Curso de Análisis Real para Economía
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Exercise 1. *Let* $U$ *be a universe. For any collection* $\mathcal{X}$ *of subsets of* $U$*, define*

$$\bigcup_{X\in\mathcal{X}} X = \{x \in U \mid \exists X \in \mathcal{X} : x \in X\} \quad \textit{and} \quad \bigcap_{X\in\mathcal{X}} X = \{x \in U \mid \forall X \in \mathcal{X}, x \in X\}.$$

*Argue that if* $\mathcal{X} = \varnothing$*, then* $\cup_{X\in\mathcal{X}} X = \varnothing$ *and* $\cap_{X\in\mathcal{X}} X = U$*.*

*Answer:* Let's prove first that $\cup_{X\in\mathcal{X}} X = \varnothing$. Since $\varnothing \subseteq \cup_{X\in\varnothing} X$ by definition of $\varnothing$, all we need to show is that $\cup_{X\in\varnothing} X \subseteq \varnothing$. If this weren't the case, there would be $x \in \cup_{X\in\varnothing} X$ such that $x \notin \varnothing$. The fact that $x \in \cup_{X\in\varnothing} X$ would imply that

$$\exists X \in \varnothing : x \in X,$$

which contradicts the definition of $\varnothing$.

For the other result, note that by definition, $\cup_{X\in\varnothing} X \subseteq U$, so all we really need to prove is that $U \subseteq \cup_{X\in\varnothing} X$. If this isn't the case, we have that for some $x \in U$,

$$x \notin \bigcap_{X\in\varnothing} X,$$

which means that

$$\neg\left(\forall X \in \mathcal{X}, x \in X\right).$$

But this means that

$$\exists X \in \mathcal{X} : x \notin X,$$

which again contradicts the definition of $\varnothing$. □

Exercise 2. *Formulate and prove generalized De Morgan's laws that apply to general collections of sets.*

*Answer:* The most general version of the laws is the following: For any «index» $a$ in a set $A$, let $X_a \subseteq U$ be a set in universe $U$. Then,

$$\left(\bigcup_{a\in A} X_a\right)^c = \bigcap_{a\in A} X_a^c \tag{$*$}$$

and

$$\left(\bigcap_{a\in A} X_a\right)^c = \bigcup_{a\in A} X_a^c. \tag{$**$}$$

For $(*)$, suppose first that $x \in (\cup_{a\in A} X_a)^c$ . That means that $x \in U$, but $x \notin \cup_{a\in A} X_a$. The latter is the same as saying that there is no $a \in A$ such that $x \in X_a$ . But this means that for all $a \in A$, $x \in X_a^c$ , since we had established that $a \in U$. Thus, we have that $x \in \cap_{a\in A} X_a^c$, as desired. This proves that

$(\cup_{a\in A}X_a)^c \subseteq \cap_{a\in A}X_a^c$ . To complete the proof of $(*)$, we can use shorter writing:

$$\begin{aligned}
x \in \bigcap_{x\in X} X_a^c &\Rightarrow \forall a \in A, a \in X_a^c \\
&\Rightarrow \forall a \in A, (x \in U \wedge x \notin X_a) \\
&\Rightarrow [x \in U \wedge (\forall a \in A, x \notin X_a)] \\
&\Rightarrow [x \in U \wedge \neg(\exists a \in A : x \in X_a)] \\
&\Rightarrow \left( x \in U \wedge x \notin \bigcup_{a\in A} X_a \right) \\
&\Rightarrow x \in \left( \bigcup_{a\in A} X_a \right)^c .
\end{aligned}$$

The arguments for $(**)$ are almost identical. ☐

EXERCISE 3. *Given sets* $A, B \subseteq X$, *prove that*

1. $A \cap B \subseteq A$ *and* $A \subseteq A \cup B$*;*
2. $(A \cap B = \varnothing \Leftrightarrow A \subseteq B^c)$, $(A \cap B = A \Leftrightarrow A \subseteq B)$ *and* $(A \cup B = A \Leftrightarrow B \subseteq A)$.

*Answer:* 1. Suppose that $x \in A \cap B$. By definition, $x \in A$ and $x \in B$. That $x \in B$ is irrelevant. We proved that $x \in A$, so $x \in A \cap B \Rightarrow x \in A$, which means that $A \cap B \subseteq A$.

The second statement is equally straightforward:

$$x \in A \Rightarrow (x \in A \vee x \in B) \Rightarrow x \in A \cup B,$$

so $A \subseteq A \cup B$.

2. For the first statement, suppose first that $A \cap B = \varnothing$, and let $x \in A$. If we had that $x \in B$, then we would conclude that $x \in A \cap B = \varnothing$, which is impossible. It must be that $x \notin B$, or, equivalently, that $x \in B^c$. We conclude, then, that $x \in A \Rightarrow x \in B$, which means that $A \subseteq B^c$.

For the opposite direction, suppose that $A \subseteq B^c$. By part 1, $A \cap B \subseteq A$, which in this case means that $A \cap B \subseteq B^c$. If it were possible that $x \in A \cap B$, we would have that $x \in B$ and $x \in B^c$. Since this cannot be true, we conclude that $x \in A \cap B$ is impossible, which means that $A \cap B = \varnothing$.

The arguments for the other two statements are similar. ☐