,2});$$
6. every agent $i \geqslant 1$ is better-off, in that
$$-\tilde{q}_1\cdot\tilde{\vartheta}_{i,1} - \tilde{q}_2\cdot\tilde{\vartheta}_{i,2} + \tau_i + \pi(\tilde{\alpha})\cdot u_{i,1}(e_{i,1} + \tilde{\vartheta}_{i,1}) + [1-\pi(\tilde{\alpha})]\cdot u_{i,2}(e_{i,2} + \tilde{\vartheta}_{i,2})$$
is higher than
$$c_{i,0} + \pi(\alpha)\cdot u_{i,1}(c_{i,1}) + [1-\pi(\alpha)]\cdot u_{i,2}(c_{i,2}).$$

Intuitively, the allocation is constrained inefficient if a planner can effect a Pareto improvement by forcing agent 0 out of the financial markets: instead of letting him choose optimal holdings of the two assets, a portfolio $\tilde{\vartheta}_0$ is allocated to him. Then competitive assets markets open for *every other* agent in the economy, and assets prices are determined endogenously. Agent 0's choices are limited to his action. Yet, at the allocation induced (endogenously) by the policy, every agent is strictly better-off.

What makes this definition weak is that we are allowing for agent 0 to receive a lump-sum transfer beyond the resulting price value of the portfolio imposed on him. It is important to note that this transfer does *not* affect the agent's incentives in choosing his action.[6] Yet, for institutional reasons it may be impossible for the planner to effect such transfer. We shall say that the allocation is constrained inefficient *in the strong sense*, if such Pareto improvement is possible even when τ_0 is required to be null.

5.2. Genericity of weak constrained inefficiency

Fix the competitive equilibrium, $\{\bar{\vartheta}, \bar{\alpha}, \bar{q}\}$. Our first goal is to show that, generically on the date-1 endowments of individuals, the allocation $(\bar{\alpha}, \bar{c})$, where $\bar{c}_{i,s} = e_{i,s} + \bar{\vartheta}_{i,s}$ for $s = 1, 2$, is constrained inefficient in the weak sense.

Since the weak definition of constrained inefficiency allows for transfers across all agents, we can again use the fact that preferences are quasilinear to write an expression for social welfare,

$$W = -\alpha + \pi(\alpha)\cdot\sum_{i=0}^{I}u_{i,1}(c_{i,1}) + [1-\pi(\alpha)]\cdot\sum_{i=0}^{I}u_{i,2}(c_{i,2}). \tag{13}$$

5 The *feasibility* condition that $\sum_{i=0}^{I}c_{i,s} = \sum_{i=0}^{I}e_{i,s}$, for both $s = 1, 2$, will hold throughout our analysis. Also, note that in the following analysis $c_{0,0}$ is taken to be *gross* of the action.

6 So, the planner is *not* bribing him to choose a higher action. To emphasize this fact, we have not added τ_0 to the maximand of Program (12). This is also the reason why quasilinear preferences make our argument stronger: the policy's transfers affect only the agents' welfare, and not their incentives to choose their actions or portfolios as all income effects are absorbed by the linear term, namely the agent's present consumption.

Recall that the problem arises from the inefficient action chosen by agent 0 when maximizing his utility in the competitive setting. We consider an exogenous perturbation in the holdings of securities of agent 0. The idea is to restrict his insurance opportunities to make him more vulnerable to the risks associated with each state. Such perturbation, $(\mathrm{d}\vartheta_{0,1}, \mathrm{d}\vartheta_{0,2})$, around the competitive equilibrium values induces changes in all other endogenous variables: from Eq. (7), it follows that the action chosen by agent 0 will change; this will induce different consumption and investment decisions by all agents $i \geqslant 1$; and, in order to guarantee market clearing, the prices of assets will need to accommodate too. If, starting from the equilibrium allocation we can find that there exists a perturbation such that $\mathrm{d}W > 0$, we can conclude that the allocation is constrained inefficient in the weak sense: the higher value of the social welfare function W implies the existence of the required profile of lump-sum transfers after which every agent in the economy is made strictly better-off. With quasilinear preferences this is all we need to show for constrained inefficiency, a task significantly less demanding than having to find a welfare improvement for each of the $I+1$ individuals.

First, let us state the following lemma which will be useful in the proof of our main result.

Lemma 2. *Except on a closed and negligible set of individual endowments,*

$$\sum_{i=1}^{I}[u_{i,1}(\bar{c}_{i,1}) - u_{i,2}(\bar{c}_{i,2})] \neq 0 \tag{14}$$

at competitive equilibrium.

The proof of this Lemma can be found in the Appendix. We are now ready to prove our main result.

Theorem 1. *Except on a closed and negligible set of individual endowments, the competitive equilibrium allocation is weakly constrained inefficient.*

Proof. Using Lemmas 1 and 2, we can fix a generic set of individual endowments where the first statement of Lemma 1 and $\sum_{i=1}^{I}[u_{i,1}(\bar{c}_{i,1}) - u_{i,2}(\bar{c}_{i,2})] \neq 0$ hold true. Fix a profile of endowments in that set, and define the function

$$(q, \vartheta, \alpha) \mapsto \begin{pmatrix} \left[u_{0,1}(e_{0,1}+\vartheta_{0,1}) - u_{0,2}(e_{0,2}+\vartheta_{0,2})\right]\cdot\pi'(\alpha) - 1 \\ \pi(\alpha)\cdot u'_{1,1}(e_{1,1}+\vartheta_{1,1}) - q_1 \\ [1-\pi(\alpha)]\cdot u'_{1,2}(e_{1,2}+\vartheta_{1,2}) - q_2 \\ \vdots \\ \pi(\alpha)\cdot u'_{I,1}(e_{I,1}+\vartheta_{I,1}) - q_1 \\ [1-\pi(\alpha)]\cdot u'_{I,2}(e_{I,2}+\vartheta_{I,2}) - q_2 \\ \sum_{i=0}^{I}\vartheta_{i,1} \\ \sum_{i=0}^{I}\vartheta_{i,2} \end{pmatrix},$$

which simply removes the first two components of function $\mathcal{F}$. Denote this mapping by $\mathcal{G}(q, \vartheta, \alpha)$.

By construction of the space of endowments (Lemma 1), at any root (q, ϑ, α) of this mapping the matrix $D_{q,\vartheta^1,\ldots,\vartheta^I,\alpha}\mathcal{G}$ is non-singular. By the Implicit Function Theorem, there exist smooth functions Q, Θ_i for $i \neq 0$, and A, defined on an open neighbourhood of ϑ_0, such that $\mathcal{G}(\bar{q}, \bar{\vartheta}_0, \bar{\vartheta}^1, \ldots, \bar{\vartheta}^I, \bar{\alpha}) = 0$ if, and only if, $\bar{q} = Q(\bar{\vartheta}_0)$, $\bar{\vartheta}_i = \Theta_i(\bar{\vartheta}_0)$ and $\bar{\alpha} = A(\bar{\vartheta}_0)$.

Using these functions, we can define, given the individual endowments, the function $\mathcal{W}(\bar{\vartheta}_0)$ by

$$-A(\bar{\vartheta}_0) + \pi(A(\bar{\vartheta}_0))\cdot\sum_{i=0}^{I}u_{i,1}(e_{i,1}+\Theta_{i,1}(\bar{\vartheta}_0)) + [1-\pi(A(\bar{\vartheta}_0))] \cdot\sum_{i=0}^{I}u_{i,2}(e_{i,2}+\Theta_{i,2}(\bar{\vartheta}_0))$$

in a neighbourhood of ϑ_0.

In order to prove the theorem, we now simply need to show that $D\mathcal{W}(\vartheta_0) \neq 0$. By direct differentiation, the latter equals

$$-DA + \sum_{i=0}^{I}[u_{i,1}(\bar{c}_{i,1}) - u_{i,2}(\bar{c}_{i,2})]\cdot\pi'(\bar{\alpha})\cdot DA + \pi(\bar{\alpha}) \cdot\sum_{i=0}^{I}u'_{i,1}(\bar{c}_{i,1})\cdot D\Theta_{i,1} + [1-\pi(\bar{\alpha})]\cdot\sum_{i=0}^{I}u'_{i,2}(\bar{c}_{i,2})\cdot D\Theta_{i,2}.$$

If $\mathcal{F}(q, \vartheta, \alpha) = 0$, then by Eqs. (5) and (8), the latter writes as

$$-DA + \sum_{i=0}^{I}[u_{i,1}(c_{i,1}) - u_{i,2}(c_{i,2})]\cdot\pi'(\alpha)\cdot DA + q_1\cdot\sum_{i=0}^{I}D\Theta_{i,1} + q_2\cdot\sum_{i=0}^{I}D\Theta_{i,2}.$$

By construction, from the last two terms in function $\mathcal{G}$ we have that

$$\sum_{i=0}^{I}D\Theta_{i,1} = \sum_{i=0}^{I}D\Theta_{i,2} = 0.$$

Then,

$$D\mathcal{W} = -DA + [u_{i,0}(c_{i,0}) - u_{i,0}(c_{i,0})]\cdot\pi'(\alpha)\cdot DA + \sum_{i=1}^{I}[u_{i,1}(c_{i,1}) - u_{i,2}(c_{i,2})]\cdot\pi'(\alpha)\cdot DA,$$

which is, by Eq. (7),

$$D\mathcal{W} = \sum_{i=1}^{I}[u_{i,1}(c_{i,1}) - u_{i,2}(c_{i,2})]\cdot\pi'(\alpha)\cdot DA. \tag{15}$$

Since $\sum_{i=1}^{I}[u_{i,1}(c_{i,1}) - u_{i,2}(c_{i,2})] \neq 0$, by Lemma 2, and $\pi' \neq 0$, the result requires simply that $DA \neq 0$.

Now, function A is defined implicitly from the equality

$$\pi'(A(\bar{\vartheta}_0))\cdot[u_{0,1}(e_{0,1}+\bar{\vartheta}_{0,1}) - u_{0,2}(e_{0,2}+\bar{\vartheta}_{0,2})] = 1,$$

it follows from the implicit function theorem and Assumption 1 that

$$DA = \frac{\pi'(\bar{\alpha})}{\pi''(\bar{\alpha})[u_{0,1}(e_{0,1}+\bar{\vartheta}_{0,1}) - u_{0,2}(e_{0,2}+\bar{\vartheta}_{0,2})]} \times\begin{pmatrix} -u'_{0,1}(e_{0,1}+\bar{\vartheta}_{0,1}) \\ u'_{0,2}(e_{0,2}+\bar{\vartheta}_{0,2})\end{pmatrix} \neq 0, \tag{16}$$

which completes the proof. □

If follows from Eqs. (15) and (16) that in a linear approximation to $\mathcal{W}(\bar{\vartheta}_0)$ around ϑ_0, one obtains

$$\mathrm{d}W = D\mathcal{W}(\vartheta_0)\cdot\mathrm{d}\vartheta_0 = \sum_{i=1}^{I}[u_{i,1}(c_{i,1}) - u_{i,2}(c_{i,2})]\cdot\pi'(\alpha)\cdot\mathrm{d}\alpha,$$

where

$$\mathrm{d}\alpha = DA\cdot\mathrm{d}\vartheta_0 = \frac{\pi'(\alpha)\cdot\left[u'_{0,2}(c_{0,2})\cdot\mathrm{d}\vartheta_{0,2} - u'_{0,1}(c_{0,1})\cdot\mathrm{d}\vartheta_{0,1}\right]}{\pi''(\alpha)\cdot[u_{0,1}(c_{0,1}) - u_{0,2}(c_{0,2})]} \tag{17}$$

for an *exogenous* policy perturbation $d\vartheta_0 = \bar{\vartheta}_0 - \vartheta_0$. The direction of the Pareto improving policy thus depends on the sign of expression

$$\sum_{i=1}^{I}[u_{i,1}(\bar{c}_{i,1}) - u_{i,2}(\bar{c}_{i,2})].$$

Positive values imply that the society, excluding agent 0, is better-off in state 1 than in state 2. As a result, $d\alpha$ must be positive: the competitive action is too low and the Pareto improving policy involves inducing agent 0 to increase the action chosen at equilibrium. Looking at Eq. (17), this can be achieved with $d\vartheta_{0,2} < 0$ and $d\vartheta_{0,1} > 0$. That is, in order to induce agent 0 to take a higher action at equilibrium, a planner would like to restrict his insurance opportunities in a way that he is better-off in the state associated with a higher action, namely state 1, and worse-off in the other state. The agent who causes the externality is, therefore, "forced" to internalize the externality through considerations of his own welfare.[7]

6. Catastrophe, relief aid and genericity of strong constrained inefficiency

If the policy that aims at effecting a Pareto improvement is restricted to not include agent 0 in the profile of date-0 transfers, we can no longer use the social welfare function W. For agent 0, the policy needs to increase

$$U = -\alpha - q_1 \cdot \tilde{\vartheta}_{0,1} - q_2 \cdot \tilde{\vartheta}_{0,2} + \pi(\alpha) \cdot u_{0,1}(e_{0,1} + \tilde{\vartheta}_{0,1}) + [1 - \pi(\alpha)] \cdot u_{0,2}(e_{0,2} + \tilde{\vartheta}_{0,2}), \quad (18)$$

while, simultaneously, increasing

$$- q_1 \cdot \sum_{i=1}^{I} \vartheta_{i,1} - q_2 \cdot \sum_{i=1}^{I} \vartheta_{i,2} + \pi(\alpha) \cdot \sum_{i=1}^{I} u_{i,1}(e_{i,1} + \vartheta_{i,1}) + [1 - \pi(\alpha)] \cdot \sum_{i=1}^{I} u_{i,2}(e_{i,2} + \vartheta_{i,2}).$$

This would suffice, as the higher value of the latter sub-aggregate implies the existence of the required sub-profile of lump-sum transfers, $(\tau_i)_{i=1}^{I}$, after which all the agents $i \geqslant 1$ are made strictly better-off.

The following two assumptions will allow us to prove that, in a generic sense, the competitive equilibrium allocation is strongly constrained inefficient. As for the techniques used in obtaining this result, we apply the methods developed by Citanna et al. (1998).[8]

Assumption 3 (*Catastrophe*)**.** At the consumption allocation $(\hat{c}_i)_{i=0}^{I}$, the whole society prefers state 1 to state 2, in the sense that

$$\mu_0 = u_{0,1}(\hat{c}_{0,1}) - u_{0,2}(\hat{c}_{0,2}) > 0 \quad (19)$$

and

$$\mu_{\neg 0} = \sum_{i=1}^{I}[u_{i,1}(\hat{c}_{i,1}) - u_{i,2}(\hat{c}_{i,2})] > 0. \quad (20)$$

Assumption 4 (*Relief Aid*)**.** There exists an external source of funding that aids agents $i = 1, \ldots, I$ in their purchases of the elementary security for state 2. This fund covers, lump sum, a total of ρ/I units of the asset for each agent $i \geqslant 1$, so that the agent receives

$$q_2 \cdot \frac{\rho}{I}$$

in the first period.

Since this relief aid takes the form of lump-sum transfers (of the correct value at equilibrium) and date-0 consumption enters the agents' utility functions linearly, these transfers have no impact on the characterization of equilibrium we have used. Also, we do *not* assume that the relief aid is not an extra policy tool. Instead, it already exists in the original economy before the asset reallocation, so that, when doing welfare comparisons the transfers are received by the agents $i \geqslant 1$, *with and without the policy intervention*. That said, we also recognize that the assumption limits the interest of our result, in the sense that our welfare comparisons will not consider the well-being of the external agent funding the relief aid transfers.[9]

Under the latter assumption, we can write the aggregate utility of agents $i = 1, \ldots, I$ as

$$V = -q_1 \cdot \sum_{i=1}^{I} \vartheta_{i,1} - q_2 \cdot \left(\sum_{i=1}^{I} \vartheta_{i,2} - \rho\right) + \pi(\alpha) \cdot \sum_{i=1}^{I} u_{i,1}(e_{i,1} + \vartheta_{i,1}) + [1 - \pi(\alpha)] \cdot \sum_{i=1}^{I} u_{i,2}(e_{i,2} + \vartheta_{i,2}). \quad (21)$$

6.1. *Local subspaces of functions*

Under the quasi-linearity assumption, the equilibrium allocation is $\hat{c}$ and the equilibrium action is $\hat{\alpha}$. In order to perform genericity analysis on the spaces of utilities and the probability function, we parameterize a local subspace of functions as follows. Given some $\beta > 0$, let $b : \mathbb{R} \to [0, 1]$ be a $\mathbf{C}^1$ function such that

$$b(x) = \begin{cases} 0, & \text{if } |x| > \beta; \\ 1, & \text{if } |x| < \beta/2. \end{cases}$$

We refer to this function as a *bump*. Also for $\bar{\delta} > 0$, consider the «perturbed» mappings

$$(\alpha, \delta_0) \mapsto \pi(\alpha) + b(\alpha - \bar{\alpha}) \cdot \delta_0 \cdot (\alpha - \bar{\alpha})^2,$$

and, for each $i \geqslant 1$ and $s = 1, 2$,

$$(c_{i,s}, \delta_{i,s}) \mapsto u_{i,s}(c_{i,s}) + b(c_{i,s} - \bar{c}_{i,s}) \cdot \delta_{i,s} \cdot (c_{i,s} - \bar{c}_{i,s})^2.$$

For simplicity of notation, we will write these perturbed functions as $\pi(\cdot; \delta_0)$ and $u_{i,s}(\cdot; \delta_{i,s})$.

Note that if β and $\bar{\delta}$ are small enough, these mappings are increasing and concave, so long as $|\delta_0| < \bar{\delta}$ and $|\delta_{i,s}| < \bar{\delta}$ for all $i = 1, \ldots, I$ and $s = 1, 2$. This implies that we can use each of these parameters to perturb the corresponding function on an

[7] Of course, in the case where $\sum_{i=1}^{I}[u_{i,1}(\bar{c}_{i,1}) - u_{i,2}(\bar{c}_{i,2})] < 0$, the society excluding agent 0 is better-off in state 2, so that the competitive action is inefficiently high: in this case, Pareto optimality prescribes $d\alpha < 0$ which can be achieved with $d\vartheta_{0,2} > 0$ and $d\vartheta_{0,1} < 0$. The intuition is, again, that one would like to make agent 0 better-off in the state associated with a lower action and worse-off in the other state so that it is optimal for him to choose a lower action than before.

[8] See, also, Villanacci et al. (2013).

[9] In the case of an environmental catastrophe such as an oil spill, the assumption amounts to saying that the rest of the world subsidizes the preparations of the local population to deal with the emergency, say in terms of the necessary equipment and also in terms of supplementary income. This assumption is not entirely untenable, in particular since we exclude the oil producer from this subsidy. From a technical point of view, the assumption gives us access to two *independent* policy instruments, which are necessary since the goal is now to improve two different utility levels. Also, if the rest of the world is sufficiently affected by the catastrophe, their well-being can also be maintained or improved in spite of a potentially more expensive relief contribution.

one-dimensional open neighbourhood of the original function. For technical reasons, we restrict attention to the open subset

$$\Delta = \{\delta \in (-\bar\delta, \bar\delta)^5 \mid u''_{1,1}(\hat c_{1,1}) + \delta_{1,1} \neq u_1''^2(\hat c_1^2) + \delta_1^2 \text{ and } u''_{1,2}(\hat c_{1,2}) + \delta_{1,2} \neq u''_{2,2}(\hat c_{2,2}) + \delta_{2,2}\}. \tag{22}$$

Under Assumption 2, it is immediate that $0 \in \Delta$.

The following lemma is immediate.

Lemma 3 (*Invariance of Equilibrium to Perturbations*). *Perturbations to the probability function and preferences of agents $i = 1, \ldots, I$ do not affect the equilibrium: since $\mathcal{F}(\bar q, \bar\vartheta, \bar\alpha) = 0$, tuple $(\bar q, \bar\vartheta, \bar\alpha)$ continues to be an equilibrium when the probability function is $\pi(\cdot; \delta_0)$ and preferences are $u_{i,s}(\cdot; \delta_{i,s})$, as long as $|\delta_0| < \bar\delta$ and $|\delta_{i,s}| < \bar\delta$ for all $i = 1, \ldots, I$ and $s = 1, 2$.*

The lemma exploits one key property of our construction: that the perturbations do *not* affect the first derivatives of the perturbed functions at the equilibrium values. What they do, however, is to affect their second derivatives, which is the second key property — one that we will use below.

6.2. A characterization of strong constrained inefficiency

In order to make the concept of strong constrained inefficient easier to analyse, define the following function, where, for simplicity, we assume that $I = 2$,

$$(q, \vartheta, \alpha, \delta) \mapsto \begin{pmatrix} U \\ V \\ [u_{0,1}(e_{0,1} + \vartheta_{0,1}) - u_{0,2}(e_{0,2} + \vartheta_{0,2})] \cdot \pi'(\alpha; \delta_0) - 1 \\ \pi(\alpha; \delta_0) \cdot u'_{1,1}(e_{1,1} + \vartheta_{1,1}; \delta_{1,1}) - q_1 \\ [1 - \pi(\alpha; \delta_0)] \cdot u'_{1,2}(e_{1,2} + \vartheta_{1,2}; \delta_{1,2}) - q_2 \\ \pi(\alpha; \delta_0) \cdot u_1'^2(e_1^2 + \vartheta_1^2; \delta_1^2) - q_1 \\ [1 - \pi(\alpha; \delta_0)] \cdot u'_{2,2}(e_{2,2} + \vartheta_{2,2}; \delta_{2,2}) - q_2 \\ \sum_{i=0}^{I} \vartheta_{i,1} \\ \sum_{i=0}^{I} \vartheta_{i,2} \end{pmatrix}, \tag{23}$$

where U and V are as in Eqs. (18) and (21), respectively, with $\pi(\cdot; \delta_0)$ and $u_{i,s}(\cdot; \delta_{i,s})$ instead of $\pi(\cdot)$ and $u_{i,s}(\cdot)$.[10]

Denoting this mapping by $\mathcal{H}(q, \vartheta, \alpha; \delta)$, the following lemma follows as an implication of the definition of strong constrained inefficiency, and of the construction of functions U and V.

Lemma 4. *If $\{\bar\vartheta, \bar\alpha, \bar q\}$ is a competitive equilibrium and the partial Jacobian of $\mathcal{H}$ with respect to (q, ϑ, α) has full row rank, then the equilibrium allocation $(\bar\alpha, \bar c)$, where $\bar c_{i,s} = e_{i,s} + \bar\vartheta_{i,s}$ for $s = 1, 2$, is constrained inefficient in the strong sense.*

Proof. If the Jacobian

$$D_{q,\vartheta,\alpha}\mathcal{H}(\bar q, \bar\vartheta, \bar\alpha; \delta)$$

has full row rank, by the inverse function theorem it follows that $\mathcal{H}(\cdot; \delta)$ maps a neighbourhood of $(\bar q, \bar\vartheta, \bar\alpha)$ *onto* a neighbourhood of $\mathcal{H}(\bar q, \bar\vartheta, \bar\alpha; \delta)$. It then follows that, for a small enough $d > 0$, we can find $(\tilde q, \tilde\vartheta, \tilde\alpha)$ such that

$$\mathcal{H}(\tilde q, \tilde\vartheta, \tilde\alpha; \delta) = \mathcal{H}(\bar q, \bar\vartheta, \bar\alpha; \delta) + (d, d, 0, \ldots, 0)^\top.$$

At equilibrium, by definition, the third to last entries of $\mathcal{H}(\bar q, \bar\vartheta, \bar\alpha; \delta)$ equal 0. Substituting, this means that

$$\begin{pmatrix} U(\tilde q, \tilde\alpha, \tilde\vartheta; \delta) \\ V(\tilde q, \tilde\alpha, \tilde\vartheta; \delta) \\ [u_{0,1}(e_{0,1} + \tilde\vartheta_{0,1}) - u_{0,2}(e_{0,2} + \tilde\vartheta_{0,2})] \cdot \pi'(\tilde\alpha; \delta_0) - 1 \\ \pi(\alpha; \delta_0) \cdot u'_{1,1}(e_{1,1} + \tilde\vartheta_{1,1}; \delta_{1,1}) - \tilde q_1 \\ [1 - \pi(\alpha; \delta_0)] \cdot u'_{1,2}(e_{1,2} + \tilde\vartheta_{1,2}; \delta_{1,2}) - \tilde q_2 \\ \pi(\alpha; \delta_0) \cdot u_1'^2(e_1^2 + \tilde\vartheta_1^2; \delta_1^2) - \tilde q_1 \\ [1 - \pi(\alpha; \delta_0)] \cdot u'_{2,2}(e_{2,2} + \tilde\vartheta_{2,2}; \delta_{2,2}) - \tilde q_2 \\ \sum_{i=0}^{I} \tilde\vartheta_{i,1} \\ \sum_{i=0}^{I} \tilde\vartheta_{i,2} \end{pmatrix} = \begin{pmatrix} U(\bar q, \bar\alpha, \bar\vartheta; \delta) + d \\ V(\bar q, \bar\alpha, \bar\vartheta; \delta) + d \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first entry of this equality and the fact that $d > 0$ imply that agent 0 is made better-off, which is the fifth requirement of the definition of weak constrained inefficiency. The second entry implies that the aggregate V of utilities is higher too, which in turn implies that every *other* agent can be made better-off by an appropriate choice of lump-sum transfers, hence guaranteeing the sixth requirement in the definition (without violating the fourth requirement). Noting that the third entry implies the first-order condition of Program (12), the first requirement is also satisfied. The second requirement follows from the fourth to the antepenultimate entries, as each successive pair of them implies the first-order conditions of Program (4), for agents $i = 1, \ldots, I$. Finally, the last two entries imply the third requirement of the definition.

6.3. Genericity of strong constrained inefficiency

The characterization of constrained inefficiency by Lemma 4 seemingly allows us to prove a second generic result.

Theorem 2. *If the economy satisfies Assumptions 1 to 4, then, on an open and dense set of individual endowments, probability functions and individual preferences, the competitive equilibrium allocation is strongly constrained inefficient.*

Fixing all preferences and endowments such that the assumptions hold true, the challenge is to show that mapping

$$(q, \vartheta, \alpha, \delta, \gamma) \mapsto \begin{pmatrix} \mathcal{F}(q, \vartheta, \alpha; \delta) \\ D_{q,\vartheta,\alpha}\mathcal{H}(q, \vartheta, \alpha; \delta)^\top \gamma \\ \frac{1}{2}(\gamma \cdot \gamma - 1) \end{pmatrix},$$

where $\gamma \in \mathbb{R}^{2+2(I+1)+1}$, is transverse to 0. Let $\mathcal{M}$ denote the previous mapping.

Note that the entries of vector γ can be named according to the rows of the Jacobian matrix $D_{q,\vartheta,\alpha}\mathcal{H}$. Recalling the definition of mapping $\mathcal{H}$, these rows are:

1. the utility level of agent 0, U;
2. the aggregate utility of all other agents, V;

[10] When writing a function, the notation $f(x; \omega)$ is used to emphasize that x is the function's argument and ω is just a parameter. Hopefully, this makes it clear that $\partial f(x; \omega)$ refers to the derivative of the function with respect to argument x, when the parameter takes the value ω.

$$\begin{pmatrix} \pi' \cdot \sum_{i=1}^{2}[(u_{i,1}-u_{i,2})\cdot\gamma_V+(u'_{i,1}\cdot\gamma_{i,1}-u'_{i,2}\cdot\gamma_{i,2})]+(\pi''+\delta_0)\cdot(u_{0,1}-u_{0,2})\cdot\gamma_0 \\ \pi'\cdot u'_{0,1}\cdot\gamma_0+\gamma_1 \\ -\pi'\cdot u'_{0,2}\cdot\gamma_0+\gamma_2 \\ (u''_{1,1}+\delta_{1,1})\cdot\gamma_{1,1}+\gamma_1 \\ (u''_{1,2}+\delta_{1,2})\cdot\gamma_{1,2}+\gamma_2 \\ (u''^{2}_{1}+\delta^2_1)\cdot\gamma^2_1+\gamma_1 \\ (u''_{2,2}+\delta_{2,2})\cdot\gamma_{2,2}+\gamma_2 \\ -\vartheta_{0,1}\cdot(\gamma_U-\gamma_V)-\gamma_{1,1}-\gamma^2_1 \\ -\vartheta_{0,2}\cdot(\gamma_U-\gamma_V)+\rho\cdot\gamma_V-\gamma_{1,2}-\gamma^2_1 \end{pmatrix} \qquad (*)$$

Box II.

3. the first-order condition with respect to his action, in the maximization of agent 0;
4. for each i other than 0 and each s, the first-order condition with respect to $\vartheta_{i,s}$, in the maximization of agent i; and
5. for each s, the market clearing condition of the corresponding elementary security.

It is then convenient to use a mnemonic to denote the entries of the vector, as follows:

$$\gamma = (\gamma_U, \gamma_V, \gamma_0, \gamma_{1,1}, \gamma_{1,2}, \gamma^2_1, \gamma_{2,2}, \gamma_1, \gamma_2).$$

Lemma 5 (*Simplifying the System*)**.** *Suppose that* $\mathcal{M}(q, \vartheta, \alpha, \delta, \gamma) = 0$. *Then,*

1. $(q, \vartheta, \alpha) = (\bar{q}, \bar{\vartheta}, \bar{\alpha})$;
2. *the second component of* $\mathcal{M}$, *namely* $D_{q,\vartheta,\alpha}\mathcal{H}(\bar{q}, \bar{\vartheta}, \bar{\alpha}; \delta)^{\top}\gamma$, *simplifies to Eq.* (∗) *given in* Box II.
3. $\gamma_1 \neq 0$ *and* $\gamma_2 \neq 0$;
4. $\gamma_0 \neq 0$, *and for both* $i = 1, 2$ *and both* $s = 1, 2$, $\gamma_{i,s} \neq 0$; *and*
5. *either* $\gamma_{1,1} \neq \gamma^2_1$ *or* $\gamma_{1,2} \neq \gamma_{2,2}$.

Proof. The first statement follows immediately from Lemma 3, as $\mathcal{M} = 0$ implies that $\mathcal{F} = 0$. For the second statement, it follows by direct computation that $D_{q,\vartheta,\alpha}\mathcal{H}(\bar{q}, \bar{\vartheta}, \bar{\alpha}; \delta)^{\top}\gamma$ equals the sum of expression (∗) and

$$\begin{pmatrix} [-1+\pi'\cdot(u_{0,1}-u_{0,2})]\cdot\gamma_U \\ (-q_1+\pi\cdot u'_{0,1})\cdot\gamma_U \\ [-q_2+(1-\pi)\cdot u'_{0,2}]\cdot\gamma_U \\ (-q_1+\pi\cdot u'_{1,1})\cdot\gamma_V \\ [-q_2+(1-\pi)\cdot u'_{1,2}]\cdot\gamma_V \\ (-q_1+\pi\cdot u'^2_1)\cdot\gamma_V \\ [-q_2+(1-\pi)\cdot u'_{2,2}]\cdot\gamma_V \\ 0 \\ 0 \end{pmatrix}.$$

At $(\bar{q}, \bar{\vartheta}, \bar{\alpha})$, this latter vector vanishes.

For the third statement, suppose, by way of contradiction, that $\gamma_1 = 0$. It follows from the second equation of the system $D_{q,\vartheta,\alpha}\mathcal{H}(\bar{q}, \bar{\vartheta}, \bar{\alpha}; \delta)^{\top}\gamma = 0$ that $\gamma_0 = 0$. In the third equation, this implies that $\gamma_2 = 0$ too. The fourth to seventh equations then imply that

$$\gamma_{1,1} = \gamma_{1,2} = \gamma_{2,1} = \gamma_{2,2} = 0.$$

By Lemma 1, the previous-to-last equation implies that $\gamma_U = \gamma_V$. Then, Assumption 4 implies, via the last equation, that $\gamma_V = 0$. Summing up, $\gamma = 0$, which is impossible since $\gamma \cdot \gamma = 1$, as $\mathcal{M} = 0$.

For the fourth statement, note that if $\gamma_0 = 0$, the second equation implies that $\gamma_1 = 0$, which we just showed to be impossible. The same occurs if $\gamma_{i,s} = 0$: then, from the fourth to seventh equations it would follow that $\gamma_s = 0$, contradicting the third statement.

For the last statement, suppose to the contrary that $\gamma_{1,1} = \gamma_{2,1}$ and $\gamma_{1,2} = \gamma_{2,2}$. The fourth and fifth equations imply, given Assumption 4, that $\gamma_U = \gamma_V = 0$. From the last four equations we then have that

$$u''_{1,1} + \delta_{1,1} = u''_{2,1} + \delta_{2,1} \text{ and } u''_{1,2} + \delta_{1,2} = u''_{2,2} + \delta_{2,2},$$

which is impossible by the definition of the space Δ of perturbations, Eq. (22). □

Lemma 6 (*Transversality*)**.** *Mapping* $\mathcal{M}$ *is transverse to 0.*

Proof. Suppose that $\mathcal{M} = 0$. It follows from Lemma 3 that its Jacobian can be written as

$$\begin{pmatrix} D_{q,\vartheta,\alpha}\mathcal{F} & 0 \\ \mathbf{M} & \Omega \end{pmatrix},$$

where Ω is the partial Jacobian of mapping

$$(q, \vartheta, \alpha, \delta, \gamma) \mapsto \begin{pmatrix} D_{q,\vartheta,\alpha}\mathcal{H}(q, \vartheta, \alpha; \delta)^{\top}\gamma \\ \frac{1}{2}(\gamma\cdot\gamma-1) \end{pmatrix}$$

with respect to (δ, γ).[11]

Then, given Lemma 1, all we need to show is that matrix Ω has full row rank. To make it easier to see that this is indeed the case, it is convenient to re-organize this matrix. Note that the first rows of this matrix correspond to the entries of the product $D_{q,\vartheta,\alpha}\mathcal{H}^{\top}\gamma$.[12] By construction, these rows correspond to the arguments with respect to which mapping $\mathcal{H}$ has been differentiated, which we have been writing in the order

$$(\alpha, \vartheta_{0,1}, \vartheta_{0,2}, \vartheta_{1,1}, \vartheta_{1,2}, \vartheta_{2,1}, \vartheta_{2,2}, q_1, q_2).$$

It is now convenient, in fact, to write the last two rows of the matrix in the fourth and fifth positions, which amounts to taking the derivatives of $\mathcal{H}$ in the order

$$(\alpha, \vartheta_{0,1}, \vartheta_{0,2}, q_1, q_2, \vartheta_{1,1}, \vartheta_{1,2}, \vartheta_{2,1}, \vartheta_{2,2}).$$

Of course, the rank of Ω is not affected by this operation. As for its columns, it will also be convenient to write them in an unusual order:

$$(\delta_0, \gamma_1, \gamma_2, \gamma_U, \gamma_V, \delta_{1,1}, \delta_{1,2}, \delta_{2,1}, \delta_{2,2}, \gamma_{1,1}, \gamma_{1,2}, \gamma_{2,1}, \gamma_{2,2}).$$

11 Matrix **M** is the partial Jacobian of this same mapping with respect to (q, ϑ, α). We need not concern ourselves with its computation.

12 These rows are followed by only one other: the derivatives of the last row of $\mathcal{M}$.

$$\begin{pmatrix} \mu_0\gamma_0 & 0 & 0 & 0 & \pi'\mu_{\neg 0} & 0 & 0 & 0 & 0 & \pi' u'_{1,1} & -\pi' u'_{1,2} & \pi' u'_{2,1} & -\pi' u'_{2,2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta_{0,1} & \theta_{0,1} & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\theta_{0,2} & \theta_{0,2}+\rho & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & \pi\gamma_{1,1} & 0 & 0 & 0 & \pi h_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & (1-\pi)\gamma_{1,2} & 0 & 0 & 0 & (1-\pi)h_{1,2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \pi\gamma_{2,1} & 0 & 0 & 0 & \pi h_{2,1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1-\pi)\gamma_{1,2} & 0 & 0 & 0 & (1-\pi)h_{2,2} \\ 0 & \gamma_1 & \gamma_2 & \gamma_U & \gamma_V & 0 & 0 & 0 & 0 & \gamma_{1,1} & \gamma_{1,2} & \gamma_{2,1} & \gamma_{2,2} \end{pmatrix},$$

Box III.

One remaining argument, γ_0, will not be necessary for our differentiations.

Permuted in this way, when $\mathcal{M} = 0$ matrix Ω reads as given in Box III where μ_0 and $\mu_{\neg 0}$ come from Eqs. (19) and (20), respectively, and $h_{i,s}$ is used to denote $u''_{i,s} + \delta_{i,s}$, for brevity.

Consider first the leading principal minor of order 5, namely the matrix

$$\begin{pmatrix} \mu_0\cdot\gamma_0 & 0 & 0 & 0 & \pi'\cdot\mu_{\neg 0} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\theta_{0,1} & \theta_{0,1} \\ 0 & 0 & 0 & -\theta_{0,2} & \theta_{0,2}+\rho \end{pmatrix}.$$

Under Assumptions 3 and 4, this matrix is non-singular, thanks to the fact that $\gamma_0 \neq 0$, as per Lemma 5. Now, add the next four columns and rows. This adds, the columns

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pi\cdot\gamma_{1,1} & 0 & 0 & 0 \\ 0 & (1-\pi)\cdot\gamma_{1,2} & 0 & 0 \\ 0 & 0 & \pi\cdot\gamma_{2,1} & 0 \\ 0 & 0 & 0 & (1-\pi)\cdot\gamma_{1,2} \end{pmatrix}.$$

Given that for both $i = 1, 2$ and both $s = 1, 2$, $\gamma_{i,s} \neq 0$, again as per Lemma 5, it follows that this whole 9×9 leading principal minor is non-singular.

It only remains to show that when we add the last row of the matrix and the four remaining columns, the whole matrix maintains its full row rank. Given the last statement in Lemma 5, we can assume, with no loss of generality, that $\gamma_{1,1} \neq \gamma_{2,1}$. Now, perform the following operations:

1. add the columns corresponding to $\gamma_{1,1}$;
2. subtract the column corresponding to $\gamma_{2,1}$;
3. subtract $h_{1,1}/\gamma_{1,1}$ times the column corresponding to $\delta_{1,1}$;
4. add $h_{2,1}/\gamma_{2,1}$ times the column corresponding to $\delta_{2,1}$; and
5. subtract

   $$\frac{\pi'\cdot(u'_{1,1}-u'_{2,1})}{(u_{0,1}-u_{0,2})\cdot\gamma_0}$$

   times the column corresponding to δ_0, which can be done thanks to Assumption 3 and the fourth statement in Lemma 5.

Note that the result of these operations is vector

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \gamma_{1,1}-\gamma_{2,1} \end{pmatrix}.$$

Since $\gamma_{1,1} \neq \gamma_{2,1}$, the last entry of this resulting vector is non-zero. This implies that the matrix, as a whole, can also span the last dimension of its co-domain, and, given the previous results, that it has full row rank. □

We are now ready to prove that, generically on endowments, preferences and the probability function, the competitive equilibrium allocation is constrained inefficient in the strong sense too, under the extra assumptions.

Proof of Theorem 2. Since $M \pitchfork 0$, by the transversality theorem we will have that $\mathcal{M}(\cdot, \delta) \pitchfork 0$, generically on $\delta \in \Delta$. Now, $\mathcal{M}(\cdot, \delta)$ has

$$3 + 2I + 2 + 2 + 2(I + 1) + 1 + 1$$

entries, and only

$$2 + 2(I + 1) + 1 + 2 + 2(I + 1) + 1$$

arguments. It follows that $D_{q,\vartheta,\alpha,\gamma}\mathcal{M}$ cannot have full row rank, and hence it must be that, generically on $\delta \in \Delta$, $\mathcal{M}(q, \vartheta, \alpha, \gamma, \delta) \neq 0$.

Now, if that is the case, it is immediate that whenever

$$\mathcal{F}(q, \vartheta, \alpha, e,) = 0 \text{ and } D_{q,\vartheta,\alpha}\mathcal{H}(\bar{q}, \bar{\vartheta}, \bar{\alpha}, e)^{\mathsf{T}}\gamma = 0,$$

we also have that $\gamma = 0$. This implies, by construction, that when $\mathcal{F}(q, \vartheta, \alpha, e, u, \pi) = 0$, matrix $D_{q,\vartheta,\alpha}\mathcal{H}(\bar{q}, \bar{\vartheta}, \bar{\alpha}, e)$ has full row rank. By Lemma 4, it follows that the equilibrium allocation is constrained inefficient, in the strong sense, generically on $\delta \in \Delta$.

This result implies the claim, as long as we endow the space of preference and probability functions with a metrizable topology: for any given initial functions, any open neighbourhood of these functions will intersect the lower-dimensional subspace Δ, since $0 \in \Delta$. Since the equilibrium allocation is strongly constrained inefficient generically on Δ, we can find an array of functions in the intersection of the open neighbourhood and Δ where this is the case. This implies denseness, as needed. □

$$\mathrm{d}\alpha = \frac{\pi'(\bar{\alpha}) \cdot \left[\bar{\vartheta}_{0,2} \cdot u''_{1,2}(e_{1,2} + \bar{\vartheta}_{1,2}) \cdot \mathrm{d}\vartheta_{0,2} - \bar{\vartheta}_{0,1} \cdot u''_{1,1}(e_{1,1} + \bar{\vartheta}_{1,1}) \cdot \mathrm{d}\vartheta_{0,1}\right]}{\pi''(\bar{\alpha}) \cdot \left\{[\kappa_1(\bar{\vartheta}_{0,1}) \cdot \bar{\vartheta}_{0,1} - \kappa_2(\bar{\vartheta}_{0,2}) \cdot \bar{\vartheta}_{0,2}] + [u_{0,1}(e_{0,1} + \bar{\vartheta}_{0,1}) - u_{0,2}(e_{0,2} + \bar{\vartheta}_{0,2})]\right\}}. \tag{28}$$

Box IV.

7. More sophisticated behaviour by agent 0

Our analysis so far has assumed that all agents, including 0, take assets prices as given. This is the application to our context of the standard definition of competitive equilibrium from Public Economics. In Financial Economics, on the other hand, when an agent has the ability to affect the probability distribution of shocks in the economy, it is usually considered that she recognizes the effect that her decisions *with respect to those probabilities* affect the willingness to pay of other agents for the existing financial assets and, therefore, the equilibrium level of assets prices. In a sense, the point is that the kind of externality that agent 0 induces in our economy is *too salient* for him to take everybody else's decisions as given.

7.1. Stackelberg–Walras equilibrium

We now recognize this possibility and study whether our previous results are robust to this more sophisticated behaviour by agent 0. In order to do this, we need to decompose asset prices into the part of them that depends on the probability distribution and the part that is determined by trade, given the probabilities. Effectively, it is as if decisions were made sequentially in the first period: individual 0 first chooses the action he is to take, and then, taking that action as given, he and all the other agents trade financial assets in a competitive manner.

Luckily, our formulation allows us to nest these two decisions in just one problem for agent 0. In order to do this, we first introduce two functions that are of common use in financial economics. For each state s, and each level of trading of the corresponding elementary security for that state by agent 0, $\vartheta_{0,s}$, let $(\tilde{c}_{i,s})_{i=1}^I$ denote the solution to the following maximization problem:

$$\max_{(c_i)_{i=1}^I} \left\{ \sum_{i=1}^I u_{i,s}(c_i) : \sum_{i=1}^I c_{i,s} = \sum_{i=0}^I e_{i,s} - \vartheta_{0,s} \right\}. \tag{24}$$

Obviously, the solution to this problem depends on $\vartheta_{0,s}$, and so we can define

$$\kappa_s(\vartheta_{0,s}) = u'^1_s(\tilde{c}^1_s),$$

a function to which we will refer as the (ex-post) *pricing kernel* for state s.

The utility of this function is that If $\{\bar{\vartheta}, \bar{\alpha}, \bar{q}\}$ is a competitive equilibrium (in the sense defined before), it follows from Eq. (5), that the equilibrium prices decompose into the product of the pricing kernel for each state and its corresponding probability:

$$\begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix} = \begin{pmatrix} \pi(\bar{\alpha}) \cdot \kappa_1(\bar{\vartheta}_{0,1}) \\ [1 - \pi(\bar{\alpha})] \cdot \kappa_2(\bar{\vartheta}_{0,2}). \end{pmatrix}$$

Since we want to maintain the assumption that all agents are competitive in the financial markets, we will now consider the situation in which agent 0 recognizes the direct effect that his choice of action has on the vector of asset prices, via the probabilities, but acts as if his choice of portfolio did not affect the pricing kernels. The behaviour of all other agents in the economy remains unchanged, as well as the market clearing condition. For the sake of definiteness, we refer to this situation as *Stackelberg–Walras Equilibrium*, defined as follows[13]: it is a tuple $\{\bar{\vartheta}, \bar{\alpha}, \bar{q}\}$, where $\bar{\vartheta} = [(\bar{\vartheta}_{i,s})_{s=1,2}]_{i=0}^I$ is the allocation of the two assets and $\bar{q} = (\bar{q}_1, \bar{q}_2)$ is the vector of asset prices, such that:

1. action $\bar{\alpha}$ and portfolio $(\bar{\vartheta}_{0,s})_{s=1,2}$ solve the following Program:

$$\max_{\alpha, \vartheta_{0,1}, \vartheta_{0,2}} \{-\alpha - q_1(\alpha; \bar{\vartheta}_{0,1})\vartheta_{0,1} - q_2(\alpha; \bar{\vartheta}_{0,2})\vartheta_{0,2} + \pi(\alpha)u_{0,1}(e_{0,1} + \vartheta_{0,1}) + [1 - \pi(\alpha)]u_{0,2}(e_{0,2} + \vartheta_{0,2})\} \tag{25}$$

   where $q_1(\alpha; \vartheta_{0,1}) = \pi(\alpha) \cdot \kappa_1(\vartheta_{0,1})$ and $-q_2(\alpha; \vartheta_{0,2}) = [1 - \pi(\alpha)] \cdot \kappa_2(\vartheta_{0,2})$;
2. for each $i \geqslant 1$, portfolio $(\bar{\vartheta}_{i,s})_{s=1,2}$ solves Program (4) when the prices are $q = \bar{q}$ and the probability of state 1 is $\pi(\bar{\alpha})$; and
3. both of the securities markets clear: $\sum_{i=0}^I \bar{\vartheta}_{i,s} = 0$, for $s = 1, 2$.

Importantly, as agent 0 recognizes the effect of his action on prices, Eq. (7) is no longer a valid first-order condition of his problem; instead, now his behaviour is characterized by the requirement that

$$1 = \pi'(\bar{\alpha}) \cdot \{-[\kappa_1(\bar{\vartheta}_{0,1}) - \kappa_2(\bar{\vartheta}_{0,2})] + [u_{0,1}(e_{0,1} + \bar{\vartheta}_{0,1}) - u_{0,2}(e_{0,2} + \bar{\vartheta}_{0,2})]\} \tag{26}$$

along with Eqs. (8) evaluated at $(\bar{\alpha}, \bar{\vartheta}_{0,1}, \bar{\vartheta}_{0,2})$.

7.2. Weak constrained inefficiency

As before, if we fix a Stackelberg–Walras equilibrium, $\{\bar{\vartheta}, \bar{\alpha}, \bar{q}\}$, our goal is again to show that the allocation $(\bar{\alpha}, \bar{c})$, where $\bar{c}_{i,s} = e_{i,s} + \bar{\vartheta}_{i,s}$, is constrained inefficient in the weak sense, generically on date-1 endowments.

With the different first-order condition with respect to agent 0's action, Eq. (26), the results of Section 5.2 imply that, upon a differential perturbation,

$$\mathrm{d}W = \left\{ [\kappa_1(\bar{\vartheta}_{0,1}) \cdot \bar{\vartheta}_{0,1} - \kappa_2(\bar{\vartheta}_{0,2}) \cdot \bar{\vartheta}_{0,2}] + \sum_{i=1}^I [u_{i,1}(\bar{c}_{i,1}) - u_{i,2}(\bar{c}_{i,2})] \right\} \cdot \pi'(\bar{\alpha}) \cdot \mathrm{d}\alpha \tag{27}$$

while $\mathrm{d}\alpha$ is given in Box IV.

Generically on date-1 endowments, a policy can induce $\mathrm{d}\alpha$ such that $\mathrm{d}W > 0$.

[13] We use the term Stackelberg–Walras to emphasize the difference with the previous definition of equilibrium, where all agents make their decisions taking the others' as given. In that sense, the previous definition can be referred to as *Nash–Walras Equilibrium*, while here, strategically, agent 0 behaves as a Stackelberg leader.

8. Uninsurable idiosyncratic risk

As an alternative framework, and in order to show that these results extend to economies with idiosyncratic risk, we now study a model where the agents are subject to uninsurable idiosyncratic shocks. For the sake of brevity, we restrict attention to the weaker definition of constrained inefficiency.

Individuals are of different types, $i = 0, \ldots, I$, and within each type there is a continuum of individuals of mass m_i. For simplicity, we assume that $m_0 = 1$.

Individuals of different types differ in their period-1 preferences and in their endowments, but they face the same idiosyncratic shocks. For simplicity, let us assume that there are only three personal states, denoted by $s = 1, 2, 3$. In $s = 1$ there is no shock in the endowment of the consumption good, while in $s = 2, 3$ there is a positive and negative shock, respectively, of size z. In period 1, a fraction $\pi(\alpha)$ of the individuals find themselves in state 1, while equal fractions of size $\frac{1}{2}[1-\pi(\alpha)]$ find themselves in states 2 and 3. However, there is no aggregate uncertainty: the aggregate endowment of the economy in period 1 is $\sum_{i=0}^{I} m_i \cdot \bar{e}_i$, where $\bar{e}_i$ is the endowment of individuals of type i in state 1, where there is no shock.

Once again, the probabilities of each personal state depend on the aggregate action that agents of type 0 will choose. In particular, while the expected value of the endowment remains unchanged, a higher action increases the probability of observing no shock and decreases the probability of positive and negative shocks. A lower action, in other words, induces a mean-preserving spread in the distributions of the agents' wealth. Therefore, a risk averse agent would prefer a distribution that second-order stochastically dominates, and this is why he would choose a non-zero action.

8.1. Competitive equilibrium

Suppose there is only a risk-less asset that can be traded: it pays one unit of the consumption good at date 1. Holdings of the asset are b_i and its price is q.

8.1.1. The problem of agents of type $i \geqslant 1$

Once again, all agents of types other than 0 have to choose their holdings of the riskless bond, and therefore the consumption in period 0 and in every personal state in period 1. The problem that an agent of type $i \geqslant 1$ faces is to choose b_i so as to maximize

$$-q \cdot b_i + \pi(\alpha) \cdot u_i(e_i + b_i) + \frac{1}{2} \cdot [1 - \pi(\alpha)] \cdot [u_i(e_i + z + b_i) + u_i(e_i - z + b_i)].$$

These agents take the price of the riskless bond and the probabilities of the personal states as given. The first-order condition of this problem is, then, that

$$q = \pi(\alpha) \cdot u_i'(e_i + b_i) + \frac{1}{2} \cdot [1 - \pi(\alpha)] \times \left[u_i'(e_i + z + b_i) + u_i'(e_i - z + b_i)\right]. \quad (29)$$

8.1.2. The problem of agents of type 0

On the other hand, each agent of type 0 has to maximize

$$-\alpha - q \cdot b_0 + \pi(\alpha) \cdot u_0(e_0 + b_0) + \frac{1}{2} \cdot [1 - \pi(\alpha)] \cdot [u_0(e_0 + z + b_0) + u_0(e_0 - z + b_0)]$$

by his choice of savings, b_0, and action, α.

It is important to note that, although we have assumed that the probability of each state depends on the aggregate action exerted by all agents of type 0, when an agent j of type 0 solves his maximization problem, he sees the probability of each state as depending only on his own action.[14] In equilibrium, this does not matter as all agents of type 0 choose the same action.

Here, the first-order conditions are that[15]

$$1 = \pi'(\alpha) \cdot \left[u_0(e_0 + b_0) - \frac{1}{2} \cdot u_0(e_0 + z + b_0) - \frac{1}{2} \cdot u_0(e_0 - z + b_0)\right] \quad (30)$$

and

$$q = \pi(\alpha) \cdot u_0'(e_0 + b_{0,j}) + \frac{1}{2} \cdot [1 - \pi(\alpha)] \cdot \left[u_0'(e_0 + z + b_0) + u_0'(e_0 - z + b_0)\right]. \quad (31)$$

8.1.3. Nash–Walras equilibrium

As before, the first-order conditions of individual rationality and the market clearing requirement characterize competitive equilibrium. We denote competitive equilibria by $\{\bar{b}, \bar{\alpha}, \bar{q}\}$, where $\bar{b} = (\bar{b}_i)_{i=0}^{I}$ is a profile of savings. These values solve the first-order conditions, Eqs. (29), (30) and (31), as well as the equality $\sum_{i=0}^{I} m_i \cdot b_i = 0$.

8.2. Constrained inefficiency of competitive equilibrium

Now, consider a policy intervention that perturbs by $\mathrm{d}b_0$ the holdings of the riskless bond of all agents of type 0. The welfare effects of such policy around the competitive equilibrium point, $\mathrm{d}W$ are, as before, the sum of:

1. the direct loss due to a different action, $-\mathrm{d}\alpha$;
2. the indirect effect due to the change in probabilities,
$$\sum_{i=0}^{I} m_i \cdot \left\{u_i(\bar{c}_{i,1}) - \frac{1}{2} \cdot \left[u_i(\bar{c}_{i,2}) + u_i(\bar{c}_{i,3})\right]\right\} \cdot \pi'(\bar{\alpha}) \cdot \mathrm{d}\alpha,$$
where $\bar{c}_{i,s}$ represents the equilibrium consumption of agents of type i in state s; and
3. the indirect effect due to the reallocation of savings,
$$\pi(\bar{\alpha}) \cdot \sum_{i=0}^{I} m_i \cdot u_i'(\bar{c}_{i,1}) \cdot \mathrm{d}c_{i,1} + \frac{1}{2} \cdot [1 - \pi(\bar{\alpha})] \cdot \sum_{i=0}^{I} m_i \cdot \left[u_i'(\bar{c}_{i,2}) \cdot \mathrm{d}c_{i,2} + u_i'(\bar{c}_{i,3})\mathrm{d}c_{i,3}\right].$$

As before, taking into account the first-order conditions of the agents at the equilibrium point and the market clearing conditions, this expression simplifies to

$$\mathrm{d}W = \sum_{i=1}^{I} m_i \cdot \left\{u_i(\bar{c}_{i,1}) - \frac{1}{2} \cdot \left[u_i(\bar{c}_{i,2}) + u_i(\bar{c}_{i,3})\right]\right\} \cdot \pi'(\bar{\alpha}) \cdot \mathrm{d}\alpha, \quad (32)$$

where we can substitute

$$\mathrm{d}\alpha = -\frac{\pi'(\bar{\alpha})}{\pi''(\bar{\alpha})} \cdot \frac{u_0'(\bar{c}_{0,1}) \cdot \mathrm{d}c_{0,1} - \frac{1}{2} \cdot \left[u_0'(\bar{c}_{0,2}) \cdot \mathrm{d}c_{0,2} + u_0'(\bar{c}_{0,3}) \cdot \mathrm{d}c_{0,3}\right]}{u_0(\bar{c}_{0,1}) - \frac{1}{2} \cdot \left[u_0(\bar{c}_{0,2}) + u_0(\bar{c}_{0,3})\right]}. \quad (33)$$

[14] That is, strictly speaking: each agent j of type 0 chooses an action α_j, considering the probability $\pi(\alpha_j)$; agents of types $i \geqslant 1$, on the other hand, take as given the probability $\pi(\int \alpha_j \, \mathrm{d}j)$.

[15] As with the case of complete markets, we can make interiority assumptions so that we do not have to look for boundary solutions.

By strong concavity,

$$\sum_{i=1}^{I} m_i \cdot u_i(e_i + \bar{b}_i) - \frac{1}{2} \cdot \sum_{i=1}^{I} m_i \cdot u_i(e_i + z + \bar{b}_i) - \frac{1}{2} \cdot \sum_{i=1}^{I} m_i \cdot u_i(e_i - z + \bar{b}_i) > 0,$$

which implies that the competitive action is inefficiently low: an increase in the action would be welfare improving for the whole society. Then, Eq. (33) implies that the direction of the perturbation of the bond holdings of agent 0 that implements a higher action depends on the sign of expression

$$u_0'(e_0 + \bar{b}_0) \cdot \mathrm{d}c_{0,1} - \frac{1}{2} \cdot \left[u_0'(e_0 + z + \bar{b}_0) \cdot \mathrm{d}c_{0,2} + u_0'(e_0 - z + \bar{b}_0) \cdot \mathrm{d}c_{0,3} \right].$$

If we now assume that agents of type 0 are *prudent*, we conclude that a perturbation $\mathrm{d}b_0 < 0$ induces an improvement in social welfare: if these agents save below the equilibrium level, with convex marginal utility they will exert a higher action since, by doing so, they make the future look less "volatile". This reduction in volatility makes the aggregate welfare higher, in spite of the fact that a higher action subtracts from the aggregate welfare functions and even though it implies an imperfect operation of the financial market.

9. Concluding remarks

This paper argues that limiting the ability of an agent to diversify his risks creates incentives for him to internalize the welfare effects of catastrophic events whose probability he affects, leading to a welfare improvement for society as whole. We model an economy where, initially, all agents participate in the trading of a complete set of financial assets. The only market failure in this economy is the externality that one agent imposes on the others via the probability distribution of their risks. We first show that as in all economies with market failures, the competitive equilibrium is Pareto inefficient. A more interesting question, however, is whether a social planner, who faces constraints in terms of the policies he can apply, would indeed be able to effect a welfare improving policy. Here, we consider the case in which the social planner can only induce an asset reallocation and lump-sum transfers. We find that, in a generic sense, everybody can be made better-off if the agent that causes the externality is not allowed to choose his portfolio of financial assets in an optimal manner, while everybody else continues to trade without constraints. We first allow for a lump-sum transfer to the agent causing the externality, getting a weak definition of constrained inefficiency. For strong constrained inefficiency, we only require that lump-sum transfers are financed through "relief aid", an external source of funding. In this way, we account for the case when, for institutional reasons and despite not altering his incentives, a transfer to the agent causing the externality cannot be implemented.

One might argue that the kind of externality that agent 0 induces in our economy is too salient for him to take everybody else's actions as given. To check the robustness of our results to a more sophisticated behaviour by agent 0, we now allow for him to recognize that his decisions with respect to probabilities affect the willingness to pay of other agents for the existing financial assets and, therefore, the equilibrium level of assets prices. Effectively, it is as if decisions were made sequentially in the first period: individual 0 first chooses the action, and then, taking that action as given, he and all the other agents trade financial assets in a competitive manner. Our results survive this added level of sophistication, and the equilibrium allocation is shown to be weakly constrained inefficient, generically.

Finally, our results extend to economies with uninsurable idiosyncratic risk. Assuming that the agents causing the externality are prudent, we find that a restriction in their equilibrium level of savings induces a distribution of risks that second-order stochastically dominates and therefore an improvement in social welfare. Hence, adding another distortion to an already imperfect financial market turns out to be socially beneficial.

A potential realm where these results may have policy relevance is environmental economics. In their exploration activities, for instance, oil companies make decisions that can make environmental accidents more likely. Our results then say that the unregulated diversification of privately perceived risks leads to a Pareto inefficient social outcomes. Even if it implies, via market clearing, that the portfolios of other agents in the economy will be perturbed too, the imposition of asset trading constraints on these companies will, generically, lead to an outcome where every agent economy (including the oil companies) are strictly better off. Surely, a mechanism design approach can be used to answer the same question. We see these two approaches as complementary, future research.

Appendix

Lemma 2. *Except on a closed and negligible set of individual endowments,*

$$\sum_{i=1}^{I} [u_{i,1}(\bar{c}_{i,1}) - u_{i,2}(\bar{c}_{i,2})] \neq 0 \tag{14}$$

at competitive equilibrium.

Proof. At competitive equilibrium, the allocation of commodities satisfies the first-order conditions of Pareto efficiency *with respect to the allocation of commodities* — compare Eqs. (5) and (8) with Eq. (11). Thus, we start by defining the following function

$$\mathcal{K}(c_1, c_2, \lambda_1, \lambda_2, e_1, e_2) = \begin{pmatrix} u_{0,1}'(c_{0,1}) - \lambda_1 \\ u_{1,1}'(c_{1,1}) - \lambda_1 \\ \vdots \\ u_{I,1}'(c_{I,1}) - \lambda_1 \\ \sum_{i=0}^{I} (e_{i,1} - c_{i,1}) \\ \sum_{i=1}^{I} [u_{i,1}(c_{i,1}) - u_{i,2}(c_{i,2})] \\ u_{0,2}'(c_{0,2}) - \lambda_2 \\ u_{1,2}'(c_{1,2}) - \lambda_2 \\ \vdots \\ u_{I,2}'(c_{I,2}) - \lambda_2 \\ \sum_{i=0}^{I} (e_{i,2} - c_{i,2}) \end{pmatrix} \tag{34}$$

which includes those conditions, the resource constraints for each state in period 1 and the utility sub-aggregate $\sum_{i=1}^{I} [u_{i,1}(c_{i,1}) - u_{i,2}(c_{i,2})]$.[16] Then, at any competitive equilibrium of the economy, the 4-tuple $(c_1, c_2, \lambda_1, \lambda_2)$ is such that all entries of $\mathcal{K}$ other than the one corresponding to $\sum_{i=1}^{I} [u_{i,1}(c_{i,1}) - u_{i,2}(c_{i,2})]$, are equal to 0. With arguments in the order

$$(c_{0,1}, \ldots, c_{I,1}, e_{0,1}, \lambda_1, c_{0,2}, \ldots, c_{I,2}, \lambda_2, e_{1,1}, \ldots, e_{I,1}, e_{0,2}, \ldots, e_{I,2}),$$

the Jacobian at the Pareto efficient allocation at each state is given in Box V.

[16] We denote by λ_1 and λ_2 the Lagrange multipliers associated with the resource constraints in states 1 and 2 respectively. These can be constructed by taking the ratio of the price of each elementary security and the probability of the corresponding state, at equilibrium.

$$\begin{pmatrix} u''_{0,1} & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & u''_{1,1} & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u''_{I,1} & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & u'_{1,1} & \cdots & u'_{I,1} & 0 & 0 & 0 & u'_{1,2} & \cdots & u'_{I,2} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & u''_{0,2} & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u''_{1,2} & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & u''_{I,2} & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix}. \qquad (35)$$

Box V.

We now argue that this $[2(I+1)+3] \times [4(I+1)+2]$ matrix has full rank, in the following four steps:

1. Consider the submatrix without the last $2I+1$ columns, which we denote with $\mathcal{V}$, and partition this submatrix as

$$\left(\begin{array}{cccccc|ccccc} u''_{0,1} & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & u''_{1,1} & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & u''_{I,1} & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & \cdots & -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & u'_{1,1} & \cdots & u'_{I,1} & 0 & 0 & 0 & u'_{1,2} & \cdots & u'_{I,2} & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & u''_{0,2} & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u''_{1,2} & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & u''_{I,2} & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -1 & -1 & \cdots & -1 & 0 \end{array}\right). \qquad (36)$$

2. We now show that the top-left $(I+3) \times (I+3)$ submatrix, which we denote $\mathcal{V}_{11}$, has full row rank. First notice that, by strong concavity, the submatrix without the last row has full row rank. Then, for $\mathcal{V}_{11}$ to have full row rank, it suffices to show that there exists a column vector $\zeta \in \mathbb{R}^{I+3}$ such that $\mathcal{V}_{11}\zeta = (0, \beta)^T$ for some scalar $\beta \neq 0$. Let

$$\zeta = \begin{pmatrix} u''_{0,1}(\hat{c}_{0,1})^{-1} \\ u''_{1,1}(\hat{c}_{1,1})^{-1} \\ \vdots \\ u''_{I,1}(\hat{c}_{I,1})^{-1} \\ \sum_{i=0}^{I} u''_{i,1}(\hat{c}_{i,1})^{-1} \\ 1 \end{pmatrix}.$$

   Then, $\mathcal{V}_{11}\zeta$ is 0 everywhere, except from the last row where it is

$$\beta = \sum_{i=1}^{I} \frac{u'_{i,1}(\hat{c}_{i,1})}{u''_{i,1}(\hat{c}_{i,1})} \neq 0.$$

3. Noting that $\mathcal{V}_{11}$ is invertible, $\mathcal{V}_{21}$ is a zero matrix and $\mathcal{V}_{22}$ has full row rank (by strong concavity), and since

$$|\mathcal{V}| = |\mathcal{V}_{11}||\mathcal{V}_{22} - \mathcal{V}_{21}\mathcal{V}_{11}^{-1}\mathcal{V}_{12}|,$$

   we conclude that $\mathcal{V}$ has full row rank.

4. Finally, adding the last $2I+1$ columns the row rank of the matrix will not change, so $D\mathcal{K}$ has full row rank.

Now, since $D\mathcal{K}$ has full row rank, we conclude that $\mathcal{K} \pitchfork 0$. Then, the set of endowments at which $\mathcal{K}(\cdot, e_1, e_2) \pitchfork 0$ has full measure. Since $D\mathcal{K}(\cdot, e_1, e_2)$ has one fewer column than it has rows, $\mathcal{K}(\cdot, e_1, e_2) \pitchfork 0$ implies that, whenever all conditions for competitive equilibrium are true, Eq. (14) is also true.

References

Bierbrauer, F.J., Hellwig, M.F., 2016. Robustly coalition-proof incentive mechanisms for public good provision are voting mechanisms and vice versa. Rev. Econom. Stud. 83 (4), 1440–1464.

Braido, L., 2005. General equilibrium with endogenous securities and moral hazard. Econom. Theory 26 (1), 85–101.

Cass, D., Siconolfi, P., Villanacci, A., 2001. Generic regularity of competitive equilibria with restricted participation. J. Math. Econom. 36 (1), 61–76.

Citanna, A., Kajii, A., Villanacci, A., 1998. Constrained suboptimality in incomplete markets: a general approach and two applications. Econom. Theory 11 (3), 495–521.

Clarke, E.H., 1971. Multipart pricing of public goods. Public Choice 11 (1), 17–33.

D'Aspremont, C., Gérard-Varet, L.-A., 1979. Incentives and incomplete information. J. Public Econ. 11 (1), 25–45.

Geanakoplos, J., Polemarchakis, H., 1986. Existence, regularity and constrained suboptimality of competitive equilibrium allocations when the asset market is incomplete. Uncertain. Inf. Commun.: Essays Honor Kenneth Arrow 3, 65–96.

Geanakoplos, J., Polemarchakis, H.M., 2008. Pareto improving taxes. J. Math. Econom. 44 (7), 682–696.

Ghosal, S., Polemarchakis, H.M., 1997. Nash–Walras equilibria. Res. Econ. 51 (1), 31–40.

Gori, M., Pireddu, M., Villanacci, A., 2012. Regularity and pareto improving on financial equilibria with price-dependent borrowing restrictions. Res. Econ.

Greenwald, B., Stiglitz, J., 1986. Externalities in economies with imperfect information and incomplete markets. Q. J. Econ. 101 (2), 229–264.

Groves, T., 1973. Incentives in teams. Econometrica 617–631.

Groves, T., Ledyard, J., 1977. Optimal allocation of public goods: A solution to the "free rider" problem. Econometrica 783–809.

Kuzmics, C., Steg, J.-H., 2017. On public good provision mechanisms with dominant strategies and balanced budget. J. Econom. Theory 170, 56–69.

Polemarchakis, H., Siconolfi, P., 1997. Generic existence of competitive equilibria with restricted participation. J. Math. Econom. 28 (3), 289–311.

Stiglitz, J., 1982. The inefficiency of the stock market equilibrium. Rev. Econom. Stud. 49 (2), 241–261.

Villanacci, A., Carosi, L., Benevieri, P., Battinelli, A., 2013. Differential Topology and General Equilibrium with Complete and Incomplete Markets. Springer Science & Business Media.