Bundling without Price Discrimination*

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Abstract

This paper examines the optimal bundling strategies of a multiproduct monopoly in markets where a seller cannot monitor and, hence, restrict the purchases of buyers to a single bundle while buyers have resale opportunities. In such markets, the standard mechanism through which bundling increases seller profits, based on price discrimination, is not feasible. The profit-maximizing bundling strategy is characterized, given the restrictions on pricing policies that result from resale and a lack of monitoring. Welfare implications of optimal bundling are analyzed.

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1 Introduction

Bundling is prevalent in many economic settings. By packaging goods sellers can extend the domain of their products offered from goods to bundles. Another instance of a bundling problem is selection of product variety by a multiproduct monopoly, with a bundle interpreted as a product with multiple continuous characteristics or attributes. Further, the problem of choosing a portfolio of risky assets to offer—faced by central banks, Treasury Departments and other institutional investors—is one of optimal bundling. Here, an asset is mathematically represented as a bundle of "goods", each of which gives a monetary payment in a particular state of the world about which buyers of assets are uncertain.

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In the literature, the central motivation for bundling is that it allows sellers to price discriminate buyers (Adams and Yellen (1976); McAfee, McMillan and Whinston (1989); Wilson (1993); Armstrong (1996); Rochet and Choné (1998))[1]. The literature recognizes that the applicability of non-linear pricing relies on two key assumptions: (1) that a seller can monitor the purchases of individual buyers to enforce contracts in which each buyer obtains *precisely one bundle*; and (2) that resale markets are limited or absent. While these assumptions describe many markets well,[2] in other environments, including markets for financial assets or certain durable goods, competition from resale markets is pervasive, even if the original producer is a monopoly. Moreover, monitoring purchases is often prohibitively costly and, even given access to information on individual purchases, sellers may not be able (as a practical or legal matter) to prevent buyers from purchasing additional bundles or reselling the bundles they purchase. Consequently, in many markets, bundling based on discriminating buyers via non-linear pricing is either impractical or unfeasible. This paper uncovers and characterizes a novel mechanism that can make bundling profitable, even in markets in which the assumptions behind non-linear pricing—namely monitoring and no resale—are not satisfied. Thus, as a tool to increase profits, bundling need not rely on price discrimination.

We study the problem of bundling for a multiproduct monopoly without monitoring purchases and with resale. We examine markets for bundles of perfectly divisible goods with a continuum of buyers who can choose arbitrary combinations of offered bundles. Throughout the analysis, we adopt an abstract interpretation of a bundle as a collection of consumption goods from which buyers derive utility. This encompasses the problem of sales by a multiproduct monopoly, selection of variety of products with different characteristics, and other economic applications in which agents derive utilities from "goods", while trading in markets for bundles thereof.

[1] The seminal paper by Adams and Yellen (1976) demonstrates in a series of two-good examples that, even with additive utilities, pure bundling (selling goods in one bundle) and mixed bundling (selling a bundle as well as individual goods) can each be profit maximizing, depending on the distribution of buyer preferences. The early literature on bundling largely focused on markets with additive preferences and unit demands, in which buyers purchase either one unit of a good or a bundle of two. More recently, significant advances in the study of non-linear pricing were made by Wilson (1993), Armstrong (1996) and Rochet and Choné (1998) for markets of divisible goods. Wilson (1993) derives first-order conditions for optimal pricing and solves a few multidimensional examples. Armstrong (1996) and Rochet and Choné (1998) develop methods to solve a class of mechanism design problems with multidimensional types, which are then applied to characterize the optimal non-linear pricing for bundles. The latter two papers derive the profit-maximizing tariff in several examples. For the question of bundling, the central economic insight from Armstrong's paper is that a multiproduct monopoly should always exclude consumers with low willingness to pay. Rochet and Choné (1998) demonstrate that bunching, which occurs if groups of agents with different types are treated the same in the solution, is a robust phenomenon with non-linear tariffs.

[2] For instance, cable TV packages or cellular phone plans, which are often viewed in the literature as bundles of continuous characteristics.

First, we identify restrictions on the available pricing policies that arise in the absence of monitoring. As pointed out by McAfee, McMillan and Whinston (1989) in a model with unit demands, a multiproduct monopoly's inability to monitor sales restricts its available pricing strategies to sub-additive ones.[3] In turn, the presence of resale further imposes that pricing strategies be super-additive, thereby ruling out price discounts. As a result, this impedes the seller from charging different prices for different quantities of the product—that is, payments must be linear in quantity in each bundle market. In fact, if buyers can trade arbitrary (positive or negative) quantities of bundles, equilibrium will feature linear bundle pricing in the following stronger sense. *For any* bundling strategy employed by the monopoly, one can find implicit prices of goods, from which the seller can determine prices of bundles. Moreover, the implicit prices of goods depend on the bundling strategy only up to the span of goods trades that it generates. By choosing bundling strategies, a seller *can affect* such implicit prices of goods and, hence, his revenue. Bundling profitability results from a mechanism that is different from price discrimination via non-linear pricing. Rather, bundling allows sellers to distort allocation of goods among buyers, which under natural assumptions on preferences, induces high willingness to pay and hence implicit prices. The problem of a multiproduct monopoly studied in this paper is not one of screening and techniques different from those used to characterize optimal non-linear pricing must be employed.[4]

We characterize the optimal choice of a bundling strategy and quantity of goods for sale given a cost function for the class of (buyer) quasilinear utilities with arbitrary complementarities and substitutabilities in goods preferences. We derive conditions under which no bundling is, *for sure*, revenue superior/inferior to *any* degree of bundling in a market with many buyers; these conditions are based on the (shape) of the buyers' marginal utility. For a two-good monopoly, we completely characterize the optimal bundling strategy: when buyer marginal utility is concave, no bundling (or any bundling strategy that spans the space of all goods trades) is optimal, whereas when marginal utility is convex, pure bundling maximizes revenue. The optimality of no bundling for concave marginal utility extends to an arbitrary number of goods. Although no bundling remains suboptimal with convex marginal utility,

[3] Consider a simple example with two goods and unit demands, in which the monopoly sells two goods separately as well as a bundle of the two goods. If a seller can ensure that each buyer receives either one of the two goods or (only) a bundle thereof, then he can charge more for a bundle of goods than the sum of the prices of the individual goods. Such pricing is clearly impossible without monitoring; otherwise, each buyer could recreate the bundle at a lower cost by purchasing goods separately. The lack of monitoring has even stronger implications for optimal selling of divisible goods, as buyers have more opportunities to "create" bundles by purchasing combinations of other bundles.

[4] Analyzing the multiproduct monopoly problem without monitoring and with resale requires a different approach from the standard mechanism design in which a designer offers a menu of contracts and each type self-selects by choosing exactly one contract. Here, each buyer can (and will, in equilibrium) select a combination of contracts from the menu (of bundles) and the seller does not control the allocation.

the seller revenue is not necessary monotone in span and pure bundling need not be optimal whenever no bundling is revenue-inferior. In particular, the optimal bundling strategy may involve *partial bundling* (a bundling strategy with span that is not full) that is richer than pure bundling. The form of optimal partial bundling depends on relative convexity of the marginal utility across goods, supply of each good offered by the seller and heterogeneity in buyer endowments.

We further examine how bundling without monitoring sales affects the social appraisal of a monopoly. The welfare properties of bundling studied in this paper are distinct from those of price discrimination in that partial bundling necessarily lowers the efficiency of allocation. Conditional on the quantity produced, deadweight loss is monotone in the (effective) variety of products offered, measured by bundle span: unbundling improves efficiency. However, when the monopoly can choose both the optimal production level and the bundling strategy, the overall welfare effect is ambiguous. While bundling distorts the allocation for a given quantity, it may increase or decrease the profit-maximizing level of production (relative to the no bundling scenario), depending on the seller's marginal cost.

A key economic insight is that while arbitrage undermines the profitability of non-linear pricing, it does not undermine the profitability of bundling. The bundling we study derives from a simple mechanism that applies to possibly non-separable utility functions and arbitrary interdependence in endowment distributions. In addition, unlike non-linear pricing, in the environments studied in this paper, the bundling profitability as such (and for two-good environments, also the optimality of the particular bundling strategy) holds in the strong, *ex-post* sense. Thus, whenever bundling is profitable, sellers can robustly increase profits through bundling with little knowledge of the market. The mechanism that gives rise to bundling profitability is unique to markets with two or more commodities—this is in contrast to price discrimination that is based on non-linear pricing, which extends from single-goods markets. Unlike non-linear pricing, the class of bundling strategies can be viewed as an instance of—and suggest an explanation for—endogenous incompleteness of bundle markets in which agents derive utilities from goods.

The paper is organized as follows. Section 2 sets out the model of a multiproduct monopoly and establishes the restrictions on the available pricing policies that arise in the absence of monitoring. Section 3 characterizes optimal bundling. Section 4 discusses the welfare effects of bundling. Section 5 provides conditions that are helpful in determining the optimal bundling strategy in a model with more general preferences.

2 Multiproduct Monopoly with Fixed Inventory

We build a model of a multiproduct monopoly that is similar to Armstrong (1996). To offer a transparent presentation of our results, we first focus on the problem of a seller who sells an exogenously given inventory of $Q = (Q_1, \ldots, Q_n)$ of n divisible goods, all in strictly positive supply. Then, we allow the monopoly to simultaneously choose the optimal inventory Q in addition to a bundling strategy, given a cost function.

2.1 Market

The seller can choose from a wide variety of alternative selling strategies. One possibility is to offer each of n goods in a separate market. An alternative is to open only one market, where agents can buy shares of bundle Q. It is clear that these two extremes to which we refer as *no bundling* and *pure bundling*, can lead to different revenue for the monopoly. More possibilities exist: in general, the monopoly can package n goods into a collection of k bundles $B = (b_1, \ldots, b_k)$. If the monopoly chooses a *bundling strategy* B, k markets open, one for each bundle and total revenue is given by the sum of revenues from each bundle market. Since with bundles B, total revenue need not coincide with that generated by pure or no bundling, a question arises as to which bundling strategy B maximizes the monopoly's profits.

Buyers derive utility from consuming n goods and money: if buyer i consumes $x = (x_1, \ldots, x_n)$ of goods and the money payment to the monopoly is t, then the consumer's utility is $u(x) - t$. We assume that goods can be consumed only in non-negative amounts, so the *utility function* $u(\cdot)$ is defined on the non-negative orthant of $\mathbb{R}^n$ only. We make the following standard assumptions on $u(\cdot)$.

(**A1**) $u(\cdot)$ is continuous, concave and monotone;
(**A2**) in the strictly positive orthant, $u(\cdot)$ is twice continuously differentiable, differentiability strictly monotone, and differentiable strictly concave; and
(**A3**) for any sequence $(x_m)_{m=1}^{\infty}$ of vectors with strictly positive quantities of all goods that converges to some vector x in the boundary of $\mathbb{R}^n_+$, we have that $\lim_{m\to\infty} || \, Du(x_m)|| = \infty$.

Let us note that we do not impose any restriction on the Hessian of the utility function, except negative definiteness. In particular, non-separable utilities with substitute and complement goods are allowed. Utility functions are assumed to be the same across buyers, but buyers have heterogeneous preferences for traded bundles: there are m types of non-atomic buyers indexed by $\{1, \ldots, m\}$; each type with a mass normalized to one, and initial holdings of type i are given by $q^i = (q^i_1, \ldots, q^i_n)$. Heterogeneity results from different realizations of initial endowments of goods across agents. The seller does not know the initial holdings of the buyers, $(q^1, \ldots, q^m)$, and holds probabilistic beliefs over this profile, represented by a distribution function F, defined over $\mathbb{R}_+^{nm}$. We assume that F is absolutely continuous with respect to the Lebesgue measure and known to the seller. No other restrictions are placed on F. In particular, the marginal distributions are not required to be the same across buyers and the joint distribution can feature arbitrary interdependence of endowments, as long as correlations are not perfect, which is the case given absolute continuity.

The market operates as a standard uniform-price (Walrasian) auction: for any bundling strategy of the seller, all traders simultaneously submit demands for all bundles for each profile of prices, and bundles are allocated at a market clearing price, at which aggregate demand equals the seller's inventory. We analyze a Nash equilibrium in which non-atomic buyers bid competitively, and hence, their demands coincide with the marginal utilities from bundles. A large Walrasian auction studied in this paper is equivalent to the limit of the VCG mechanism for perfectly divisible bundles as the number of agents increases and payments coincide with the externality imposed on other buyers. Without asymmetric information or, more generally, with non-stochastic (aggregate) demand, our model becomes a standard model of a monopoly with posted prices.

We study the problem of a multiproduct monopoly that chooses bundling strategy B to maximize profits. The problem considered in this paper differs from that of Armstrong (1996) and Rochet and Choné (1998) in the following two respects. There, buyers are offered a continuum menu of bundles in $\mathbb{R}_+^n$ and each buyer chooses one, the most preferred bundle. In the present paper, the monopoly cannot prevent buyers from trading arbitrary combinations of offered bundles and the menu of offered bundles is finite, which in Section 2.3, we argue to be without loss of generality.

2.2 Revenue Function

This section characterizes the monopoly's objective function and, in particular, shows that it is well defined for any bundling strategy. The result sheds light on the economic nature of the problem of optimal bundling, which is characterized in the next section. Bundling strategy B, described by $n \times k$ matrix $B = (b_1, \ldots, b_k)$ enables the monopoly to sell its entire inventory of goods Q if, and only if, Q is in the (column) span of B, a linear subspace of $\mathbb{R}^n$ defined as

$$\langle B \rangle \equiv \{x \in \mathbb{R}^n | Bz = x \text{ for some } z \in \mathbb{R}^k\}. \tag{1}$$

Span $\langle B \rangle$ gives the set of all goods transfers that can result from trades of bundles B. The collection of all bundling strategies, for $k = 1, 2, ...$, satisfying $Q \in \langle B \rangle$, is denoted by $\mathcal{B}_Q$. We say that a bundling strategy B is *partial* if the rank of B is strictly less than n (and, hence, $\langle B \rangle \subset \mathbb{R}^n$). For any $B \in \mathcal{B}_Q$ and the corresponding k, a vector $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_k)$ exists such that

$$\bar{z}_1 b_1 + \ldots + \bar{z}_k b_k = Q. \tag{2}$$

$\bar{z}$ gives the seller's *effective inventory* of bundles B through which Q can be sold. Importantly, correspondence $\bar{z}(B)$ that assigns effective inventory, and therefore satisfies $B\bar{z}(B) = Q$ for any $B \in \mathcal{B}_Q$, is non-empty. In the case of no bundling, B is an $n \times n$ identity matrix, $\mathbb{I}_{n \times n}$, and the effective inventory coincides with Q. With pure bundling, a bundling strategy is given by one vector, Q, and the effective inventory of this bundle is $\bar{z} = 1$. In both examples, inventory $\bar{z}$ is uniquely defined. More generally, bundles need not be linearly independent and effective supply $\bar{z}(B)$ is not necessarily a singleton (a unique vector).

A large Walrasian auction gives each type i incentive to submit a competitive demand that reveals i's marginal utility of each bundle and is given by

$$z^i(p; B) \equiv \arg\max_{z \in \mathbb{R}^k} u(q^i + Bz) - p \cdot z. \tag{3}$$

The monopoly faces aggregate demand $z(p; B) \equiv \sum_i z^i(p; B)$. In a Walrasian auction, effective inventory $\bar{z} \in \bar{z}(B)$ is sold at prices p such that markets for bundles clear, $\bar{z} \in z(p; B)$. Let $\bar{p}(\bar{z})$ give all vectors of prices for which markets clear given supply $\bar{z}$. For any $B \in \mathcal{B}_Q$, revenue correspondence is defined as

$$R(B) \equiv \{ p \cdot \bar{z} \,|\, \bar{z} \in \bar{z}(B) \text{ and } p \in \bar{p}(\bar{z}) \}. \tag{4}$$

Next, we show that, despite effective inventory being a correspondence, the revenue of the seller is uniquely defined for each $B \in \mathcal{B}_Q$.

Proposition 1. *(*Revenue*) Correspondence defined in* (4) *is a function* $R : \mathcal{B}_Q \to \mathbb{R}$.

The proof of Proposition 1 offers a characterization of equilibrium allocation of commodities and prices of bundles that will be useful in subsequent analysis and, therefore, is presented in the text. Specifically, we show that for any given choice of B, the allocation of goods resulting from a Walrasian auction of bundles, which is determined through market clearing and optimization by each buyer, is the same as the allocation chosen by a benevolent social planner whose objective is to maximize the sum of buyers' utilities, but whose choices of goods transfers across buyers are restricted to those in the span $\langle B \rangle$. In certain respects, characterizing the solution to a planner optimization problem is simpler than characterizing equilibrium in a Walrasian auction. The former also reveals a number of equilibrium properties in the latter: equilibrium allocation of goods is constrained efficient and independent from how the monopoly distributes inventory Q among bundles, $\bar{z} \in \bar{z}(B)$; prices of bundles, which correspond to the shadow prices in the planner program,[5] are unique and independent from the monopoly's effective inventory $\bar{z} \in \bar{z}(B)$.

Proof: We prove Proposition 1 via two lemmas. Lemma 1 characterizes the equilibrium allocation of goods in a Walrasian auction as constrained efficient allocation in a planner's problem. Lemma 2 demonstrates the existence and uniqueness of prices, and hence revenue, for any bundling strategy $B \in \mathcal{B}_Q$. Define a set of allocations of goods that are feasible given B,

$$X(B) \equiv \left\{ (x^1, \ldots, x^m) \in \mathbb{R}_+^{nm} \,\middle|\, \sum_i x^i = \sum_i q^i + Q \text{ and } (x^i - q^i) \in \langle B \rangle \text{ for all } i \right\}. \tag{5}$$

Lemma 1. (Allocation Equivalence) *Fix $B \in \mathcal{B}_Q$ and corresponding effective inventory $\bar{z}$. Bundle allocation $(\tilde{z}^1, \ldots, \tilde{z}^m)$ such that $\sum_i \tilde{z}^i = \bar{z}$ is an equilibrium allocation of bundles in a Walrasian auction if, and only if, the implied allocation of goods $(\tilde{x}^1, \ldots, \tilde{x}^m)$, where $\tilde{x}^i = q^i + B\tilde{z}^i$, solves the planner problem,*

$$\max_{(x^1,\ldots,x^m)} \sum_i u(x^i) : (x^1, \ldots, x^m) \in X(B). \tag{6}$$

Proof: (Only if) Let $\tilde{p}$ and $(\tilde{z}^1, \ldots, \tilde{z}^m)$ be equilibrium prices and an equilibrium bundle allocation in a Walrasian auction, respectively. Since $\tilde{x}^i - q^i = B\tilde{z}^i$, transfers of goods are in the span $\langle B \rangle$ and $\sum_i \tilde{x}^i = \sum_i q^i + \sum_i B\tilde{z}^i = \sum_i q^i + Q$. It follows that $(\tilde{x}^1, \ldots, \tilde{x}^m) \in X(B)$. In addition, by optimality of each buyer's choice,

$$u(q^i + Bz^i) - \tilde{p} \cdot z^i \leq u(q^i + B\tilde{z}^i) - \tilde{p} \cdot \tilde{z}^i, \tag{7}$$

for all $z^i \in \mathbb{R}^k$. Summing (7) over all i gives

$$\sum_i u(q^i + Bz^i) \leq \sum_i u(\tilde{x}^i), \tag{8}$$

for all $(z^1, \ldots, z^m)$ such that $\sum_i Bz^i = Q$, where we used that, in a Walrasian equilibrium, $\tilde{p} = B^\mathsf{T} Du(\tilde{x}^1)$ and, hence,

$$\tilde{p} \cdot \sum_i z^i = \left(B^\mathsf{T} Du(\tilde{x}^1)\right) \cdot \sum_i z^i = Du(\tilde{x}^1)^\mathsf{T} \underbrace{\sum_i Bz^i}_{Q} = \tilde{p} \cdot \sum_i \tilde{z}^i. \tag{9}$$

[5] In an alternative interpretation of the allocation mechanism, shadow prices correspond to the externality imposed in the VCG mechanism by each buyer on other buyers.

Thus, $(\tilde{x}^1, \ldots, \tilde{x}^m)$ solves the planner problem.

(If) Since set $X(B)$ can be alternatively written as

$$X(B) = \left\{ \left(q^1 + Bz^1, ..., q^m + Bz^m\right) \in \mathbb{R}^{nm}_+ \,\middle|\, \left(z^1, ..., z^m\right) \in \mathbb{R}^{km} \text{ and } \sum_i z^i = \bar{z} \right\}, \quad (10)$$

problem (6) is equivalent to

$$\max_{(z^1,...,z^m)} \sum_i u(q^i + Bz^i) : \sum_i z^i = \bar{z}, \quad (11)$$

and, by assumption, $\tilde{z}^i$ is a solution to it. By interiority (A3) and using that Q is strictly positive in all components and lies in $\langle B \rangle$, $\tilde{x}^i$ is strictly positive in all components for all individuals. By the Kühn-Tucker Theorem, then, multipliers $\tilde{p}$ must exist such that $B^\mathsf{T} Du(e^i + B\tilde{z}^i) = \tilde{p}$ for every i. Since the utility function $u(\cdot)$ is strictly concave in the interior of the domain, we have that $\tilde{z}^i = z^i(\tilde{p}; B)$ and, by assumption, bundle markets clear. Therefore $(\tilde{z}^1, \ldots, \tilde{z}^m)$ and $\tilde{p}$ constitute an equilibrium in a Walrasian auction. *Q.E.D.*

We turn to characterization of equilibrium prices and revenue.

Lemma 2. (Existence and Uniqueness of Prices) *For any bundling strategy $B \in \mathcal{B}_Q$,*
(i) for any effective inventory $\bar{z} \in \bar{z}(B)$, there exists p such that $\bar{z} \in \sum_i z^i(p; B)$; and
(ii) for any $\bar{z}, \bar{z}' \in \bar{z}(B)$ and p and p' such that $\bar{z} \in \sum_i z^i(p; B)$ and $\bar{z}' \in \sum_i z^i(p'; B)$, $p = p'$ holds; and
(iii) revenue $R(B)$ is a singleton.

Proof: For claim (i), observe first that, for any B, set $X(B)$ defined in (5) is non-empty convex and compact, so planner program (6) has a unique solution, as utility function $u(\cdot)$ is continuous and strictly concave. Moreover, this solution is interior by A3. Since the solution satisfies $(\tilde{x}^i - q^i) \in \langle B \rangle$, vector $(\tilde{z}^1, \ldots, \tilde{z}^m)$ exists such that that $\tilde{x}^i - q^i = B\tilde{z}^i$ and $\sum_i \tilde{z}^i = \bar{z}$. By Lemma 1, $(\tilde{z}^1, \ldots, \tilde{z}^m)$ is an equilibrium allocation of bundles in a Walrasian auction, where equilibrium prices are given by shadow prices from program (11).

Claim (ii) follows from Lemma 1, and the fact that set $X(B)$ in (6) does not depend on effective inventory, hence the (unique) allocation of commodities $(\tilde{x}^1, \ldots, \tilde{x}^m)$ is the same for $\bar{z}$ and $\bar{z}'$. By interiority A3, in equilibrium of a Walrasian auction, $p = B^\mathsf{T} Du(x^1) = p'$; that is, bundle prices are the same.

Claim (iii) follows from Lemma 2 and claims (i) and (ii). From claim (i), for any $\bar{z} \in \bar{z}(B)$, a unique vector of equilibrium prices exists that clears the market. Therefore, $R(B)$ is nonempty. By claim (ii), for any $\bar{z}, \bar{z}' \in \bar{z}(B)$, the allocation of goods $(\tilde{x}^1, \ldots, \tilde{x}^m)$ is the same and we have that

$$R = p \cdot \bar{z} = B^{\mathsf{T}} Du(\tilde{x}^1) \cdot \bar{z} = Du(x^1)^{\mathsf{T}} B\bar{z} = Du(x^1)^{\mathsf{T}} B\bar{z}' = p' \cdot \bar{z}', \tag{12}$$

since $B\bar{z} = B\bar{z}' = Q$. It follows that $R(B)$ exists and is unique. *Q.E.D.*

Hereafter, we use the function $R(B)$ to represent the seller's revenue over bundling strategies $B \in \mathcal{B}_Q$ and function $p(B) \equiv \bar{p}(\bar{z}(B))$ to represent prices for bundles without any qualification.[6] Lemma 2 asserts that the price of each bundle B can be determined uniquely and—by constrained efficiency of allocation of goods—in a manner that is independent of effective inventory $\bar{z}(B)$. The next section offers a (constructive) characterization of equilibrium prices in terms of market primitives.

2.3 Linear Pricing and Equivalent Bundling

A central lesson from the literature on optimal bundling for a two-good monopoly selling to buyers with unit demands (Adams and Yellen (1976), McAffee at all (1989)) is that non-linear pricing may allow the monopoly to better discriminate among buyers and increase revenue. Such price discrimination is not feasible when buyers can trade arbitrary combinations of bundles, as is demonstrated next.

In the absence of monitoring and with resale, the equivalent characterization of the allocation of goods in the seller and planner problems (Lemma 1) implies that the equilibrium allocation of goods depends on bundle strategy B solely through its span: Let two bundling strategies $B, B' \in \mathcal{B}_Q$ span the same space of goods trades, i.e., $\langle B \rangle = \langle B' \rangle$. Since in the planner problem (6), a set of feasible allocations of goods given B, $X(B)$, depends on the bundling strategy only through the span of B (see (5)), the allocations of goods that solve the planner problem coincide for B and B'. By Lemma 1, the equilibrium holdings of goods and money in a Walrasian auction are the same under the two bundling strategies. More generally, let L be a linear subspace of $\mathbb{R}^n$, and with a slight abuse of notation, let $x(L)$ denote an allocation that results from *any* bundling strategy B with span $\langle B \rangle = L$. Let $\mathcal{L}_Q$ be a collection of all linear subspaces of $\mathbb{R}^n_+$ that contain Q, and let function $\kappa : \mathcal{L}_Q \to \mathbb{R}^n_+$ be defined as the average marginal revenue at the equilibrium allocation of goods,

$$\kappa(L) \equiv \frac{1}{m} \sum_i Du(x^i(L)). \tag{13}$$

[6] The dimensionality of price vector p depends on the rank of B. Thus, formally, $\bar{p}(B)$ is as a collection of functions $\{\bar{p}_k(B)\}_{k=1,2,\ldots}$, where $\bar{p}_k : \mathcal{B}_Q^k \to R^k$ maps a set of matrices that have rank k, $\mathcal{B}_Q^k \subset \mathcal{B}_Q$, into $\mathbb{R}^k$.

Lemma 3 demonstrates that $\kappa(\cdot)$ defines *implicit* prices of goods which, importantly, since characterized in terms of primitive marginal utility, allow us to price bundles and inventory Q.

Lemma 3. *(*Linear Pricing*) For any bundling strategy B, the equilibrium prices of bundles p are given by*

$$p^T = \kappa(\langle B \rangle) \cdot B \tag{14}$$

and revenue of the multiproduct monopoly is equal to

$$R(B) = \kappa(\langle B \rangle) \cdot Q. \tag{15}$$

Proof: By Lemma 1, $x(L)$ is the unique allocation of goods in a Walrasian auction for any bundling strategy B satisfying $\langle B \rangle = L$. Since, in the equilibrium of a Walrasian auction

$$p(B) = B^{\mathsf{T}} Du(x^i(\langle B \rangle)), \tag{16}$$

for all B and i, taking the average of (16) across all buyers gives (14). In addition, for each i,

$$R(B) = p(B) \cdot \bar{z}(B) = Du(x^i(\langle B \rangle))^{\mathsf{T}} B \bar{z}(B) = Du(x^i(\langle B \rangle)) \cdot Q, \tag{17}$$

where revenue $p(B) \cdot \bar{z}(B) \equiv p(B) \cdot \bar{z}$ for some $\bar{z} \in \bar{z}(B)$ is unambiguously defined by Proposition 1. Again, taking the average of (17) across buyers gives the result. *Q.E.D.*

Lemma 3 does not require that the utility function be separable and, hence, holds for markets with complement or substitute goods. Let us emphasize that the lemma holds beyond the environment described in Section 2.1. In particular, none of the arguments in the proof require symmetry of the utility function across buyers. Thus, the results apply directly to environments with type-dependent utilities $u^i(\cdot)$ (from consumption and not just trade) that satisfy A1-A3.

Two implications for the monopoly's bundling problem are immediate from Lemma 3. First, because the implicit goods prices depend on bundling strategy B only through span $\langle B \rangle$, any bundling strategies with the same span bring the same revenue given the realization of initial holdings. Thus, the lemma defines equivalence classes for bundles with respect to revenue and choosing optimal bundling can be recast as choosing the span. In addition, equilibrium must involve linear bundle pricing. It is not surprising that linear pricing obtains in our setting given that competitive buyers can trade arbitrary quantities of bundles. The usefulness of Lemma 3 lies in characterizing implicit goods prices as the average marginal utility:[7] although with partial bundling, buyer marginal utilities $Du\left(x^i\left(L\right)\right)$ are (typically) not equalized among buyers and do not coincide with implicit goods prices (13), implicit prices equal the *average* marginal utility. The intuition behind this result is as follows: all buyers trade at the same bundle prices and, hence, in optimum orthogonal projections of marginal utilities $Du\left(x^i\left(L\right)\right)$ onto a bundle span $\langle B\rangle$ are the same for all buyers. Any differences in $Du\left(x^i\left(L\right)\right)$ across i must then be orthogonal to $\langle B\rangle$, but since all bundles, as well as inventory Q, are contained in $\langle B\rangle$, such differences cannot be traded away and, thus, do not have any effect on bundle prices.[8]

The feature that bundling strategies with the same span bring the same revenue and the linear pricing prediction from the present paper differ markedly from the results obtained in the literature on bundling in markets with unit demands. In a two-good model, Adams and Yellen (1976) and McAffee at all (1986) demonstrate that non-linear pricing is optimal, and no bundling ($B = \mathbb{I}_{2\times 2}$) and mixed bundling ($B' = [\mathbb{I}_{2\times 2}|\mathbf{1}]$, where $\mathbf{1}$ is a unit vector) differ in revenue terms, even with additive buyer preferences. Since $\langle B\rangle = \langle B'\rangle = \mathbb{R}^2$, strategies B and B' are revenue equivalent in our model, even in markets with complementarities or substitutabilities among goods. That mixed bundling does not bring about strict improvement in revenue in the markets studied in this paper—an implication of Lemma 3—holds for any number of goods. The qualitative difference in the bundling strategies and prices thereof between models with monitoring and no resale (both unit demands and divisible goods, such as Armstrong (1996) and Rochet and Choné (1998)) and ours stems from restrictions imposed on trades. Since there, by assumption, buyer decisions are restricted to the choice as to which bundle to buy, buyers are not able to arbitrage price differentials across bundle markets, which makes non-linear pricing consistent with equilibrium. The present paper analyzes markets in which buyers can trade arbitrary combinations of bundles, and price discrimination through mixed bundling with non-linear pricing is not feasible. Linear pricing is a constraint on the pricing policy of the monopoly that results from the monopoly's inability to enforce one-bundle contracts. Still, in the next section, we shall demonstrate that bundling can increase revenue, even with linear pricing, through a different mechanism than price discrimination.

Let us note another implication of equivalent bundling. Unlike Armstrong (1996), where it is strictly optimal for the monopolist to offer a continuum menu of bundles, in our model, for an arbitrary collection of bundles, a finite sub-collection of $k \leq n$ linearly independent bundles exists that the monopoly can offer without loss in revenue. Thus, a small number

[7] No-arbitrage only implies existence—not the structure—of implicit pricing.

[8] In fact, any vector from the set $\{\kappa\left(L\right)\} + L^{\perp}$ satisfies properties (14) and (15). In particular, each vector whose average defines $\kappa\left(L\right)$ does so. The characterization of $\kappa\left(L\right)$ as an average is useful, as it allows us to determine the optimal bundling strategy by the monopoly.

of bundles are sufficient for the monopoly to earn maximum revenue. In particular, with $n = 2$, the monopoly has effectively two bundling strategies, namely: no bundling and pure bundling.

3 Optimal Bundling

3.1 Existence of an Optimal Bundling Strategy B^*

We show that within the menu of all bundling strategies in $\mathcal{B}_Q$, a bundling strategy exists that maximizes the monopoly's revenue. Existence is not straightforward because of two features of the bundling problem. First, the space of bundling strategies is not compact. Second, when the dimension of the bundling span changes, there are discontinuities in the individual demands of buyers and, hence, in the seller's revenue.

Proposition 2. *(*Existence of an Optimal Bundling Strategy*) A bundling strategy $B^* \in \mathcal{B}_Q$ exists, such that $E\left[R\left(B^*\right)\right] \geq E\left[R\left(B\right)\right]$ for all $B \in \mathcal{B}_Q$.*

Proof of this proposition is based on the following result about linear spaces. By Lemma 3, we can concentrate on the set of linear subspaces of $\mathbb{R}^n$ rather than the set of bundling strategies themselves, which allows us to deal with the problem of the non-compactness of $\mathcal{B}_Q$. For any $k \leq n$, the set of k-dimensional linear subspaces of $\mathbb{R}^n$ is an $k(n-k)$-dimensional compact manifold. This set, known as the *Grassmanian* of k-dimensional linear subspaces of $\mathbb{R}^n$, retains all the topological structure of a compact subset of $\mathbb{R}^{k(n-k)}$ locally, at any point.

Proof: For any $1 \leq k \leq n$, the space of k-dimensional spaces of trades in goods that contain Q is equivalent to (i.e., it is a rotation of; see Section 5) the set of $(k-1)$-dimensional linear subspaces of $\mathbb{R}^{k-1}$. This Grassmanian is a compact manifold (of dimension $(k-1)(n-k)$). By the Theorem of the Maximum, the allocation function, $x(L)$, is continuous on it. By Lemma 3, the expected revenue function, $E\left(\kappa\left(L\right)\right) \cdot Q$, is continuous on the Grassmanian too. It follows that a linear space exists that maximizes the seller's revenue on this manifold. Denote by R_k^* the maximized revenue over the Grassmanian. Since n is finite, the seller's program reduces to finding the maximum of $\{R_1^*, \ldots, R_n^*\}$. *Q.E.D.*

Incidentally, the same argument implies that a bundling strategy exists that minimizes expected revenue.

3.2 Optimal Bundling with Fixed Inventory

This section characterizes the optimal bundling strategy for a monopoly that sells inventory Q of n goods. Lemma 4 asserts that which among the forms of bundling that the monopoly can employ dominates in terms of revenue depends on the *shape* of the common *marginal* utility function, $Du(\cdot)$. Specifically, any degree of bundling is revenue-superior, revenue-inferior or revenue-equivalent to no bundling, depending on whether the marginal utility functions are convex, concave or linear on the relevant part of the domain. More formally, let $\mathcal{X}$ be a convex set that contains all the equilibrium allocations of goods for initial holdings (given by the image of $x(L)$) on the support of F. For the purpose of the lemma, we say that $Du(\cdot)$ is strictly convex (concave) if each component function $\partial u/\partial x_h$ is strictly convex (concave) in x_h and weakly in $x_{h'}$, $h' \neq h$.

Lemma 4. *(*Optimal Bundling Strategy*) The following describes optimal bundling strategies:*
(i) If $Du(\cdot)$ is strictly convex over $\mathcal{X}$, then any partial bundling strategy strictly dominates no bundling in terms of revenue F-almost surely (and not lower surely).
(ii) If $Du(\cdot)$ is strictly concave over $\mathcal{X}$, then any bundling strategy with full span strictly dominates any partial bundling in terms of revenue F-almost surely (and not lower surely).
(iii) If $Du(\cdot)$ is linear over $\mathcal{X}$, then all bundling strategies are revenue-equivalent.

Since a two-good monopoly chooses effectively between no bundling and pure bundling, Lemma 4 completely characterizes the optimal bundling strategy in this case, which is worth highlighting as a corollary.

Corollary 1. *(*Optimal Bundling for a Two-Good Monopoly*) Consider a problem of a two-good monopoly. If $Du(\cdot)$ is strictly convex (concave) on $\mathcal{X}$, the optimal bundling strategy involves pure bundling (no bundling).*

Before presenting the proof of Lemma 4, we provide a simple example that explains the key economic intuition.[9] Note that the result holds in the strong, *ex-post* sense. For transparency of the arguments, in all examples presented in the paper we can, thus, consider deterministic initial holdings that are not Pareto efficient.

Example 1. *Suppose that a two-good monopoly sells inventory $Q = (1,1)^\mathsf{T}$ to two buyers whose utility function is given by*

$$u(x_1, x_2) = \ln(x_1) + \ln(x_2), \tag{18}$$

and whose initial holdings of goods are given by $q^1 = (1,0)^\mathsf{T}$ and $q^2 = (0,1)^\mathsf{T}$. The two-good monopoly has effectively two bundling choices: no bundling and pure bundling. If the

[9] The mechanism behind the result in Lemma 4 is similar to that in Weil (1990) who, in a financial economy, studies the impact of market incompleteness on asset prices.

monopoly does not bundle, given identical quasilinear utilities, the equilibrium allocation of goods is Pareto efficient, $x^1 = x^2 = (1, 1)$*; the marginal utility for each good and buyer, given by* $1/x^i_h$*, is the same and equal to* 1 *at the equilibrium allocation; and the seller's revenue is* 2*. Under pure bundling, each buyer obtains half of the bundle* Q *and the equilibrium allocation of goods is given by* $x^1 = \left(\frac{3}{2}, \frac{1}{2}\right)$ *and* $x^2 = \left(\frac{1}{2}, \frac{3}{2}\right)$*. The average marginal utility for each good is* $\frac{1}{2}\left(\frac{2}{3} + 2\right) = 1\frac{1}{3}$ *and revenue equals* $2\frac{2}{3}$*. Hence, pure bundling dominates no bundling in terms of revenue obtained. It is straightforward to show that when marginal utility is linear, then the two bundle strategies produce the same revenue and when the it is concave, then no bundling maximizes revenue.*

In the example, absent bundling, each buyer purchases only the good for which his initial endowment is zero, so that the marginal utilities of buyers coincide for each good in the Pareto efficient allocation. Under pure bundling, for a buyer to obtain the desired good, he must purchase the bundle that contains (the same quantity of) the other good as well. Thus, by introducing a wedge in consumption bundling creates a wedge in marginal utility between the two buyers in each market. With a convex $Du(\cdot)$, the wedge increases the willingness to pay of the buyer with lower equilibrium consumption above the Pareto efficient level by more than it reduces the willingness to pay of the buyer who consumes more. Therefore, bundling induces higher equilibrium average marginal utility from goods relative to no bundling (Pareto efficient level). Since the bundle contains a good for which the average willingness to pay remains high after trade, the equilibrium value of a bundle is also high.

Notice that in Example 1, revenue increases only if equilibrium allocation is inefficient. In general, even with Pareto inefficient endowments (which occurs $F-$almost surely given that F is absolutely continuous with respect to the Lebesgue measure) and partial bundling, final allocation may still be Pareto efficient. However, for any given partial bundling strategy, the realizations of endowments that give efficient outcomes are non-generic—the equilibrium allocation is $F-$almost surely Pareto inefficient.

Proof: If the seller chooses a bundling strategy for which the column span is the complete space of goods trades, $\mathbb{R}^n$, then the resulting allocation of goods is efficient and each individual consumes $\frac{1}{n}(Q + \sum_i q^i)$, and the resulting revenue for the seller equals

$$R(\mathbb{I}_{n\times n}) = Du\left(\frac{1}{n}(Q + \sum_i q^i)\right) \cdot Q. \tag{19}$$

Now, consider a strategy B whose span is not $\mathbb{R}^n$. It is immediate that $\langle B\rangle$ is of a dimension lower than n, so the set of endowment profiles, $(q^1, \ldots, q^m)$, for which

$$\frac{1}{n}(Q + \sum_i q^i) - q^1 \in \langle B \rangle \tag{20}$$

has zero Lebesgue measure (as a subset of $\mathbb{R}^n$). Since F is absolutely continuous with respect to the Lebesgue measure, it follows that, for at least two types of individuals, $x^i(\langle B \rangle) \neq x^j(\langle B \rangle)$ almost surely.

For claim (i), notice that since function $Du(\cdot)$ is strictly convex and

$$\frac{1}{m}\sum_i x^i(\langle B \rangle) = \frac{1}{n}(Q + \sum_i q^i), \tag{21}$$

one has that

$$\kappa(\langle B \rangle) = \frac{1}{m}\sum_i Du(x^i(\langle B \rangle)) > Du\left(\frac{1}{n}(Q + \sum_i q^i)\right) = \kappa(\mathbb{R}^n). \tag{22}$$

Then, it follows from Lemma 3 that $R(B) = \kappa(\langle B \rangle) \cdot Q > \kappa(\mathbb{R}^n) \cdot Q = R(\mathbb{I}_{n\times n})$, F-almost surely. In the (zero F-probability) set where all consumers equate consumption, the two revenue levels are equal. The arguments for claims (ii) and (iii) can be mimicked. *Q.E.D.*

More generally, beyond two-good markets, Lemma 4 asserts that with n goods no bundling is almost always dominated by (dominates) any partial bundling whenever marginal utility is convex (concave). We are led to ask: Is the seller revenue monotonically increasing in reducing product variety, as measured by a bundle span? That is, for bundling strategies B and B' such that $\langle B \rangle \subseteq \langle B' \rangle$, does revenue satisfy $R(B) \geq R(B')$? In markets with $n > 2$, pure bundling need not be the revenue maximizing partial bundling strategy, even with strictly convex marginal utility. This is shown in Example 2.

Example 2. *Consider a problem of a three-good monopoly selling inventory $Q = (1,1,1)^\mathsf{T}$ of goods to two types of buyers whose utility function is given by*

$$u(x) = -\frac{1}{2}(2 - x_1)^2 - \frac{1}{2}(2 - x_2)^2 + \ln(x_3). \tag{23}$$

The initial holdings of goods are $q^1 = (1,0,0)^\mathsf{T}$ and $q^2 = (0,1,0)^\mathsf{T}$. In the symmetric market, the equilibrium allocation of goods under pure bundling is given by $x^1 = (1.5, 0.5, 0.5)^\mathsf{T}$ and $x^2 = (0.5, 1.5, 0.5)^\mathsf{T}$, the implicit prices of goods are $(1,1,2)$ and the seller's revenue is 4. Now, consider the following partial bundling strategy (which is not necessarily revenue-

maximizing):

$$B = (b_1, b_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{24}$$

With bundling strategy (24)*, a market for good 2 (bundle b_1) opens and buyers equalize marginal utility in this market. Bundle b_2 is more attractive to buyer 2 as it contains the first and third goods, and in equilibrium, buyers obtain $z^1 = (1, 0.4)$ and $z^2 = (0, 0.6)$. The implied allocation of goods is $x^1 = (1.4, 1, 0.4)^\intercal$ and $x^2 = (0.6, 1, 0.6)^\intercal$, the implicit prices of goods are* $(1, 1, 2.08)$ *and the seller's revenue is* 4.08*. Since the revenue under B is strictly greater than that under pure bundling, the quadratic components in function* (23) *can be perturbed so that the marginal utilities of the first two components are strictly convex while B still yields strictly higher revenue than pure bundling. Observe that the goods allocations are inefficient under both selling strategies and deadweight losses are* 0.5 *and* 0.2 *under pure bundling and B, respectively. Thus, the partial bundling strategy B dominates pure bundling, not only in terms of revenue, but also efficiency.*

In Example 2, pure bundling introduces a wedge in the final consumption of the first two goods, whereas the allocation of the third good is Pareto efficient. Given that the utilities from the first two goods are quadratic, distortion brings no increase in revenue relative to no bundling— the average marginal utility remains intact. In contrast, while the two-bundle strategy (24) improves the efficiency of the allocation of the first two goods, it introduces a wedge in the allocation of the third good. Given the strict convexity of the marginal utility of the third good, the wedge increases the implicit price of the third good and the seller's revenue.

As a more general insight, the optimal selling strategy should distort consumption, relative to the Pareto efficient allocation, (1) in the direction for which the convexity of marginal utility—and, hence, the potential increase in average willingness to pay—is greatest; for goods (2) for which supply is the largest; and (3) the probability of the greatest heterogeneity in initial endowments is the greatest, *ceteris paribus.*

Example 2 demonstrates that even for convex marginal utility, revenue need not be monotone in a bundle span. The next example shows that for CARA utilities, in models that preserve symmetries in utilities and inventories across markets, seller revenue is monotone in bundle span and optimal selling involves pure bundling. This allows highlighting the role that heterogeneity plays in breaking the co-monotonicity of the seller revenue and span.

Example 3. *Consider a multiproduct monopoly selling inventory $Q = \lambda(1, ..., 1)^\intercal$ for some*

$\lambda > 0$ to buyers with CARA utilities,

$$u(x) = \sum_{h=1}^{n} -e^{-\alpha x_h} \tag{25}$$

and F is absolutely continuous. Then, the expected revenue is decreasing in a bundle span. In particular, pure bundling is revenue-maximizing. CARA utility provides a useful benchmark, as in this case, the objective of the seller is the (scaled) negative of that of the social planner (6)*, and thus restricting the choice of the planner by reducing span cannot reduce revenue.*

Let us conclude the characterization of optimal bundling with some final remarks about Lemma 4. First, note that the result assumes that buyers have common utilities *over holdings of goods*; since buyers may differ in their initial goods holdings, they may have different effective utilities over bundles (or trades of these goods, or trades of bundles), trade differently, and may end up with different final holdings of goods. Secondly, since the inequalities are strict with F-probability 1 in claims (i) and (ii) of the lemma, it follows that the result is robust to sufficiently small asymmetries in buyer utility functions.[10] The assumptions of the three claims are to hold over some convex subset of their domains, which is large enough to include all the relevant equilibrium allocations of goods; we introduce this qualification, for otherwise, the class of preferences under consideration may be vacuous.[11] If distribution F has bounded support, we can always find a bounded set of outcomes $\mathcal{X}$ to qualify the assumptions on the shape of marginal utilities. Finally, notice that the first two claims hold true if all marginal utilities weakly satisfy the assumptions of convexity or concavity, and one of them does so strictly over the set $\mathcal{X}$. However, we cannot draw any general conclusions for the case when marginal utility functions for some goods are strictly convex while for others strictly concave, nor for the case when there is no clear second-order behavior over the relevant set of final goods allocations. Section 5 provides a computation method of finding the optimal bundle that applies to these more general markets as well.

3.3 Optimal Bundling with Endogenous Inventory

This section addresses the main question of the paper. We consider the bundling problem of a multiproduct monopoly that optimally chooses inventory Q along with bundling strategy B. We endow the monopoly with cost function $C(Q)$. The monopoly's objective is to

[10] This is clearly the case for a given bundling strategy and, by the compactness argument used in the proof of Proposition 2, extends to all partial bundling strategies.

[11] While a global assumption would not be problematic for the claim (i), given assumptions (A1)-(A3), a strictly concave marginal utility function does not exist wherein the marginal utilities are always strictly positive and concave or linear.

maximize profit

$$\pi(Q,B) \equiv R(Q,B) - C(Q), \tag{26}$$

where $R(Q,B)$ is the revenue obtained from selling inventory Q using bundling strategy B. When Q is chosen from a compact set, applying arguments similar to those in Proposition 1, profit function π exists and is uniquely defined. With monotone costs $C(Q)$, the choice of production-bundling plan (Q^*, B^*) can be restricted to $Q^* \in \langle B^* \rangle$ without loss of generality.[12] We assume that $C(Q)$ and buyer preferences are such that an optimal plan exists (Q^*, B^*), and Q^* is unique.[13]

As with a fixed inventory, when Q^* is optimally chosen, bundling strategies with the same span are revenue equivalent. Theorem 1 characterizes the optimal production-bundling choice.

Theorem 1. *(*Optimal Production-Bundling Plan*) Let (Q^*, B^*) be an optimal bundling strategy. If $Du(\cdot)$ is strictly concave, any $B^* \in \mathcal{B}_{Q^*}$ that has full span is optimal; if $Du(\cdot)$ is linear, any $B^* \in \mathcal{B}_{Q^*}$ is optimal; if $Du(\cdot)$ is strictly convex, B^* is a partial bundling strategy. For a two-good monopoly, the optimal bundling strategy is $B^* = (\lambda Q^*)$, for any $\lambda \neq 0$.*

Proof: Assume that the marginal utility $Du(\cdot)$ is strictly concave. Then, a plan (Q^*, B^*) such that B^* does not have full column rank cannot be optimal as, by Lemma 4, any other plan (Q^*, B) with $\langle B \rangle = \mathbb{R}^n$ gives a strictly higher revenue F-almost surely. Given fixed Q^*, any bundling strategy with full span results in the same Pareto efficient allocation and hence implicit prices and revenue of a monopoly as B^* and hence is optimal. If marginal utility is linear, by Lemma 4, an optimal (Q^*, B^*) gives the same profit as (Q^*, B), for any $B \in \mathcal{B}_{Q^*}$. With a strictly convex $Du(\cdot)$, by Lemma 4, production-bundling plan (Q^*, B) with any partial B gives strictly higher expected revenue with the same cost and, thus, (Q^*, B^*) is not optimal. For a two-good monopoly, the partial bundling that allows selling Q^* is given by $\lambda^* Q^*$ for some $\lambda^* \neq 0$. Given fixed Q^*, the allocation in the planner problem is constrained efficient and hence the same for any $\lambda \neq 0$. Thus, implicit prices of goods and revenue do not depend on the choice of $\lambda \neq 0$. *Q.E.D.*

As the next section argues, the joint choice of production levels and bundling strategies has important implications for welfare.

[12] Alternatively, one can assume that Q^* is arbitrary and the monopoly sells only the part of Q^* that is in the span of B (and disposes freely of the remaining part). If cost is monotone in all goods, then such a strategy is suboptimal.

[13] Recall that even in the case of a one-good monopoly, the optimal quantity in the production-bundling plan may not exist with certain convexity of marginal utility.

4 Regulation of a Multiproduct Monopoly

A central concern in policy and market design is that market power distorts quantity relative to the Pareto efficient outcome. As this paper shows, apart from inefficiently reducing production levels, a multiproduct monopoly can exercise market power by bundling goods, which distorts the allocation of Q among market participants. To restore efficiency, both forms of market power should be regulated. A natural question arises as to how the additional degrees of freedom in the seller's strategy affect welfare. To better understand the welfare effect of bundling itself, relative to the effect of quantity reduction, we begin by analyzing a monopoly with a fixed inventory Q.

Our fixed-inventory model has the following implications for the appraisal of monopolistic bundling practices and regulation of multiproduct companies. In order to improve Pareto efficiency of market outcomes, the optimal regulation of a monopoly should require that it offer a full-span portfolio of bundles. This recommendation can be strengthened: introduction of an additional bundle is never detrimental to buyer welfare, even if it does not fully complete a bundling strategy. Let $DWL(B)$ denote a deadweight loss resulting from bundling strategy B. Since all monetary transfers sum to zero, by Lemma 1, for any pair $B, B' \in \mathcal{B}_Q$, such that $\langle B \rangle \subseteq \langle B' \rangle$,

$$DWL(B) - DWL(B') = \max_{(x^1,\ldots,x^m)\in X(B')} \sum_i u(x^i) - \max_{(x^1,\ldots,x^m)\in X(B)} \sum_i u(x^i). \tag{27}$$

Since $X(B) \subseteq X(B')$, it follows that a deadweight loss is (weakly) decreasing in the span of a bundling strategy, $DWL(B) \geq DWL(B')$.

Optimal bundling necessarily distorts efficiency in markets where buyer marginal utility is convex: revenue maximization requires partial bundling, which ($F-$almost surely) introduces a wedge in buyer marginal utility and consumption in equilibrium. Indeed, the very mechanism through which bundling provides an effective means to increasing seller revenue, in this case, is by introducing inefficiency in the allocation of Q. Nevertheless, even for fixed supply, while the exercise of market power through bundling introduces a revenue-efficiency trade-off, the monopoly revenue is not necessarily monotone in deadweight loss and the seller benefit need not be associated with buyer loss (Example 2).[14]

As we now argue, the apparent recommendation to increase the variety of bundles should be considered with caution when a multiproduct monopoly chooses optimal inventory Q. In markets where the buyer marginal utility is concave, a competition-enhancing policy is

[14] With linear marginal utilities, revenue is invariant to bundling—any $B \in \mathcal{B}_Q$ is revenue maximizing; but among all such bundling strategies, only those that have a full span yield the efficient allocation.

essentially the same as in the case of a standard single-good monopoly.[15] Does the availability of two instruments amplify the overall negative welfare effect? Example 4 demonstrates that the joint welfare effect of the two instruments through which monopoly may increase revenue is not unambiguous—the two forms of market power can offset or reinforce each other, depending on marginal cost and, hence, the optimal production scale.

Example 4. *Consider a two-good monopoly, choosing inventory $Q = (Q_1, Q_2)$ to maximize profits given constant marginal cost $C(Q) = (Q_1 + Q_2) \times c$. The monopoly sells goods to two buyers whose utility function is*

$$u(x_1, x_2) = \ln(x_1) + \ln(x_2), \tag{28}$$

and deterministic initial holdings of goods are $q^1 = (2, 1)^\top$ and $q^2 = (1, 2)^\top$. Since marginal utility (28) *is strictly convex, by Corollary 1, the optimal bundling strategy involves pure bundling. The profit function of a monopoly is strictly concave, symmetric around the 45°-line and attains the maximum at $Q_1^* = Q_2^* = Q_h$ for any value of production. This allows focus on the determination of quantity in the optimal production-bundling plan on either good market. It can be shown that, for any level of inventory $Q = (Q_h, Q_h) \in \mathbb{R}^2_+$, pure bundling yields higher total revenue than no bundling. However, there is a quantity threshold in each market h such that the marginal revenue under pure bundling exceeds that of no bundling for quantity Q_h below the threshold, whereas the reverse is true for quantities above the threshold. Thus, the marginal benefit from bundling—created by distorting the equilibrium consumption of two traders—decreases more sharply compared to no bundling. Intuitively, given convex buyer marginal utility, differences in consumption translate into a gap in marginal utility, but the gap weakens as consumption grows (with production) to eventually vanish as $\lim_{Q_h \to \infty} \left\| Du\left(\frac{1}{2}Q + q^1\right) - Du\left(\frac{1}{2}Q + q^2\right) \right\| = 0$. The optimal quantity from bundling or no bundling in each market is determined by equality of the respective marginal revenue and marginal cost c. In fact, for a range of cost parameters that are high enough, the possibility of bundling increases production and brings it closer to efficiency. Thus, with a large marginal cost, the direct adverse effects of bundling on welfare are partially offset by a reduced distortion in the scale of production. In turn, for small marginal costs, not only does bundling adversely affect welfare by preventing the efficient allocation of a given quantity, but it also aggravates the inefficient reduction of production beyond that implied by the monopoly's standard response to its price impact.*

[15] Clearly, the optimal production level for a multi-product monopoly in one market may depend on the optimal production plan in other markets, unless buyer utility and seller cost functions are separable. In this case, the equilibrium production level coincides with the level chosen by the monopoly that maximizes revenue market-by-market.

5 Computation of B^*

Since finding an optimal bundling strategy with $n > 2$ may be quite challenging, in this section we show that a seller's program that finds B^* can be simplified using characterization of the bundling choices by linear spaces: any $k-$dimensional linear subspace containing Q can be spanned by a basis that comprises Q and $k-1$ linearly independent n-vectors orthogonal to Q. Technically, this means that we can restrict attention to bundling strategies

$$B = (Q, b_2, \ldots, b_k), \tag{29}$$

where: (i) we have that $Q \cdot b_j = 0$ for every $j = 2, \ldots, k$; and (ii) the $n \times (k-1)$ matrix $(b_2, \ldots, b_k)$ has rank $k-1$. For any such bundling strategy, effective inventory is given by $\bar{z}(B) = (1, 0, \ldots, 0)^\mathsf{T}$. This observation simplifies computation of the revenue derivatives with respect to perturbations in the bundling strategy. This means that the problem of choosing an optimal bundling strategy is equivalent to a problem wherein the monopoly sells shares of its inventory of goods, Q, while also creating markets for trades of other bundles that are in zero net supply. In this setting, the seller's revenue equals the price that results in the market for shares of his inventory.

We now study the effects of perturbations on the bundling strategy on revenue, by computing the *derivative* of the latter with respect to the former. For this purpose, we consider only perturbations to the bundling strategy that do not affect the dimension of the space of goods trades. For simplicity of notation, we will write the h-th canonical vector in $\mathbb{R}^k$ by $\mathbf{e}_h$, so that $\bar{z}(B) = \mathbf{e}_1$. Define function ϕ, for bundling strategies of the form (29), as

$$\phi(z^1, \ldots, z^m, p, B) \equiv \begin{pmatrix} \vdots \\ B^\mathsf{T} Du(q^i + Bz^i) - p \\ \vdots \\ \mathbf{e}_1 - \sum_i z^i \end{pmatrix}, \tag{30}$$

defined over bundle trades such that all commodity holdings are strictly positive (that is, $q^i + Bz^i \gg 0$ for all i). This function is useful in that its roots capture the market-clearing prices and individual bundle demands.[16] The function is continuously differentiable and its

[16] That is, $\phi(z^1, \ldots, z^m, p, B) = 0$ at, and only at, $(z^1(p(B); B), \ldots, z^I(p(B); B), p(B), B)$.

partial Jacobean, $D_{z^1,\ldots,z^m,p}\phi(z^1,\ldots,z^m,p,B)$, is, by direct computation,

$$\begin{pmatrix} B^\top D^2u^1B & 0 & \ldots & 0 & -\mathbb{I}_{k\times k} \\ 0 & B^\top D^2u^2B & \ldots & 0 & -\mathbb{I}_{k\times k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & B^\top D^2u^mB & -\mathbb{I}_{k\times k} \\ -\mathbb{I}_{k\times k} & -\mathbb{I}_{k\times k} & \ldots & -\mathbb{I}_{k\times k} & 0 \end{pmatrix}, \tag{31}$$

where the arguments in the Hessian matrices have been omitted for simplicity of presentation, so that D^2u^i denotes the matrix $D^2u(q^i + Bz^i)$.

Importantly, since the bundling strategy B has full column rank, this partial Jacobean is invertible, [17] so it follows that for a perturbation $\mathrm{d}b_h$, orthogonal to Q, we have that

$$\begin{pmatrix} \vdots \\ \mathrm{d}z^i \\ \vdots \\ \mathrm{d}p \end{pmatrix} = -(D_{z^1,\ldots,z^m,p}\phi)^{-1}(D_{b_h}\phi)db_h. \tag{32}$$

By direct computation, matrix $(D_{z^1,\ldots,z^m,p}\phi)^{-1}$ in (32) is

$$\begin{pmatrix} \Gamma & \Theta \\ \Theta^\top & \Delta \end{pmatrix}, \tag{33}$$

where $\Delta = -(\sum_i (B^\top D^2u^iB)^{-1})^{-1}$, matrix Γ is

$$\begin{pmatrix} (B^\top D^2u^1B)^{-1}(\mathbb{I} + \Delta(B^\top D^2u^1B)^{-1}) & \ldots & (B^\top D^2u^1B)^{-1}\Delta(B^\top D^2u^mB)^{-1} \\ \vdots & \ddots & \vdots \\ (B^\top D^2u^mB)^{-1}\Delta(B^\top D^2u^1B)^{-1} & \ldots & (B^\top D^2u^mB)^{-1}(\mathbb{I} + \Delta(B^\top D^2u^mB)^{-1}) \end{pmatrix}, \tag{34}$$

[17] To see that this is indeed the case, notice that each matrix $B^\top D^2u^iB$ is negative definite, since u is strictly concave and B has full column rank. This implies that the sub-matrix obtained by deleting the last super-row and the last super-column, being diagonal, is itself negative definite. Then, to verify that the whole matrix is nonsingular, it suffices to observe that

$$D_{z^1,\ldots,z^m,p}\phi(z^1,\ldots,z^I,p,B) \begin{pmatrix} -(B^\top D^2u^1B)^{-1}\sum_i B^\top D^2u^iB \\ \vdots \\ -(B^\top D^2u^mB)^{-1}\sum_i B^\top D^2u^iB \\ -\sum_i B^\top D^2u^iB \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbb{I}_{k\times k} \end{pmatrix}.$$

and matrix Θ is

$$\begin{pmatrix} (B^\top D^2 u^1 B)^{-1}\Delta \\ \vdots \\ (B^\top D^2 u^m B)^{-1}\Delta \end{pmatrix}. \tag{35}$$

Matrix $D_{b_h}\phi$ in (32) is computed as

$$D_{b_h}\phi = \begin{pmatrix} \vdots \\ z_n^i B^\top D^2 u + \mathbf{e}_n \otimes (Du^i)^\top \\ \vdots \\ 0 \end{pmatrix}, \tag{36}$$

where the sign $\otimes$ denotes the Kronecker product of the two matrices. By substitution, it follows from equation (32) that

$$\mathrm{d}p = \frac{1}{m}\mathcal{H}(\{B^\top D^2 u^i B\}_{i=1}^m) \sum_i (B^\top D^2 u^i B)^{-1}(z_h^i B^\top D^2 u^i + \mathbf{e}_h \otimes (Du^i)^\top)\mathrm{d}b_h, \tag{37}$$

where $\mathcal{H}$ denotes the harmonic average of the array of matrices.

By construction, revenue is equivalent to the first price in the markets, so we can simply compute $\mathrm{d}R$, with respect to a perturbation $\mathrm{d}b_h$ such that $\mathrm{d}b_h \cdot Q = 0$, by looking at the first row of matrix

$$\frac{1}{m}\mathcal{H}(\{B^\top D^2 u B\}_{i=1}^m) \sum_i z_h^i (B^\top D^2 u^i B)^{-1} B^\top D^2 u. \tag{38}$$

Directions of improvement of revenue are given by perturbations to the vector b_h such that their inner product with the first row of matrix (38) is positive. A critical point of the revenue function with respect to bundling strategies that contain exactly k linearly independent bundles will occur when the first row in matrix (38) is zero in all elements, for all perturbations (orthogonal to Q) of the h-th bundle in the strategy, for each of the $k-1$ additional bundles in B. Whether a critical point is a local maximizer or minimizer, or an inflection point, is not obvious at all. To begin, it is important to notice that the standard tools of convex analysis do not apply here, as the domain of this sub-program, namely the Grassmanian of $k-1$ dimensional subspaces of $\mathbb{R}^{n-1}$ does not have the structure of a convex set. Moreover, as this manifold has an empty boundary, all of its points are "interior" and, hence, both extrema of the revenue function will satisfy the first-order condition of critical points. For instance, if there are three goods, the space of two-dimensional linear spaces that contain Q can be identified as the (one-dimensional) unit circle in the two-dimensional plane; any continuous function defined on that set will attain its maximum and its minimum, and so, if it is smooth, the function will have at least two critical points, whose character cannot be identified from first-order conditions only.

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