A continuous extension that preserves concavity, monotonicity and Lipschitz continuity*

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Abstract

The following is proven here: let $W : X \times C \longrightarrow \mathbb{R}$, where X is convex, be a continuous and bounded function such that for each $y \in C$, the function $W(\cdot, y) : X \longrightarrow \mathbb{R}$ is concave (resp. strongly concave; resp. Lipschitzian with constant M; resp. monotone; resp. strictly monotone) and let $Y \supseteq C$. If C is compact, then there exists a continuous extension of W, $U : X \times Y \longrightarrow \left[\inf_{X\times C} W, \sup_{X\times C} W\right]$, such that for each $y \in Y$, the function $U(\cdot, y) : X \longrightarrow \mathbb{R}$ is concave (resp. strongly concave; resp. Lipschitzian with constant M_y; resp. monotone; resp. strictly monotone).

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There are classical extension results in real, abstract and convex analysis.

Urysohn's lemma says that, given two closed disjoint subsets of a normal space, there exists a continuous function defined on the whole space, mapping each set into a different constant.

Tietze's extension theorem states that any continuous and bounded function defined from a closed subset of a metric space into the real line has a continuous extension to the whole space, with the same bounds as the original function.

Regarding Lipschitz continuity, Kirszbaum's theorem (see Federer (1969), 2.10.43) states that given a Lipschitzian function defined on a subset of a metric space, there exists an extension to the whole space that has the same Lipschitz constant.

In convex analysis, the classical extension result studies conditions under which a convex and bounded real valued function defined on the interior of a set can be extended to a continuous and convex function defined on the whole of the set.

A problem similar to the one addressed by Tietze's theorem is studied here. Suppose that $X \subseteq \mathbb{R}^J$ and $C \subseteq Y \subseteq \mathbb{R}^K$, where $J, K < \infty$. Suppose that X. Let $W : X \times C \longrightarrow \mathbb{R}$ be continuous, bounded and such that for each $y \in C$, $W(\cdot, y) : X \longrightarrow \mathbb{R}$ is concave. I study conditions on C under which one can ensure that there exists $U : X \times Y \longrightarrow \mathbb{R}$ such that:

1. U is continuous;
2. for every $y \in Y$, $U(\cdot, y) : X \longrightarrow \mathbb{R}$ is concave;
3. for every $(x, y) \in X \times C$, $U(x, y) = W(x, y)$;
4. for every $(x, y) \in X \times Y$, $\inf_{(x', y') \in X \times C} W(x', y') \leqslant U(x, y) \leqslant \sup_{(x'', y'') \in X \times C} W(x'', y'')$.

This result is important, since preferences of individuals that interact strategically are usually modelled by functions like U. For example, the simplest proof of existence of Nash equilibria in a finite game, which invokes Kakutani's fixed point theorem, assumes that each individual's preferences are representable by functions like U, with x representing that individual's action and y representing his opponent's actions. If one were given a set of profiles of actions in such an environment and one wanted to rationalize that set as Nash equilibria for some unobserved preferences, then, for each individual, one would need to apply standard revealed preference analysis on his own actions, for each observed value of the actions of his opponents, and would then need a continuous extension like the one introduced here.

By adapting the proof of Tietze's extension theorem given by Bridges (1998), I find that compactness of C suffices. Moreover, compactness allows me to show that if for each $y \in C$, $W(\cdot, y)$ is strongly concave (resp. Lipschitzian with constant M – independent of y; resp monotone; resp. strictly monotone), then U can further be found that satisfies that for each $y \in Y$, $U(\cdot, y)$ is strongly concave (resp. Lipschitzian with constant M_y; resp. monotone; resp. strictly monotone).

There is some related literature. Stadje (1987) shows that if $A \subseteq (a, b)$ has full Lebesgue measure with respect to (a, b) and $W : A \longrightarrow \mathbb{R}$ is measurable and mid-convex, then there exists a convex extension of W to (a, b). Neither continuity nor Lipschitz continuity are studied by Stadje.

Matoušková (2000) shows that if Y is a compact Hausdorff space, $A \subseteq Y$ is closed, and $W : A \longrightarrow \mathbb{R}$ is continuous and Lipschitzian, then there exists a continuous extension of W, $U : Y \longrightarrow \mathbb{R}$ which is Lipschitzian, with the same constant as W, and has the same sup norm.[1] No concavity or monotonicity properties are studied by Matoušková.

On the other hand, Howe (1986) gives necessary and sufficient conditions under which for a finite collection $\{W^l\}_{l=1}^{L}$ of continuous and concave functions, $W^l : \mathbb{R}_+^J \longrightarrow \mathbb{R}_+$, there exist $M \in \mathbb{N}$, a linear function $L : \mathbb{R}^J \longrightarrow \mathbb{R}^M$ and $U : \mathbb{R}_+^M \longrightarrow \mathbb{R}_+$ continuous and concave,

[1] The target set need not be $\mathbb{R}$, but any metric space with lower semicontinuous metric.

and for each $l \in \{1, ..., L\}$, there exists $y^l \in \mathbb{R}_+^M$ such that $W^l(x) = U\left(Lx + y^l\right)$, for every $x \in \mathbb{R}_+^J$.

Because this last problem is similar to the one studied here, the differences deserve to be pointed out. The first and more obvious one is that no smoothness or monotonicity properties are dealt with by Howe. The second one, which is fundamental, is that I am not assuming that C is finite. Besides, I take as given the set Y, and, therefore, cannot use its dimension as a variable. Moreover, I do not require concavity of the function U, but only of its cross sections ($U(\cdot, y)$ for each $y \in Y$), and Y need not even be convex.

In what follows, given a set $Y \subseteq \mathbb{R}^K$, I define the point-to-set distance function dis : $Y \times \mathcal{P}(Y) \setminus \{\varnothing\} \longrightarrow \mathbb{R}_+; (y, C) \longmapsto \inf_{\widehat{y} \in C} \|y - \widehat{y}\|$, where $\mathcal{P}(Y)$ represents the power set of Y. For simplicity of notation, I use $\|\cdot\|$ for the Euclidean norm without being specific about the dimension of the space being considered. Similarly, $B_d(z)$ denotes the open ball of radius d around z in the Euclidean space in which z lies. Given $Z \subseteq \mathbb{R}^K$, I denote by Z^0 its interior and by $\overline{Z}$ its closure.

The main result obtained here is the following:

Theorem 1 *Let $X \subseteq \mathbb{R}^J$ and $Y \subseteq \mathbb{R}^K$, where $J, K \in \mathbb{N}$, be nonempty. Suppose that X is convex and $C \subseteq Y$ is compact. Suppose that $W : X \times C \longrightarrow \mathbb{R}$ is continuous and bounded, and that for each $y \in C$, $W(\cdot, y)$ is concave. Then, there exists $U : X \times Y \longrightarrow \mathbb{R}$, continuous, such that*

1. *for each $(x, y) \in X \times C$, $U(x, y) = W(x, y)$;*

2. *for each $y \in Y$, $U(\cdot, y)$ is concave;*

3. $\inf_{(x,y) \in X \times Y} U(x, y) = \inf_{(x,y) \in X \times C} W(x, y)$ *and* $\sup_{(x,y) \in X \times Y} U(x, y) = \sup_{(x,y) \in X \times C} W(x, y)$.

Proof. If W is constant, the result is trivial. Else, let

$$l : \left[\inf_{(x,y)\in X\times C} W(x,y), \sup_{(x,y)\in X\times C} W(x,y) \right] \longrightarrow [1,2]$$

be the affine increasing bijection. Both l and l^{-1} are concave, continuous and strictly increasing. Define $f = l \circ W : X \times C \longrightarrow [1,2]$. By construction, f is continuous, $\inf_{(x,y)\in X\times C} f(x,y) = 1$ and $\sup_{(x,y)\in X\times C} f(x,y) = 2$, and for each $y \in C$, $f(\cdot, y)$ is concave.

Since C is closed, it follows (Moore, 1999b, 7.54) that $\forall y \in Y \backslash C$, $dis(y, C) > 0$.

Define the function $F : X \times Y \longrightarrow \mathbb{R}$ by

$$F(x,y) = \begin{cases} f(x,y) & \text{if} \quad y \in C \\ \frac{\inf_{\widehat{y}\in C} f(x,\widehat{y}) \|y-\widehat{y}\|}{dis(y,C)} & \text{otherwise} \end{cases}$$

It is obvious that $\forall (x,y) \in X \times C$, $F(x,y) = f(x,y)$. Moreover, F has the following properties:

Property 1: $\forall (x,y) \in X \times Y$, $F(x,y) \in [1,2]$

Proof of property 1: Fix $(x,y) \in X \times (Y \backslash C)$. Clearly, $\forall \widehat{y} \in C$, $f(x,\widehat{y}) \|y - \widehat{y}\| \leqslant 2 \|y - \widehat{y}\|$, from where,

$$\inf_{\widehat{\widehat{y}}\in C} f\left(x, \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\| \leqslant f(x,\widehat{y}) \|y - \widehat{y}\| \leqslant 2 \|y - \widehat{y}\|$$

so, $\forall \widehat{y} \in C$, $F(x,y) \leqslant 2 \|y - \widehat{y}\| / dis(y,C)$. However, since $\|\cdot\|$ is continuous and C is compact, it follows that $\exists \widehat{y} \in C$ such that $\|y - \widehat{y}\| = dis(y,C)$, and, therefore, that $F(x,y) \leqslant 2$. Moreover, $\forall \widehat{y} \in C$, $1 \leqslant \|y - \widehat{y}\| / dis(y,C) \leq f(x,\widehat{y}) \|y - \widehat{y}\| / dis(y,C)$, from where $F(x,y) \geqslant 1$.

Property 2: $\forall y \in Y$, $F(\cdot, y)$ is concave.

Proof of property 2: This is obvious for $y \in C$, so fix $y \in Y \backslash C$. Let $x, x' \in X$ and $\lambda \in [0,1]$. By compactness of C and continuity of f and $\|\cdot\|$, $\exists \widehat{y} \in C$ such that

$$F(\lambda x + (1-\lambda)x', y) = \frac{f(\lambda x + (1-\lambda)x', \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)}$$

Fix one such $\widehat{y} \in C$. Since $f(\cdot, \widehat{y})$ is concave, $\|y - \widehat{y}\| > 0$ and $dis(y, C) > 0$.

$$\begin{aligned} F(\lambda x + (1-\lambda)x', y) &\geqslant \lambda \frac{f(x, \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)} + (1-\lambda) \frac{f(x', \widehat{y}) \|y - \widehat{y}\|}{dis(y, C)} \\ &\geqslant \lambda \frac{\inf_{\widehat{\widehat{y}} \in C} f\left(x, \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)} + (1-\lambda) \frac{\inf_{\widehat{\widehat{y}} \in C} f\left(x', \widehat{\widehat{y}}\right) \left\|y - \widehat{\widehat{y}}\right\|}{dis(y, C)} \\ &= \lambda F(x, \widehat{y}) + (1-\lambda) F(x', \widehat{y}) \end{aligned}$$

Property 3: F is continuous.

Proof of property 3: Continuity of F at $(x, y) \in X \times C^0$ follows by construction, and follows easily from the theorem of the maximum, at $(x, y) \in X \times (Y \backslash C)$, since f is continuous, C is compact and $(x, y) \longmapsto dis(y, C)$ is continuous (Moore, 1999b, 7.53). So, it only remains to show that F is continuous at each $(x, y) \in X \times \left(C \cap \overline{(Y \backslash C)}\right)$.[2] Fix $(x, y) \in X \times \left(C \cap \overline{(Y \backslash C)}\right)$ and $\varepsilon \in (0, 1)$. This result is shown in a series of claims:

Claim 1: $(\exists r \in \mathbb{R}_{++}) (\forall (\widetilde{x}, \widetilde{y}) \in (B_r(x) \times B_r(y)) \cap (X \times C)) : |f(x, y) - f(\widetilde{x}, \widetilde{y})| < \varepsilon$

Proof of claim 1: Since $(x, y) \in X \times C$ and f is continuous, there exists $\overline{r} \in \mathbb{R}_{++}$ such that

$$(\forall (\widetilde{x}, \widetilde{y}) \in B_{\overline{r}}(x, y) \cap (X \times C)) : |f(x, y) - f(\widetilde{x}, \widetilde{y})| < \varepsilon$$

[2] It is obvious that $C^0 \cup (Y \backslash C) \cup \left(C \cap \overline{(Y \backslash C)}\right) \subseteq Y$. To see that $Y \subseteq C^0 \cup (Y \backslash C) \cup \left(C \cap \overline{(Y \backslash C)}\right)$, let $y \in Y$. Suppose that $y \notin Y \backslash C$. If $y \notin C^0$, then, $\forall \varepsilon \in \mathbb{R}_{++}$, $B_\varepsilon(y) \cap (Y \backslash C) \neq \emptyset$, which implies that $y \in \overline{(Y \backslash C)}$.

Fix one such $\overline{r}$, and define $r = \frac{\overline{r}}{2} \in \mathbb{R}_{++}$. By triangle inequality, $B_r(x) \times B_r(y) \subseteq B_{\overline{r}}(x,y)$, which proves the claim.

For the following claims, take r as given by claim 1.

Claim 2: $\left(\forall \widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)\right) : dis(\widetilde{y}, C) = dis(\widetilde{y}, B_r(y) \cap C)$

Proof of claim 2: Fix $\widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)$. For each $\widehat{y} \in C \backslash B_r(y)$,

$$\begin{aligned}
\|\widetilde{y} - \widehat{y}\| &\geqslant \|y - \widehat{y}\| - \|y - \widetilde{y}\| \\
&> \frac{3r}{4} \\
&> 2\|\widetilde{y} - y\| \\
&\geqslant 2 \inf_{\widehat{\widehat{y}} \in B_r(y) \cap C} \left\|\widetilde{y} - \widehat{\widehat{y}}\right\| \\
&= 2dis(\widetilde{y}, B_r(y) \cap C) \\
&\geqslant dis(\widetilde{y}, B_r(y) \cap C)
\end{aligned}$$

which establishes the result.

Claim 3: $(\forall \widetilde{x} \in B_r(x) \cap X) \left(\forall \widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)\right)$:

$$\inf_{\widehat{y} \in C} f(\widetilde{x}, \widehat{y}) \|\widetilde{y} - \widehat{y}\| = \inf_{\widehat{y} \in B_r(y) \cap C} f(\widetilde{x}, \widehat{y}) \|\widetilde{y} - \widehat{y}\|$$

Proof of claim 3: Fix $\widetilde{x} \in B_r(x) \cap X$ and $\widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)$. For each $\widehat{y} \in C \backslash B_r(y)$, since $f(\widetilde{x}, \widehat{y}) \geqslant 1$ and $f(\widetilde{x}, y) \leqslant 2$, it follows from the set of inequalities in the proof of claim 2 that

$$f(\widetilde{x}, \widehat{y}) \|\widetilde{y} - \widehat{y}\| > \frac{3r}{4} > f(\widetilde{x}, y) \|\widetilde{y} - y\| \geqslant \inf_{\widehat{\widehat{y}} \in B_r(y) \cap C} f\left(\widetilde{x}, \widehat{\widehat{y}}\right) \left\|\widetilde{y} - \widehat{\widehat{y}}\right\|$$

which establishes the claim.

Claim 4: $(\forall \widetilde{x} \in B_r(x) \cap X) \left(\forall \widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)\right) : F(x,y) - \varepsilon \leqslant F(\widetilde{x}, \widetilde{y}) \leqslant F(x,y) + \varepsilon$

Proof of claim 4: Fix $\widetilde{x} \in B_r(x) \cap X$ and $\widetilde{y} \in B_{r/4}(y) \cap (Y \backslash C)$. For each $\widehat{y} \in B_r(y) \cap C$, since $(\widetilde{x}, \widehat{y}) \in (B_r(x) \times B_r(y)) \cap (X \times C)$, it follows from claim 1 that $f(x,y) - \varepsilon < f(\widetilde{x}, \widehat{y}) < f(x,y) + \varepsilon$. So, since $f(x,y) - \varepsilon > 0$ and $0 < dis(\widetilde{y}, C) \leqslant \|\widetilde{y} - \widehat{y}\|$, one has that

$$\begin{aligned}
(f(x,y) - \varepsilon)\, dis(\widetilde{y}, C) &\leqslant (f(x,y) - \varepsilon) \|\widetilde{y} - \widehat{y}\| \\
&\leqslant f(\widetilde{x}, \widehat{y}) \|\widetilde{y} - \widehat{y}\| \\
&\leqslant (f(x,y) + \varepsilon) \|\widetilde{y} - \widehat{y}\|
\end{aligned}$$

and, therefore,

$$\begin{aligned}
(f(x,y) - \varepsilon)\, dis(\widetilde{y}, C) &\leqslant \inf_{\widehat{y} \in B_r(y) \cap C} f(\widetilde{x}, \widehat{y}) \|\widetilde{y} - \widehat{y}\| \\
&\leqslant (f(x,y) + \varepsilon) \inf_{\widehat{y} \in B_r(y) \cap C} \|\widetilde{y} - \widehat{y}\| \\
&= (f(x,y) + \varepsilon)\, dis(\widetilde{y}, B_r(y) \cap C)
\end{aligned}$$

Claims 3 and 2, then, imply that

$$(f(x,y) - \varepsilon)\, dis(\widetilde{y}, C) \leqslant \inf_{\widehat{y} \in C} f(\widetilde{x}, \widehat{y}) \|\widetilde{y} - \widehat{y}\| \leqslant (f(x,y) + \varepsilon)\, dis(\widetilde{y}, C)$$

and, therefore, since $dis(\widetilde{y}, C) > 0$, that $f(x,y) - \varepsilon \leqslant F(\widetilde{x}, \widetilde{y}) \leqslant f(x,y) + \varepsilon$, which proves the claim.

Hence, to establish continuity at (x,y), define $\delta = \frac{r}{4} \in \mathbb{R}_{++}$. It follows from claims 1 and 4 that $\forall (\widetilde{x}, \widetilde{y}) \in B_\delta(x,y) \cap (X \times Y)$, $|F(x,y) - F(\widetilde{x}, \widetilde{y})| \leqslant \varepsilon$, which proves the property.

Now, define $U = l^{-1} \circ F : X \times Y \longrightarrow \mathbb{R}$, which is well defined given property 1. Given property 3, since l^{-1} is continuous, so is U. Also, if $(x, y) \in X \times C$, by construction, $U(x, y) = l^{-1}(F(x, y)) = l^{-1}(f(x, y)) = l^{-1}(l(W(x, y))) = W(x, y)$. Moreover, since l^{-1} is increasing and concave, property 2 implies that for each $y \in Y$, $U(\cdot, y)$ is concave. Finally, since

$$l^{-1} : [1, 2] \longrightarrow \left[\inf_{(x,y) \in X \times C} W(x, y), \sup_{(x,y) \in X \times C} W(x, y) \right]$$

it is obvious that $\inf_{(x,y) \in X \times Y} U(x, y) \geqslant \inf_{(x,y) \in X \times C} W(x, y)$, whereas, by definition, $\inf_{(x,y) \in X \times Y} U(x, y) \leqslant \inf_{(x,y) \in X \times C} W(x, y)$. A similar reasoning establishes the result for the supremum. ■

Although compactness of C was used in the proof of properties 1 and 2, this is by no means necessary. Property 1 can be established assuming only closedness as in Bridges (1998, 3.2.13), whereas concavity of $F(\cdot, y)$ for $y \in Y \backslash C$ could be argued as follows: fix $y \in Y \backslash C$, and consider the family

$$\mathcal{F}_y = \{ g : X \longrightarrow \mathbb{R} | (\exists \widehat{y} \in C) : g(\cdot) = f(\cdot, \widehat{y}) \|y - \widehat{y}\| \}$$

of concave and bounded-below functions; it follows from Moore (1999a, 5.72) that $\widehat{g}_y : X \longrightarrow \mathbb{R}; x \longmapsto \inf_{g \in \mathcal{F}_y} g(x)$ is concave, which implies, since $dis(y, C) > 0$, that $F(\cdot, y)$ is concave.

Compactness of C, however, allows the following:

Corollary 1 *Let $X \subseteq \mathbb{R}^J$ and $Y \subseteq \mathbb{R}^K$, where $J, K \in \mathbb{N}$, be nonempty. Suppose that X is convex and $C \subseteq Y$ is compact. Suppose that $W : X \times C \longrightarrow \mathbb{R}$ is continuous and bounded, and that for each $y \in C$, $W(\cdot, y)$ is strongly concave (resp. Lipschitzian with constant M; resp. monotone; resp. strictly monotone). Then, there exists $U : X \times Y \longrightarrow \mathbb{R}$, continuous, such that*

1. *For each $(x, y) \in X \times C$, $U(x, y) = W(x, y)$*

2. *For each* $y \in Y$, $U(\cdot, y)$ *is strongly concave (resp. Lipschitzian with constant* M_y*; resp. monotone; resp. strictly monotone.)*

3. $\inf_{(x,y)\in X\times Y} U(x,y) = \inf_{(x,y)\in X\times C} W(x,y)$ *and* $\sup_{(x,y)\in X\times Y} U(x,y) = \sup_{(x,y)\in X\times C} W(x,y)$.

Proof. If X is a singleton, the result follows trivially from theorem 1. Else, recall all the definitions given in the proof theorem 1.

For strong concavity, it suffices to show that for each $y \in C$, $f(\cdot, y)$ is strongly concave, that for each $y \in Y \backslash C$, $F(\cdot, y)$ is strongly concave and that l^{-1} is strictly increasing. By strong concavity, $\sup_{(x,y)\in X\times C} W(x,y) > \inf_{(x,y)\in X\times C} W(x,y)$, from where both l and l^{-1} are strictly increasing. Fix $y \in C$, let $x, x' \in X$, $x \neq x'$ and let $\lambda \in (0,1)$; by strong concavity of $W(\cdot, y)$ and concavity and strict monotonicity of l,

$$\begin{aligned} l\left(W\left(\lambda x + (1-\lambda)x', y\right)\right) &> l\left(\lambda W(x,y) + (1-\lambda) W(x',y)\right) \\ &= \lambda l\left(W(x,y)\right) + (1-\lambda)\, l\left(W(x',y)\right) \end{aligned}$$

Now, fix $y \in Y \backslash C$, let $x, x' \in X$, $x \neq x'$ and let $\lambda \in (0,1)$; by compactness of C and continuity of f and $\|\cdot\|$, $\exists \widehat{y} \in C$ such that

$$F\left(\lambda x + (1-\lambda)x', y\right) = \frac{f\left(\lambda x + (1-\lambda)x', \widehat{y}\right)\|y - \widehat{y}\|}{dis(y, C)}$$

Fix one such $\widehat{y} \in C$. Since $f(\cdot, \widehat{y})$ is strongly concave, $\|y - \widehat{y}\| > 0$ and $dis(y, C) > 0$.

$$\begin{aligned} F\left(\lambda x + (1-\lambda)x', y\right) &> \lambda \frac{f(x, \widehat{y})\|y - \widehat{y}\|}{dis(y,C)} + (1-\lambda)\frac{f(x', \widehat{y})\|y - \widehat{y}\|}{dis(y,C)} \\ &\geqslant \lambda \frac{\inf_{\widehat{\widehat{y}}\in C} f\left(x, \widehat{\widehat{y}}\right)\left\|y - \widehat{\widehat{y}}\right\|}{dis(y,C)} + (1-\lambda)\frac{\inf_{\widehat{\widehat{y}}\in C} f\left(x', \widehat{\widehat{y}}\right)\left\|y - \widehat{\widehat{y}}\right\|}{dis(y,C)} \\ &= \lambda F(x, \widehat{y}) + (1-\lambda) F(x', \widehat{y}) \end{aligned}$$

I now show that if for each $y \in C$, $W(\cdot, y)$ is Lipschitzian with constant M (independent of y), then for each $y \in Y$, $U(\cdot, y)$ is Lipschitzian with some constant M_y. If W is constant, the result is trivial. Hence, I assume that the affine bijection l has slope $a > 0$. It follows by construction that for each $y \in C$, $U(\cdot, y)$ is Lipschitzian with constant $M_y = M$. Since for each $y \in C$, $W(\cdot, y)$ is Lipschitzian with constant M, one has that $f(\cdot, y)$ is Lipschitzian with constant aM. Fix $y \in Y \backslash C$ and $x, x' \in X$. By definition of F, compactness of C and continuity of f and $\|\cdot\|$, as before, there exist $\widehat{y}, \widehat{\widehat{y}} \in C$ such that

$$\begin{aligned} F(x,y) &= \frac{f(x,\widehat{y}) \|y-\widehat{y}\|}{dis(y,C)} \\ F(x',y) &= \frac{f\left(x',\widehat{\widehat{y}}\right) \left\|y-\widehat{\widehat{y}}\right\|}{dis(y,C)} \end{aligned}$$

Fix such $\widehat{y}, \widehat{\widehat{y}} \in C$. By definition, $f(x,\widehat{y}) \|y-\widehat{y}\| \leqslant f\left(x,\widehat{\widehat{y}}\right) \left\|y-\widehat{\widehat{y}}\right\|$ and $f\left(x',\widehat{\widehat{y}}\right) \left\|y-\widehat{\widehat{y}}\right\| \leqslant f(x',\widehat{y}) \|y-\widehat{y}\|$, whereas, since both $f(\cdot,\widehat{y})$ and $f\left(\cdot,\widehat{\widehat{y}}\right)$ are Lipschitzian with constant aM, $|f(x,\widehat{y}) - f(x',\widehat{y})| \leqslant aM \|x-x'\|$ and $\left|f\left(x,\widehat{\widehat{y}}\right) - f\left(x',\widehat{\widehat{y}}\right)\right| \leqslant aM \|x-x'\|$, so, therefore,

$$\begin{aligned} \left|\frac{f(x,\widehat{y}) \|y-\widehat{y}\|}{dis(y,C)} - \frac{f(x',\widehat{y}) \|y-\widehat{y}\|}{dis(y,C)}\right| &\leqslant aM\frac{\|x-x'\| \|y-\widehat{y}\|}{dis(y,C)} \\ \left|\frac{f\left(x,\widehat{\widehat{y}}\right) \left\|y-\widehat{\widehat{y}}\right\|}{dis(y,C)} - \frac{f\left(x',\widehat{\widehat{y}}\right) \left\|y-\widehat{\widehat{y}}\right\|}{dis(y,C)}\right| &\leqslant aM\frac{\|x-x'\| \left\|y-\widehat{\widehat{y}}\right\|}{dis(y,C)} \end{aligned}$$

Define $M_y = \frac{M}{dis(y,C)} \max_{y' \in C} \|y-y'\|$, which exists and satisfies $M_y > 0$, because C is compact and $\|\cdot\|$ is continuous. Clearly,

$$\begin{aligned}\left|\frac{f(x,\widehat{y})\|y-\widehat{y}\|}{dis(y,C)}-\frac{f(x',\widehat{y})\|y-\widehat{y}\|}{dis(y,C)}\right| &\leqslant aM_y\|x-x'\| \\ \left|\frac{f\left(x,\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|}{dis(y,C)}-\frac{f\left(x',\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|}{dis(y,C)}\right| &\leqslant aM_y\|x-x'\|\end{aligned}$$

Now, if $f(x,\widehat{y})\|y-\widehat{y}\|\leqslant f\left(x',\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|$, it follows that $F(x,y)\leqslant F(x',y)\leqslant\frac{f\left(x',\widehat{\widehat{y}}\right)\|y-\widehat{\widehat{y}}\|}{dis(y,C)}$, from where $|F(x,y)-F(x',y)|\leqslant aM_y\|x-x'\|$. If, on the other hand, $f\left(x',\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|<f(x,\widehat{y})\|y-\widehat{y}\|$, then $F(x',y)<F(x,y)\leqslant\frac{f\left(x,\widehat{y}\right)\|y-\widehat{y}\|}{dis(y,C)}$, so, again, $|F(x,y)-F(x',y)|\leqslant aM_y\|x-x'\|$. Hence, it follows that $F(\cdot,y)$ is Lipschitzian with constant aM_y, and, therefore, that $U(\cdot,y)=\left(l^{-1}\circ F\right)(\cdot,y)$ is Lipschitzian with constant M_y.

Finally, I show that if for each $y\in C$, $W(\cdot,y)$ is monotone (resp. strictly monotone), then for each $y\in Y$, $U(\cdot,y)$ is monotone (resp. strictly monotone). If there do not exist $x,x'\in X$ such that $x\gg x'$ (resp. $x>x'$) the result is trivial. Else, fix $x,x'\in X$, $x\gg x'$ (resp. $x>x'$) and $y\in Y$. Since for each $\widehat{y}\in C$, $W(x,\widehat{y})>W(x',\widehat{y})$, it follows that both l and $l^{-1}$are strictly increasing and, hence, that for each $\widehat{y}\in C$, $f(\cdot,\widehat{y})$ is monotone (resp. strictly monotone). Then, if $y\in C$, the result is trivial and I now assume that $y\in Y\backslash C$. By compactness of C and continuity of f, there exists $\widehat{y}\in C$ such that $F(x,y)=f(x,\widehat{y})\|y-\widehat{y}\|/dis(y,C)$. Fix one such $\widehat{y}\in C$. Since $f(\cdot,\widehat{y})$ is monotone (resp. strongly monotone), $\|y-\widehat{y}\|>0$ and $dis(y,C)>0$, one has that

$$F(x,y)=\frac{f(x,\widehat{y})\|y-\widehat{y}\|}{dis(y,C)}>\frac{f(x',\widehat{y})\|y-\widehat{y}\|}{dis(y,C)}\geqslant\inf_{\widehat{y}\in C}\frac{f\left(x',\widehat{\widehat{y}}\right)\left\|y-\widehat{\widehat{y}}\right\|}{dis(y,C)}=F(x',y)$$

showing that $F(\cdot,y)$ is monotone (resp. strictly monotone). Since l^{-1} is strictly increasing, it follows that $U(\cdot,y)$ is monotone (resp. strictly monotone). ■

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