Exercise 4. *Let $K \in \mathbb{N}$ be fixed, and define the function $\delta : \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$ by*

$$\delta(x, y) = \sum_{k=1}^{K} |x_k - y_k|.$$

Argue that δ satisfies the following properties (which mean that it is a metric for $\mathbb{R}^K$):

1. *for all $x, y \in \mathbb{R}^K$, $\delta(x, y) \geqslant 0$;*
2. *for all $x, y \in \mathbb{R}^K$, $\delta(x, y) = \delta(y, x)$;*
3. *$\delta(x, y) = 0$ when, and only when, $x = y$; and*
4. *for all $x, y, z \in \mathbb{R}^K$, $\delta(x, y) \leqslant \delta(x, z) + \delta(z, y)$.*

Answer: That $\delta(x, y) = \delta(y, x) \geqslant 0$ for all $x, y \in \mathbb{R}^K$ is obvious, as is the fact that $\delta(x, y) = 0$ when, and only when, $x = y$. Only triangle inequality requires an argument. Let $x, y, z \in \mathbb{R}^K$ be fixed, and note that

$$\begin{aligned} \delta(x, y) &= \sum_k |x_k - y_k| \\ &= \sum_k |x_k - z_k + z_k - y_k| \\ &\leqslant \sum_k (|x_k - z_k| + |z_k - y_k|) \\ &= \sum_k |x_k - z_k| + \sum_k |z_k - y_k| \\ &= \delta(x, z) + \delta(z, y), \end{aligned}$$

where the inequality comes from applying the triangle inequality for the absolute value. □

Exercise 5. *Prove that if $x \in \mathbb{R}_{++}$ and $y \in \mathbb{R}$ is such that $|y - x| < x$, then $y \in \mathbb{R}_{++}$. Also prove that if $x \in \mathbb{R}_{--}$ and $y \in \mathbb{R}$ is such that $|y - x| < -x$, then $y \in \mathbb{R}_{--}$.*

Answer: Suppose first that $x \in \mathbb{R}_{++}$ and $|y - x| < x$. If $y \geqslant x$, we are done. Else, $|y - x| = x - y$, so, by assumption, $x - y < x$, which implies that $y > 0$.

If $x \in \mathbb{R}_{--}$ and $|y - x| < -x$, the argument is similar. When $y \leqslant x$, there is nothing to prove. Else, $y > x$ and $|y - x| = y - x$. But, then, $y - x < -x$ implies that $y < 0$. □

Exercise 6. *Does the sequence $(1/\sqrt{n})_{n=1}^{\infty}$ have a limit? Is it Cauchy? How about the sequence $(3n/(n + \sqrt{n}))_{n=1}^{\infty}$?*

Answer: Consider first the sequence $(1/\sqrt{n})_{n=1}^{\infty}$. This sequence converges to 0: for any $\varepsilon > 0$, define $n^* = \lceil 1/\varepsilon^2 \rceil + 1 \in \mathbb{N}$. Then, for any $n \geqslant n^*$,

$$\left| \frac{1}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} \leqslant \frac{1}{\sqrt{n^*}} < \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}} = \varepsilon.$$

To see that it is Cauchy, let n^* defined as before, given $\varepsilon > 0$. Then, for any $n, n' \geqslant n^*$, with $n \leqslant n'$

$$\left| \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n'}} \right| = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n'}} \leqslant \frac{1}{\sqrt{n}} \leqslant \frac{1}{\sqrt{n^*}} < \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}} = \varepsilon.$$

To see that $(3n/(n+\sqrt{n}))_{n=1}^{\infty} \to 3$, let $\varepsilon > 0$, and define

$$n^* = \left\lceil \frac{9}{\varepsilon^2} \right\rceil + 1, \tag{$*$}$$

and note that, if $n \geqslant n^*$, then

$$\left| \frac{3n}{n+\sqrt{n}} - 3 \right| = \frac{3n}{n+\sqrt{n}} - 3 = \frac{3\sqrt{n}}{n+\sqrt{n}} = \frac{3}{1+\sqrt{n}} < \frac{3}{\sqrt{n}} \leqslant \frac{3}{\sqrt{n^*}} < \frac{3}{\sqrt{9/\varepsilon^2}} = \varepsilon.$$

To see that it is Cauchy, let n^* be defined as in Eq. $(*)$, given $\varepsilon > 0$. For any, $n, n' \geqslant n^*$, if $n' \geqslant n$ it is immediate that

$$\left| \frac{3n}{n+\sqrt{n}} - \frac{3n'}{n'+\sqrt{n'}} \right| < \left| \frac{3n}{n+\sqrt{n}} - 3 \right|,$$

so it follows that

$$\left| \frac{3n}{n+\sqrt{n}} - \frac{3n'}{n'+\sqrt{n'}} \right| < \varepsilon. \qquad \square$$