Testable restrictions of general equilibrium in production economies [1]

Andrés Carvajal [2]

November 9, 2005

[1] This work was carried while the author was at the Cowles Foundation, Yale University. Hospitality and financial support are gratefully acknowledged. I thank Herakles Polemarchakis, Caterina Calsamiglia, Donald Brown and Shubhash Ray for helpful conversations about this topic. The usual disclaimer applies.

[2] Department of Economics, Royal Holloway College, University of London; andres.carvajal@rhul.ac.uk

Abstract

This note shows that the testability result obtained by Donald Brown and Rosa Matzkin [Econometrica 64, 1249-1262] for exchange economies survives the introduction of standard, aggregate production, even without the observation of production levels.

Key words: testable restrictions; general equilibrium; constant returns to scale; no free lunch.

JEL classification numbers: D11, D21, D51

In a seminal contribution, [1] showed that the hypothesis of Walrasian equilibrium in an exchange economy can be refuted upon observation of finitely many pairs of prices and profiles of individual endowments (i.e. that there are nontrivial restrictions on the equilibrium manifold). The crux of the argument there was that, in the presence of nonnegativity constraints, there exists a fundamental tension between the two principles of the Walrasian model, namely individual rationality and market clearing.[1]

A natural question to ask is whether the same conclusion applies to economies in which production takes place. In this case, if data on production is available, then it readily follows that the general equilibrium hypothesis can be refuted. What is less obvious is that the same conclusion applies if only prices and endowments are observed. In this setting, conditions that are equivalent to the rationalizability of a data set seem easy to obtain (subject to the conditions imposed on the technology) but one might expect that these conditions be tautologies, and hence that refutability fail, because: (i) If profits are not observed, individual incomes are undetermined and hence the restrictions imposed by individual rationality may be weakened; and (ii) In the presence of production, nonnegativity constraints on consumption are less informative: production may transform endowments so as to allow consumption allocations *outside* the original Edgeworth boxes.

Introducing production, however, adds profit maximization as an element of the model and this additional structure may as well be a source of refutability for the hypothesis.

This note studies refutability of the Walrasian hypothesis for economies with aggregate production. Two technological assumptions are considered: constant returns to scale and *no free lunch*. Under constant returns to scale, profit maximization implies zero profits for the firm and, hence, its introduction does not leave individual outcomes undetermined, effectively assuming away the first difficulty mentioned above. The second difficulty, however, remains, and the result obtained here shows that the structure added by profit maximization makes up for weakened nonnegativity constraints. When the only assumption is that one cannot obtain positive output without using some input, the result shows that the equilibrium hypothesis is refutable upon observation of the (constant) ownership structure in the economy: profit maximization makes up for the loss structure that induces both difficulties.

In both cases, the argument extends the example provided by [1], to argue that there exist data which are inconsistent with the equilibrium hypothesis. This goes to say that the relationship existing between prices and endowments, first discovered by [1] for exchange economies, is profound enough to be maintained in seemingly less restrictive settings.

For the hypothesis of Pareto-efficiency in economies with (produced) public goods, [10] solved the testability problem, imposing conditions on the technology which are identical to the ones initially imposed here. As it turns out, [10] finds

[1] This work has been extended to economies with public goods, [10], dynamic exchange economies with incomplete markets, [6], exchange economies with random preferences, [2], and exchange economies with externalities, [3]. For a survey of this literature, see [4].

examples of data which are inconsistent with the efficiency hypothesis and arise simply from the demand side of that economy (see [4] too). Here, demand considerations do not suffice, and it is the combination of individual rationality and profit maximization that implies refutability.

1 Constant returns to scale

Fix a society $\mathcal{I}$, with cardinality $I \in \mathbb{N}$, and fix $L \in \mathbb{N}$, the number of commodities. Let $\mathcal{C}$ be the set of all non-empty, closed, negative monotonic,[2] convex cones in $\mathbb{R}^L$, and let $\mathcal{U}$ be the class of all continuous, strongly concave, strictly monotone functions from $\mathbb{R}^L_+$ into $\mathbb{R}$.

For each individual $i \in \mathcal{I}$, preferences will be denoted by $u^i \in \mathcal{U}$ and endowments by $e^i \in \mathbb{R}^L_{++}$. An aggregate technology with constant returns to scale is denoted by $C \in \mathcal{C}$. The $(L-1)$-dimensional, strictly positive simplex will be denoted by $\mathcal{S}^{L-1}_{++}$. For $(p, m) \in \mathcal{S}^{L-1}_{++} \times \mathbb{R}_{++}$, denote $B(p, m) = \left\{ x \in \mathbb{R}^L_+ \middle| p \cdot x \leq m \right\}$.

Definition 1. *For $\left(\left(u^i, e^i\right)_{i \in \mathcal{I}}, C \right) \in \left(\mathcal{U} \times \mathbb{R}^L_{++} \right)^I \times \mathcal{C}$, let $W\left(\left(u^i, e^i\right)_{i \in \mathcal{I}}, C \right)$ be the set of prices $p \in \mathcal{S}^{L-1}_{++}$ such that there exist optimal supply, $y \in \arg\max_{y' \in C} p \cdot y'$, and a profile of optimal demands,*

$$\left(x^i\right)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} \arg \max_{x \in B(p, p \cdot e^i)} u^i(x),$$

such that markets clear: $\sum_{i \in \mathcal{I}} \left(x^i - e^i\right) = y$.

It is important to notice that, in the previous definition, $p \cdot y = 0$ and, hence, profits can be ignored in each individual budget constraint.

1.1 Characterization

Suppose that one observes a finite data set of prices and profiles of endowments. The refutability problem studies whether it is always possible to find a profile of preferences and a technology for which each one of the observed prices is consistent with the corresponding profile of endowments via the equilibrium concept. For this, one treats the technology and the profile of individual preferences as the unobserved and invariant fundamentals whose existence is being tested. The profile of endowments is assumed to be an observable, exogenous variable and the prices are the endogenous, observable variable.[3] The question is, then, whether there exist conditions on the observable variables, which are implied by the existence of unobservable fundamentals and the equilibrium concept. If the conditions are not tautologies, then the theory is refutable. If they characterize existence of fundamentals, then they exhaust the implications of the model.

Let $T \in \mathbb{N}$.

[2] A set $Y \subseteq \mathbb{R}^L$ is said to be negative monotonic if $(y \in Y \text{ and } y' \leq y) \Longrightarrow y' \in Y$.
[3] Supply and individual demands are assumed to be unobservable.

Definition 2. *A sequence* $\left(\left(e^{it}\right)_{i\in\mathcal{I}}, p^t\right)_{t=1}^{T}$ *in* $\mathbb{R}^{LI}_{++}\times\mathcal{S}^{L-1}_{++}$*, is CRS-rationalizable if there exists* $\left(\left(u^i\right)_{i\in\mathcal{I}}, C\right)\in\mathcal{U}^I\times\mathcal{C}$ *such that for every* $t\in\{1,...,T\}$*,* $p^t\in W\left(\left(u^i, e^{it}\right)_{i\in\mathcal{I}}, C\right)$.

The following theorem offers a characterization of rationalizability which is mediated by existential quantifiers.

Theorem 1. *Sequence* $\left(\left(e^{it}\right)_{i\in\mathcal{I}}, p^t\right)_{t=1}^{T}$ *is CRS-rationalizable if, and only if, there exist* $(y^t)_{t=1}^{T}$ *in* $\mathbb{R}^L$*, and* $\left(\left(v^{it}, \lambda^{it}, x^{it}\right)_{i\in\mathcal{I}}\right)_{t=1}^{T}$ *in* $\mathbb{R}\times\mathbb{R}_{++}\times\mathbb{R}^L_{+}$*, that satisfy the following conditions:*

1. *For every* $t,t'\in\{1,...,T\}$*,* $p^t\cdot y^{t'}\geq p^t\cdot y^t=0$*;*
2. *For every* $i\in\mathcal{I}$ *and every* $t,t'\in\{1,...,T\}$*,* $v^{it'}\leq v^{it}+\lambda^{it}p^t\cdot\left(x^{it'}-x^{it}\right)$*, with strict inequality if* $x^{it}\neq x^{it'}$*;*
3. *For every* $i\in\mathcal{I}$ *and every* $t\in\{1,...,T\}$*,* $p^t\cdot\left(x^{it}-e^{it}\right)=0$*;*
4. *For every* $t\in\{1,...,T\}$*,* $\sum_{i\in\mathcal{I}}\left(x^{it}-e^{it}\right)=y^t$*.*

Proof. This follows straightforwardly from [11, Theorem 6] and [7, Theorem 2], by an argument identical to the one used by [1]. □

Because of its existential quantifiers, the previous characterization fails to provide a test of the Walrasian hypothesis (at least a direct one). Standard quantifier-elimination results, introduced to economics by [1], allow us to prove that there exists a quantifier-free characterization of rationalizability:

Corollary 1. *There exists a finite system, (CRS−R), of polynomial inequalities in* $\left(\mathbb{R}^{LI}_{++}\times\mathcal{S}^{L-1}_{++}\right)^T$*, such that a sequence* $\left(\left(e^{it}\right)_{i\in\mathcal{I}}, p^t\right)_{t=1}^{T}$ *is CRS-rationalizable if, and only if, it satisfies* $(CRS-R)$*.*

Proof. This follows from the Tarsky-Seidenberg theorem, since the conditions of the theorem (including domain restrictions) constitute a semialgebraic set: see [8] or [4]. □

Since the corollary does not rule out the possibility of system $(CRS-R)$ being a tautology, it is silent about whether the equilibrium hypothesis can be refuted. Put another way, the corollary only says that the set of rationalizable data sets is semialgebraic (given I, L and T), but fails to imply the existence of nonrationalizable data sets, since $\left(\mathbb{R}^{LI}_{++}\times\mathcal{S}^{L-1}_{++}\right)^T$ is semialgebraic.

1.2 Nonrationalizable data

To see that the equilibrium hypothesis can be refuted, it suffices to notice that the data

$$\left(\left(\left(e^{11}, e^{21}\right), p^1\right), \left(\left(e^{12}, e^{22}\right), p^2\right)\right)$$

of Figure 1 is not rationalizable:

$[Figure\ 1]$

To see why, notice that if individual 1 is to satisfy conditions 2 and 3 of the theorem (2 implies WARP), without loss of generality, x^{11} has to lie in the thicker part of individual 1's corresponding budget line in Figure 2. Now, given nonnegativity constraints on consumption, this implies that production y^1 must lie in the thicker part of the (zero-profit) hyperplane $\left\{ y \in \mathbb{R}^2 \middle| p^1 \cdot y = 0 \right\}$, as depicted in Figure 3,which, then, implies $p^2 \cdot y^1 > 0$ and violates condition 1 of the theorem (profit maximization).

$[Figure\ 2]$

$[Figure\ 3]$

Indeed, in this example individual rationality of consumer 1 requires that at prices p^1 commodity 1 be used to produce commodity 2 in a ratio equal to the relative price of commodity 1 ($y_1^1 < 0$ and $-y_2^1/y_1^1 = p_1^1/p_2^1$). Then, at prices p^2, where the relative price of commodity 2 has increased and given that the previous rate of transformation was technically feasible, there would exist feasible production bundles consistent with positive profits (at p^2: $p^2 \cdot y^1 = y_1^1 \left(p_1^2 - p_2^2 p_1^1/p_2^1\right) > 0$), which would be inconsistent with profit maximization and constant returns to scale.[4]

2 No free lunch

A production set $Y \subseteq \mathbb{R}^L$ satisfies no free lunch if there cannot be production without the use of some input: $y > 0$ implies $y \notin Y$. Let $\mathcal{Y}$ be the set of all non-empty, closed, convex, negative monotonic sets in $\mathbb{R}^L$ that satisfy no free lunch.

Since the technology may now allow for non-zero profits, our definition of individual must include information about profit shares, which will be denoted by $\left(\theta^i\right)_{i \in \mathcal{I}}$, and the equilibrium set (W) must now account account for the effects of profits on the budget constraints. Let $\mathcal{S}_+^{I-1}$ be the nonnegative $(I-1)$-dimensional simplex in $\mathbb{R}^I$.

[4]There is another way to read the example: If the data are going to be consistent with profit maximization, then production levels must have $y_2^1 \leq 0$ and $y_1^2 \leq 0$, which only goes to reinforce the violation of WARP by consumer 1.

Definition 3. *For* $\left(\left(u^i, e^i\right)_{i\in\mathcal{I}}, \theta, Y\right) \in \left(\mathcal{U} \times \mathbb{R}^L_{++}\right)^I \times \mathcal{S}^{I-1}_{+} \times \mathcal{Y}$, *let* $W\left(\left(u^i, e^i\right)_{i\in\mathcal{I}}, \theta, Y\right)$ *be the set of prices* $p \in \mathcal{S}^{L-1}_{++}$ *such that there exist optimal supply,* $y \in \arg\max_{y'\in Y} p \cdot y'$, *and a profile of optimal demands,*

$$\left(x^i\right)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \arg \max_{x\in B(p,p\cdot e^i+\theta^i p\cdot y)} u^i\left(x\right),$$

such that markets clear: $\sum_{i\in\mathcal{I}}\left(x^i - e^i\right) = y$.

2.1 Characterization

Again, suppose that one observes prices and profiles of endowments, and assume, furthermore, that the ownership structure θ is also known. As before, the question is whether there exist nontautological conditions on the observable variables implied by the existence of unobservable fundamentals and the equilibrium concept.

Definition 4. *A data set* $\left(\left(\left(e^{it}\right)_{i\in\mathcal{I}}, p^t\right)_{t=1}^{T}, \theta\right) \in \left(\mathbb{R}^{LI}_{++} \times \mathcal{S}^{L-1}_{++}\right)^T \times \mathcal{S}^{I-1}_{+}$, *is rationalizable if there exists* $\left(\left(u^i\right)_{i\in\mathcal{I}}, Y\right) \in \mathcal{U}^I \times \mathcal{Y}$ *such that for every* $t \in \{1, ..., T\}$, $p^t \in W\left(\left(u^i, e^{it}\right)_{i\in\mathcal{I}}, \theta, Y\right)$.

Before stating our characterization of rationalizability, we need to strengthen the axiom of profit maximization, proposed by [5] and [11], so as to account for the no free lunch assumption.

Lemma 1. *Let* $\left(y^t, p^t\right)_{t=1}^T \in \left(\mathbb{R}^L \times \mathcal{S}^{L-1}_{++}\right)^T$. *There exists* $Y \in \mathcal{Y}$ *such that*

$$(\forall t \in \{1, ..., T\}) : y^t \in \arg\max_{y\in Y} p \cdot y$$

if, and only if,

$$(\forall t, t' \in \{1, ..., T\}) : p^t \cdot y^{t'} \leq p^t \cdot y^t,$$

and

$$\left(\exists \rho \in \mathbb{R}^L_{++}\right) (\forall t \in \{1, ..., T\}) : \rho \cdot y^t \leq 0.$$

Theorem 2. *Sequence* $\left(\left(\left(e^{it}\right)_{i\in\mathcal{I}}, p^t\right)_{t=1}^{T}, \theta\right)$ *is rationalizable if, and only if, there exist* $\rho \in \mathbb{R}^L_{++}$, $\left(y^t\right)_{t=1}^T$ *in* $\mathbb{R}^L$, *and* $\left(\left(v^{it}, \lambda^{it}, x^{it}\right)_{i\in\mathcal{I}}\right)_{t=1}^T$ *in* $\mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}^L_{+}$, *that satisfy the following conditions:*

1. *For every* $t, t' \in \{1, ..., T\}$, $p^t \cdot y^{t'} \geq p^t \cdot y^t$;
2. *For every* $t \in \{1, ..., T\}$, $\rho \cdot y^t \leq 0$;
3. *For every* $i \in \mathcal{I}$ *and every* $t, t' \in \{1, ..., T\}$, $v^{it'} \leq v^{it} + \lambda^{it} p^t \cdot \left(x^{it'} - x^{it}\right)$, *with strict inequality if* $x^{it'} \neq x^{it}$;

4. *For every* $i \in \mathcal{I}$ *and every* $t \in \{1, ..., T\}$, $p^t \cdot \left(x^{it} - e^{it}\right) = \theta^i p^t \cdot y^t$;

5. *For every* $t \in \{1, ..., T\}$, $\sum_{i \in \mathcal{I}} \left(x^{it} - e^{it}\right) = y^t$.

Proof. This follows straightforwardly from lemma 1 and [7, Theorem 2], by an argument identical to the one used by [1]. □

Again, because of its existential quantifiers, the previous characterization is not an immediate test of the equilibrium hypothesis. As before:

Corollary 2. *There exists a finite system, (R), of polynomial inequalities in* $\left(\mathbb{R}^{LI}_{++} \times \mathcal{S}^{L-1}_{++}\right)^T \times \mathcal{S}^{I-1}_{+}$ *such that a sequence* $\left(\left(\left(e^{it}\right)_{i \in \mathcal{I}}, p^t\right)_{t=1}^{T}, \theta\right)$ *is rationalizable if, and only if, it satisfies (R).*

Proof. This follows from the Tarsky-Seidenberg theorem, since the conditions of the theorem constitute a semialgebraic set. □

2.2 Nonrationalizable data

Remarkably, the same data set used by [1] and in the exercise with constant returns to scale, will fail to be rationalizable in the present setting. For the sake of definiteness, a specific numeric example is initially given.

Consider the following data set,[5]

$$\begin{array}{cc} e^{11} = (9,1) & e^{12} = (1,9) \\ e^{21} = (1,1) & e^{22} = (1,1) \\ p^1 = (100,1) & p^2 = (1,100) \\ \theta^1 = 1 & \theta^2 = 0 \end{array}$$

and suppose that it is rationalizable. Then, fix $\left(\left(x^{11}, x^{21}, y^1\right), \left(x^{12}, x^{22}, y^2\right)\right)$ such that:

$$\pi^1 := p^1 \cdot y^1 \geq p^1 \cdot y^2 \tag{WAPM1}$$

$$\pi^2 := p^2 \cdot y^2 \geq p^2 \cdot y^1 \tag{WAPM2}$$

$$\left(\begin{array}{c} p^2 \cdot x^{11} \leq p^2 \cdot e^{12} + \pi^2 \\ p^1 \cdot x^{12} \leq p^1 \cdot e^{11} + \pi^1 \end{array}\right) \Longrightarrow x^{11} = x^{12} \tag{WARP}$$

$$p^1 \cdot x^{11} = p^1 \cdot e^{11} + \pi^1 \tag{WL1,1}$$

$$p^2 \cdot x^{12} = p^2 \cdot e^{12} + \pi^2 \tag{WL1,2}$$

$$p^1 \cdot x^{21} = p^1 \cdot e^{21} \tag{WL2,1}$$

$$p^2 \cdot x^{22} = p^2 \cdot e^{22} \tag{WL2,2}$$

$$x^{11} + x^{21} = e^{11} + e^{21} + y^1 \tag{MC1}$$

[5] For notational simplicity, prices are not normalized here.

$$x^{12} + x^{22} = e^{12} + e^{22} + y^2 \tag{MC2}$$

$$x^{11}, x^{21} \geq 0 \tag{NN1}$$

$$x^{12}, x^{22} \geq 0 \tag{NN2}$$

$$y_1^1 > 0 \Longrightarrow y_2^1 < 0 \tag{NFL}$$

In this case, the following claims would be true:

Claim 1. *$y_1^1 \geq y_1^2 \geq -2$ and $y_2^2 \geq y_2^1 \geq -2$.*

Proof. From WAPM1 and WAPM2, $100y_1^1 \geq 100y_1^2 + \left(y_2^2 - y_2^1\right) \geq 100y_1^2 + \frac{1}{100}\left(y_1^1 - y_1^2\right)$, so $y_1^1 \geq y_1^2$. That $y_1^2 \geq -2$ follows from MC2 and NN2.

The other inequalities are proven similarly. □

Claim 2. *$p^2 \cdot x^{11} < p^2 \cdot e^{12} + \pi^2$ and $p^1 \cdot x^{12} < p^1 \cdot e^{11} + \pi^1$.*

Proof. From WL1,1, notice that $x_1^{11} = \frac{1}{100}\left(901 + \pi^1 - x_2^{11}\right)$, so

$$\begin{aligned}
p^2 \cdot x^{11} &= \frac{1}{100}\left(901 + \pi^1 - x_2^{11}\right) + 100x_2^{11} \\
&= 9.01 + \frac{1}{100}\pi^1 + \left(100 - \frac{1}{100}\right)x_2^{11} \\
&= 9.01 + \frac{1}{100}\left(100y_1^1 + y_2^1\right) + \left(100 - \frac{1}{100}\right)\left(2 + y_2^1 - x_2^{21}\right) \\
&\leq 208.99 + y_1^1 + 100y_2^1 \\
&\leq 208.99 + y_1^2 + 100y_2^2 \\
&< 901 + y_1^2 + 100y_2^2 \\
&= p^2 \cdot e^{12} + \pi^2
\end{aligned}$$

where the third equality follows from MC1, the first inequality from NN1 and the second one from WAPM2.

The other inequality is proven similarly. □

Claim 3. *$x^{11} \neq x^{12}$.*

Proof. By NFL, either $y_1^1 \leq 0$ or $y_2^1 < 0$.

Consider first the case $y_1^1 \leq 0$, which implies, by the first claim, that $y_1^2 \leq 0$. Then, by MC2, NN2 and the fact that $y_1^2 \leq 0$, we have that $x_1^{12} = 2 + y_1^2 - x_1^{22} \leq 2$. On the other hand, by MC1

$$x_1^{11} = 10 + y_1^1 - x_1^{21} \geq 8 - x_1^{21} \geq 6.99,$$

where the first inequality follows from the first claim and the second one from WL2,1 and NN1, which imply that $x_1^{21} = \frac{1}{100}\left(101 - x_2^{21}\right) \leq 1.01$.

Now, suppose that $y_2^1 < 0$. Then, by MC1 and NN1, $x_2^{11} = 2 + y_2^1 - x_2^{21} < 2 - x_2^{21} \leq 2$, whereas

$$x_2^{12} = 10 + y_2^2 - x_2^{22} \geq 8 - x_2^{22} \geq 6.99,$$

which follows from MC2, the first claim, WL2,2 and NN2 (the last two imply that $x_2^{22} \leq 1.01$). □

Claims 2 and 3 imply that WARP is inconsistent with the rest of conditions and, hence, that the data are not rationalizable.

2.3 Why?

The numeric example makes the case for refutability, but does not offer a clear illustration of why this result is true. A graphic explanation is offered here.

Consider the original setup of an exchange economy. The reason why the data are not rationalizable in that case is that (via nonnegativity constraints of consumer 2) the sizes of the Edgeworth boxes make it impossible for individual 1 to satisfy WARP, given the positions of her (observed) budget lines, as shown in the Figure 4.

[*Figure* 4]

With production, exchange may occur outside the original boxes and the budget lines may be elsewhere. In order to rationalize the data in an economy with production, one would like to be able to modify the Edgeworth boxes, and/or displace individual 1's budget lines, so that individual 1's budget lines intersect in the interior of one (or both) of the boxes. Graphically, the first option would look like Figure 5, while the second option would look like Figure 6.

[*Figure* 5]

[*Figure* 6]

The added structure imposed by the assumption of individual rationality, however, may suffice to make both options impossible.

First, profit maximization implies that any attempt to make the exchange at $\left(\left(e^{11}, e^{21}\right), p^1\right)$ occur at a taller (and hence narrower) box, will result in the exchange at $\left(\left(e^{12}, e^{22}\right), p^2\right)$ necessarily occurring at an even narrower box, and, furthermore, there is a bound to how much narrower the box at $\left(\left(e^{11}, e^{21}\right), p^1\right)$ can be made. This is illustrated by Figure 7: by profit maximization, y^2 must lie below the hyperplane normal to p^2 through y^1, and hence $y_1^2 < y_1^1$. (Obviously, a symmetric problem and bound will arise when attempting to make the exchange box $\left(\left(e^{12}, e^{22}\right), p^2\right)$ broader.)

[*Figure* 7]

The second choice would try to impose production levels consistent with displacements of individual 1's budget lines in opposite directions, for example by introducing losses at $\left(\left(e^{12}, e^{22}\right), p^2\right)$ and not at $\left(\left(e^{11}, e^{21}\right), p^1\right)$, as illustrated before. Profit maximization with convexity and no free lunch, however, implies that profits at both observations cannot differ by too much, as illustrated by Figure 8: given y^1, profits at $\left(\left(e^{12}, e^{22}\right), p^2\right)$ can be neither too low (given by a hyperplane below the lower hyperplane normal to p^2 in the figure) nor too high (given by a hyperplane above the higher hyperplane normal to p^2).

[*Figure* 8]

3 Concluding remarks

The results here show that the Walrasian hypothesis in economies with production can be refuted upon observation of the ownership structure of the economy and finitely many pairs of prices and endowments, and that for given, finite cardinalities, it is characterized by a finite set of polynomial inequalities. This is an extension of [1] to production economies. In terms of the introductory discussion, the results show that introducing production weakens the implications of nonnegativity constraints, yet introduces additional structure via the principle of profit maximization, which suffices for refutability. It remains to be studied whether other technological assumptions allow for the same result.

4 Appendix: Proof of the lemma

Proof. For necessity, the condition that, $\forall t, t' \in \{1, ..., T\}$, $p^t \cdot y^{t'} \leq p^t \cdot y^t$ is obvious. Now, suppose that

$$\left(\forall \rho \in \mathbb{R}^L_{++}\right) (\exists t \in \{1, ..., T\}) : \rho \cdot y^t > 0$$

Then, the system

$$\rho \cdot y^1 \leq 0, \ldots, \rho \cdot y^T \leq 0, \rho \cdot \left(-e^1\right) < 0, \ldots, \rho \cdot \left(-e^T\right) < 0,$$

where e^l, $l = 1, ..., L$, is the l^{th} canonical vector in $\mathbb{R}^L$, has no solution (on ρ). It follows from [9, Theorem 22.2] that there exist $(\alpha^t)_{t=1}^T \in \mathbb{R}^T_+$ and $\left(\beta^l\right)_{l=1}^L \in \mathbb{R}^L_+ \setminus \{0\}$ such that

$$\sum_{t=1}^T \alpha^t y^t = \sum_{l=1}^L \beta^l e^l > 0.$$

It follows that $(\alpha^t)_{t=1}^T > 0$ and, hence, by convexity, that

$$\sum_{t=1}^T \frac{\alpha^t}{\sum_{s=1}^T \alpha^s} y^t \in Y.$$

But

$$\sum_{t=1}^T \frac{\alpha^t}{\sum_{s=1}^T \alpha^s} y^t = \frac{1}{\sum_{s=1}^T \alpha^s} \sum_{l=1}^L \beta^l e^l > 0,$$

which contradicts no free lunch.

For sufficiency, as in [11], let Y be the convex hull of set $\bigcup_{t=1}^T \left(y^t - \mathbb{R}^L_+\right)$, which is nonempty, closed and convex. That $\forall t \in \{1, ..., T\}$, $y^t \in \arg\max_{y \in Y} p \cdot y$ follows from [11, Theorem 2]. To see that Y satisfies no free lunch, suppose that $y \in Y$ and $y > 0$. By construction, $\exists \left(\widetilde{y}^t\right)_{t=1}^T \in \prod_{t=1}^T \left(y^t - \mathbb{R}^L_+\right)$ and $(\alpha^t)_{t=1}^T \in \mathcal{S}^{T-1}_+$ such that $y = \sum_{t=1}^T \alpha^t \widetilde{y}^t$, so, since $\rho \in \mathbb{R}^L_{++}$,

$$0 < \rho \cdot y = \rho \cdot \sum_{t=1}^T \alpha^t \widetilde{y}^t \leq \rho \cdot \sum_{t=1}^T \alpha^t y^t = \sum_{t=1}^T \alpha^t \left(\rho \cdot y^t\right) \leq 0.$$

□

References

[1] Brown, D., Matzkin, R.: Testable restrictions on the equilibrium manifold. Econometrica **64**, 1249-1262 (1996)

[2] Carvajal, A.: Testable restrictions on the equilibrium manifold under random preferences. Journal of Mathematical Economics **40**, 121-143 (2004)

[3] Carvajal, A.: Testable restrictions of Nash-Walras equilibrium in exchange economies with externalities. Royal Holloway, University of London, Discussion paper **2004-28** (2004)

[4] Carvajal, A., Ray, I., Snyder, S.: Equilibrium behavior in markets and games: testable restrictions and identification. Journal of Mathematical Economics **40**, 1-40 (2004)

[5] Hanoch, G., Rothschild, M.: Testing the assumptions of production theory: a nonparametric approach. Journal of Political Economy **80**, 256-275 (1972)

[6] Kübler, F.: Observable restrictions of general equilibrium models with financial markets. Journal of Economic Theory **110**, 137-153 (2003)

[7] Matzkin, R., Richter, M.: Testing strictly concave rationality. Journal of Economic Theory **58**, 287-303 (1991)

[8] Mishra, B.: Algorithmic Algebra. New York: Springer-Verlag 1993.

[9] Rockafellar, R. T.: Convex Analysis. Princeton: Princeton University Press 1970.

[10] Snyder, S.: Testable restrictions of Pareto optimal public good provision. Journal of Public Economics **71**, 97-119 (1999)

[11] Varian, H.: The nonparametric approach to production analysis. Econometrica **52**, 579-598 (1984)

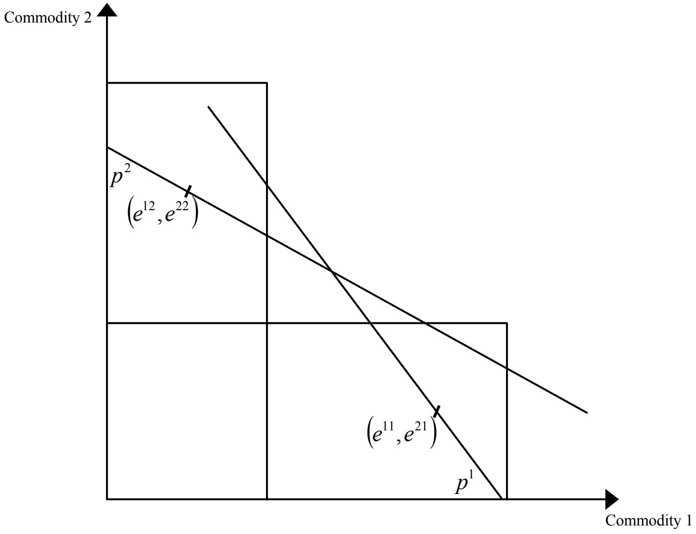


Figure 1

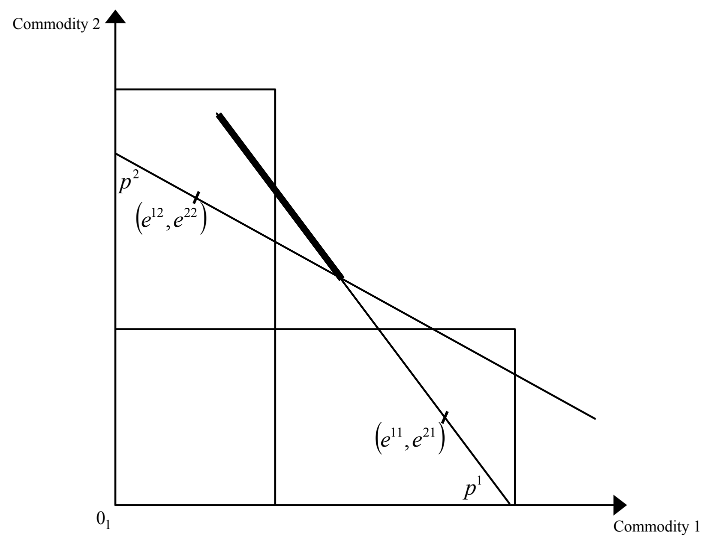
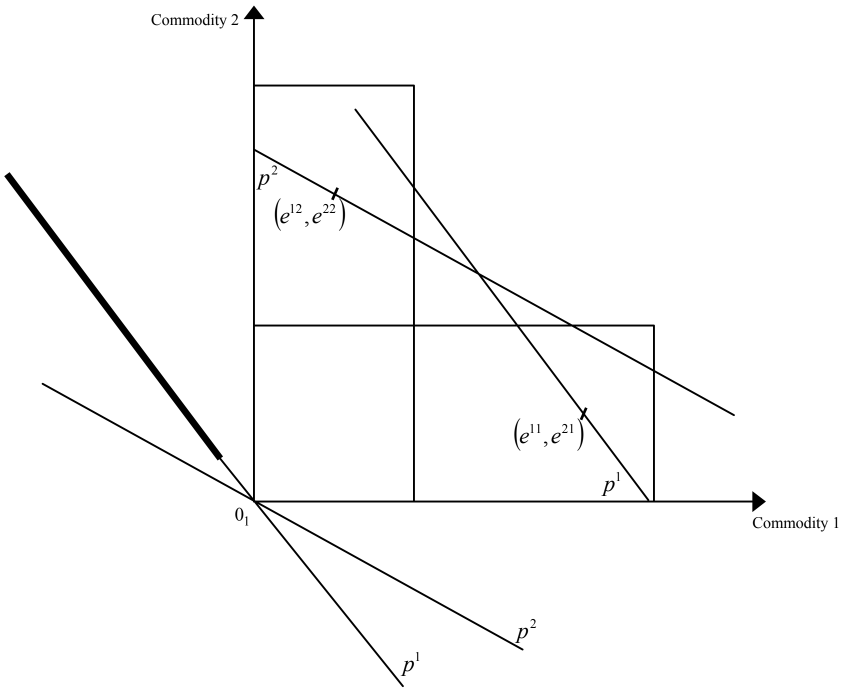


Figure 2



Commodity 2
p^2
$\left(e^{12}, e^{22}\right)$
$\left(e^{11}, e^{21}\right)$
p^1
0_1
Commodity 1
p^2
p^1

Figure 3

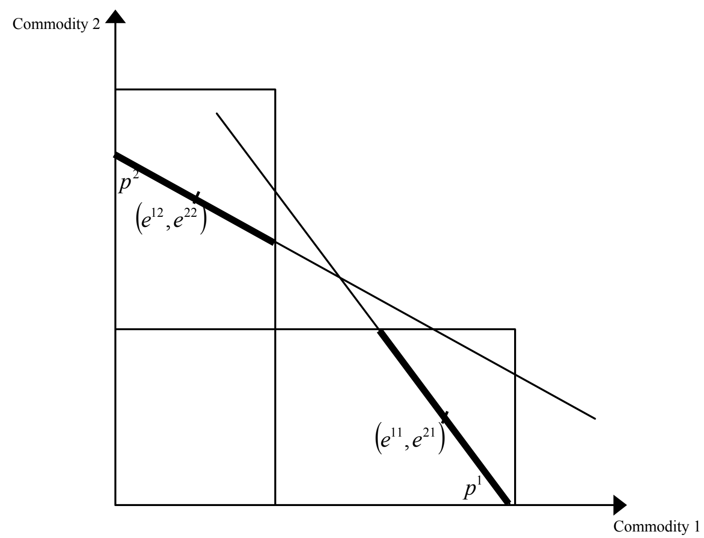


Figure 4

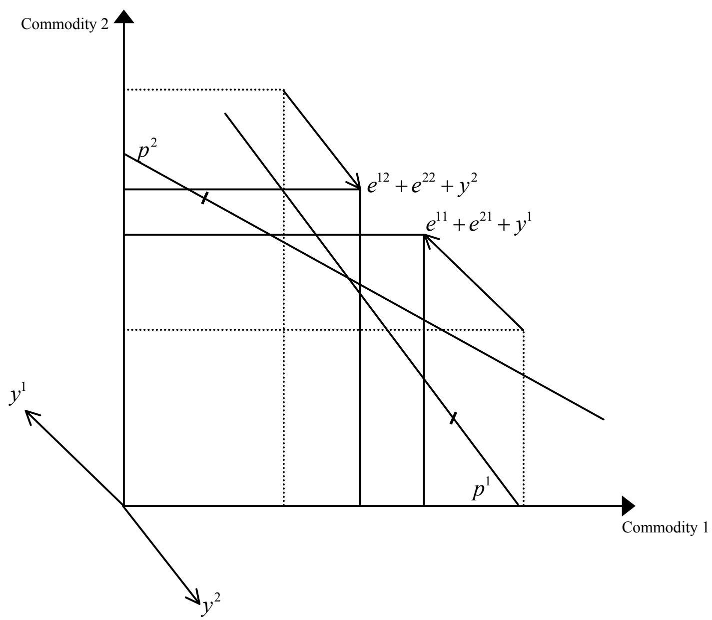


Figure 5

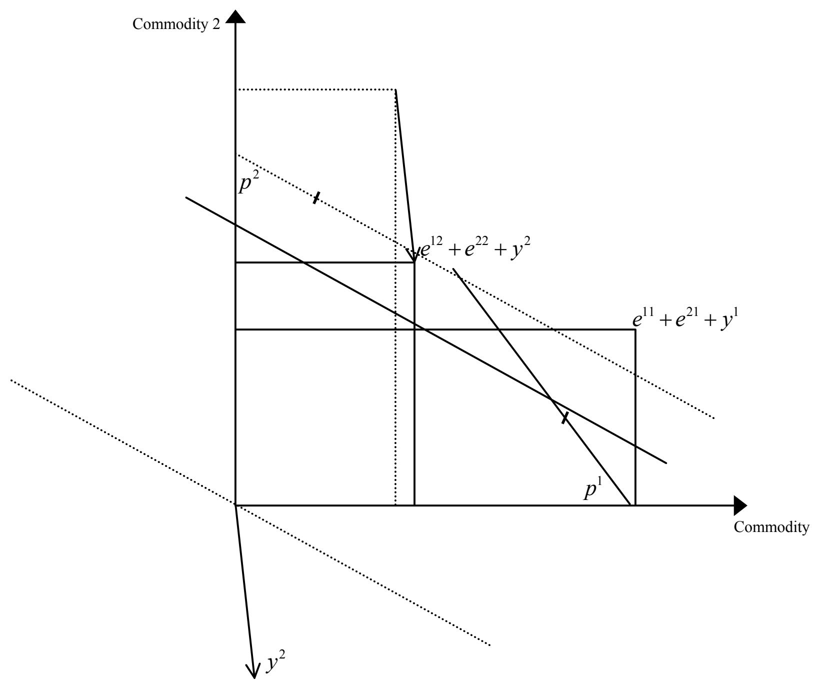


Figure 6

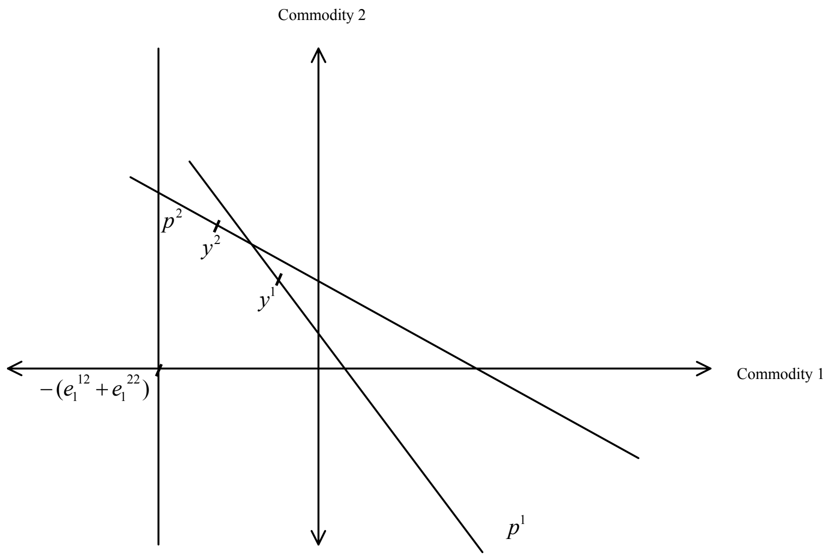


Figure 7

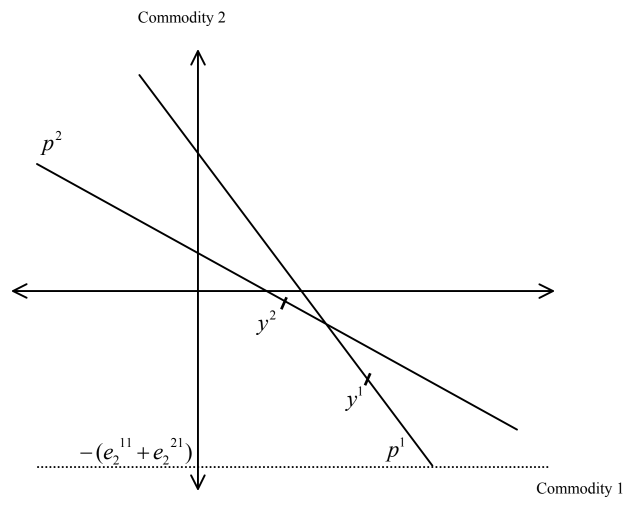


Figure 8