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Arbitrage Pricing in Non-Walrasian Financial Markets

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Abstract: This paper presents conditions under which a model of non-Walrasian trading in financial markets separates the real equilibrium outcomes from the details of the financial structure, and hence permits the pricing of non-traded derivatives by means of no-arbitrage formulæ. I demonstrate that these conditions hold in a number of standard models, including the canonical settings of Cournot and Stackelberg. In contrast, Nash equilibrium in the model of strategic market games proposed by Shapley and Shubik does not allow for the pricing of non-traded derivatives, and I explain why this is the case.

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In mathematical finance, the fundamental theorem of asset pricing allows for the determination of the prices of derivative assets even when these assets are not actively traded in the market. This theorem is normally argued under the premise that all agents intervening in the markets observe asset prices that they take as given. The logic is that if a price-taking agent could trade a derivative at a price other than the market cost of the portfolio that replicates it, she would realize that she can increase her wealth by buying the cheaper option and selling the more expensive one.

Mathematically, the property of the competitive equilibrium model that permits this analysis is that it *decouples* the real and financial parts of the equilibrium outcome, in the following sense: Given a financial structure, the model determines equilibrium asset prices and asset holdings, and, therefore, present and future consumption for all agents. Under a different financial structure, equilibrium prices and asset holdings will be different, but, *if the spaces of revenue transfers permitted by the two financial structures are the same, the resulting consumption plans will be the same, too.* Put in other words, in this model the equilibrium consumption plans (and state prices) depend on the financial structure of the economy *only up to the space of revenue transfers it allows.*[1]

Such separation of outcomes need not hold in a model with non-competitive traders. J. Peck and K. Shell [12], and L. Koutsougeras [8] have shown that state prices need not be well defined at the Nash equilibria of the strategic market game introduced by L. Shapley and M. Schubik [14]. This model, therefore, does not allow an analyst to price derivative securities using the prices of fundamental assets, even when such derivatives are traded in the market.

Furthermore, even in markets where some price-taking investors trade, and where, therefore, state prices do exist, these prices need not be useful for counter-factual analysis — specifically, for the appraisal of non-traded assets. This is because the introduction of a new asset, even if it is in zero net supply, may alter the incentives of non-competitive traders in such a way that equilibrium consumption plans and state prices change. In models where this happens, the equilibrium outcomes depend on the specific details of asset structure, and not just on its transfers space.

In a previous paper, [4], M. Weretka and I applied the model of non-Walrasian economic equilibrium introduced by him in [15], where the impact that each trader has on asset prices is determined endogenously as part of the equilibrium concept.[2] Under the assumption of complete financial markets, we argued that the principle of no-arbitrage asset pricing is consistent with the behavior of (non-Walrasian) arbitrageurs, and extended the fundamental theorem of asset pricing to this setting.

[1] An immediate example of this separation occurs when markets are complete: even though agents trade assets, equilibrium outcomes are *as if* they were trading contingent claims (and, thus, the introduction of an extra asset does not affect them).

[2] This model is, in turn, a refinement of the one used by A. Kyle [10], which applies the original ideas of S. Grossman [6] to financial markets with asymmetric information. See also J. Caballé and M. Krishnan [3], and K. Back and S. Buch [1].

Instead of considering a particular model for the determination of asset prices, in this paper I present sufficient conditions that guarantee that a model decouples the equilibrium outcomes from the details of the asset structure, and hence that it permits the pricing of untraded derivatives by means of no-arbitrage conditions. As in [4], this separation permits the extension of the fundamental theorem of asset pricing to the corresponding model. Moreover, in this setting the theorem includes a useful feature: not only does it allow for the counterfactual determination of the prices of derivatives, but also for the determination of the price impacts that each trader will have after the introduction of the derivatives.

In regards to the negative results of [12] and [8], I present a rationale by pointing out at the feature of the strategic market game model that induces such failure.

1 The economy

Consider a two-period financial economy with uncertainty. In the second period there are S states of the world, indexed by $s = 1, \ldots, S$. There are $I \geq 2$ traders, indexed by $i = 1, \ldots, I$, whose wealth in the second period, contingent on the realization of state s, is e^i_s. I denote by e^i the random variable $(e^i_1, \ldots, e^i_S)$.[3]

Each individual derives utility from her first-period consumption, c^i_0, and from her second-period random consumption, $c^i = (c^i_1, \ldots, c^i_S)$. I assume that preferences are quasilinear in c^i_0, so that the ex-ante utility of i is $c^i_0 + u^i(c^i)$, where $u^i : \mathbb{R}^S_+ \to \mathbb{R}$ is twice continuously differentiable on $\mathbb{R}^S_{++}$, strictly monotone and strictly concave. Quasilinearity gives tractability and is usually assumed in non-competitive settings. Here, it will play a subtle role besides the usual simplification of income effects: as I do not impose non-negativity constraints in first-period consumptions, I can consider perturbations to each individual's consumption in a given future state of the world, keeping her consumption in all other future states and without violating her budget constraint.[4]

An *asset* is a random variable, a, defined over state space $\{1, \ldots, S\}$. In future state of the world s, the asset pays $a_s = a(s) \in \mathbb{R}$. Letting K be the number of assets available in the economy, I denote them as $a^k = (a^k_1, \ldots, a^k_S)$, for $k = 1, \ldots, K$. I denote the *financial structure* by $A = (a^1, \ldots, a^K)$, and treat it as an $S \times K$ matrix, with its (s, k) entry, a^k_s, denoting the payoff of asset k in state s.

Given a financial structure A, a *portfolio* for individual i is $\theta^i = (\theta^i_1, \ldots, \theta^i_K)$, where θ^i_k denotes her holdings of asset k.[5] An *allocation* is a profile of portfolios $\theta = (\theta^1, \ldots, \theta^I)$ such that $\sum_i \theta^i = 0$. Asset prices are denoted by $p = (p_1, \ldots, p_K)$.[6]

[3] When performing vector and matrix operations, this and all other random variables in the paper will be treated as column vectors, unless explicitly stated otherwise.

[4] That is, when taking partial derivatives with respect to c^i_s, I implicitly assume that c^i_0 absorbs the cost of such perturbation, while every other $c^i_{s'}$ remains invariant.

[5] I treat portfolios as column vectors too.

[6] Prices, on the other hand, are taken as row vectors.

2 Models

Unlike previous literature, I do not impose a specific model of equilibrium determination. Instead, I consider a general, abstract definition of a model as a series of exogenous premises describing the perception that each trader has of how her trading position affects asset prices.

2.1 Walrasian agents

I shall say that (an) agent (of type) i is *Walrasian*, or *a price taker*, if, when she observes prices p, she considers feasible any consumption plan in the set

$$B_A^i(p) = \{(c_0^i, c^i) \mid \exists \theta^i : c_0^i = -p \cdot \theta^i \text{ and } c^i = e^i + A\theta^i\}. \tag{1}$$

I assume that the model specifies a number $n \in \{0, \ldots, I\}$ such that all agents (with type) $i > n$ are Walrasian.[7] Of course, when $n = I$ all traders are non-Walrasian.

2.2 Non-Walrasian agents

The literature recognizes different motives why some agents may have non-negligible price impact when they trade assets. Here we focus on the case where these agents recognize, perhaps due to their size relative to the rest of the market, that their volume of trade makes a significant difference on whether markets clear or not.[8] In order to model these agents in a general way, we will consider that they internalize in their decisions the changes in prices that will be necessary for them to be supplied, by the rest of the market, the portfolio that they demand.

Formally, for each individual $i = 1, \ldots, n$, the model specifies a *budget set* of consumption plans that are feasible to her, and which may depend on the portfolios of other non-competitive agents. Formally, for each i the model specifies a correspondence $B_A^i : \mathbb{R}^{K(n-1)} \twoheadrightarrow \mathbb{R} \times \mathbb{R}_+^S$, so that $B_A^i(\theta^{\neg i})$ is the set of consumption plans (c_0^i, c^i) that are feasible for i when the sub-profile of portfolios of all other non-Walrasian individuals is $\theta^{\neg i} = (\theta^j)_{j \in \{1,\ldots,n\} \setminus \{i\}}$. This correspondence is constructed by specifying a *residual inverse demand* function for the individual: letting $P_A^i : \mathbb{R}^{Kn} \to \mathbb{R}^K$, the budget set of i is

$$B_A^i(\theta^{\neg i}) = \{(c_0^i, c^i) \mid \exists \theta^i : c_0^i = -P_A^i(\theta^i, \theta^{\neg i}) \cdot \theta^i \text{ and } c^i = e^i + A\theta^i\}, \tag{2}$$

where $P_A^i(\theta) = P_A^i(\theta^i, \theta^{\neg i})$ is understood as the price vector at which, according to the model, individual i is able to constitute portfolio θ^i when the other non-Walrasian individuals'

[7] The reader may want to interpret each i as a continuum, of unitary mass, of agents with the corresponding preferences and endowments. Everything that follows in the paper remains valid under this interpretation. If the masses of the different continua are to differ, only the constructions of aggregate demand functions have to be revised, so that each type receives the appropriate weight.

[8] Indeed, one could take this feature as the definition of a non-Walrasian agent: one who recognizes, when deciding her portfolio, the market adjustments that will be necessary for markets to clear.

portfolio sub-profile is $\theta^{\neg i}$.

An observation on the notation used here may be useful: this notation is meant to be general enough to accommodate different models of interest, but the reader may want to note that not all P^i_A functions need have non-trivial dependence on all θ^j portfolios.[9]

2.3 Examples of models

Formally, a model is an array

$$\mathcal{M} = \{n, (P^i_A : \mathbb{R}^{Kn} \to \mathbb{R}^K)_{i=1,\ldots,n}\}.$$

The understanding is that all agents $i > n$ are price takers when they choose their portfolios, while each agent $i \leq n$ chooses hers subject to the constraint correspondence B^i_A, as induced by residual demand P^i_A via Eq. (2).

Example **1** (The competitive model)**.** The competitive model assumes that all traders are price takers,[10] so that, for all i, her feasible set at prices p is given by Eq. (1). Using our notation, this means simply that $n = 0$. □

Example **2** (Cournot oligopoly)**.** Suppose that the first $n < I$ individuals in the economy are Cournot competitors. Then, for each $i \leq n$, the budget set is constructed from a specific inverse demand function, as follows.

Suppose first that $\operatorname{rank}(A) = K$. For any p for which there exists $\pi \gg 0$ for which $\pi A = p$, program

$$\max_{\theta^j} \left\{-p \cdot \theta^j + u^j(e^j + A\theta^j)\right\} \tag{3}$$

has a unique solution, denoted $\theta^j_A(p)$. The aggregate demand of the Walrasian traders is

$$\Theta_A(p) = \sum_{j>n} \theta^j_A(p); \tag{4}$$

and, then, letting $\theta = (\theta^1, \ldots, \theta^n)$, the inverse demand faced by the non-Walrasian players is

$$P^i_A(\theta) = \Theta_A^{-1}\left(-\sum_{j\leq n} \theta^j\right), \tag{5}$$

which is continuously differentiable. This construction simply says that each Cournot competitor recognizes that the prices at which she can buy a portfolio are those at which the aggregate of price-taking investors are willing to sell the aggregate demand of the non-

[9] See Example 3 below. In fact, for reasons of convenience in notation, later on I sometimes include prices (which act as reference points) as an argument of the inverse demand functions — see Example 4 below.

[10] Namely, that $n = 0$.

Walrasian traders, of which her demand is part.[11]

When $\text{rank}(A) < K$, one loses uniqueness of the optimal portfolios in the previous construction. In this case, each θ_A^j and Θ_A are nonempty-valued correspondences, so one needs to define P_A^i implicitly, by requiring that

$$-\sum_{i\leq n}\theta^i \in \Theta_A(P_A^i(\theta)).$$

In order to see that this function is well defined, rewrite and partition $A = (A_F, A_R)$, so that A_F has full column rank, and $\text{rank}(A_F) = \text{rank}(A)$. $P_{A_F}^i$ is constructed using the definition for the full-rank case, given above. Since $A_R = A_F H$ for some matrix H of dimensions $\text{rank}(A) \times [K - \text{rank}(A)]$, I can construct

$$P_A^i(\theta_F, \theta_R) = [P_{A_F}^i(\theta_F + H\theta_R), P_{A_F}^i(\theta_F + H\theta_R)H]. \tag{6}$$

The intuition of this construction will appear later on in the general theory of the paper: the Cournot model allows us to first price a basis of *primitive* assets, and then to use these prices to find the ones corresponding to other (redundant, or derivative) securities. □

Example **3** (Stackelberg duopoly)**.** Suppose that $I \geq 3$ and $n = 2$. Let trader 1 behave as a Stackelberg leader and trader 2 as a Stackelberg follower, while all other agents are price takers. As in Example 2, define function Θ_A by Eq. (4), using $n = 2$.

For the Stackelberg follower, individual 2, the inverse demand is again $P_A^2(\theta^1, \theta^2) = \Theta_A^{-1}(-\theta^1 - \theta^2)$.

Define now the function

$$\theta_A^2(\theta^1) = \text{argmax}_{\theta^2}\left\{-P_A^2(\theta)\cdot\theta^2 + u^2(e^2 + A\theta^2)\right\}.$$

Then,

$$P_A^1(\theta^1) = \Theta_A^{-1}\left(-\theta^1 - \theta_A^2(\theta^1)\right)$$

is the inverse demand of the leader.[12] □

Example **4** (Market games)**.** Consider the model of strategic market games proposed by L. Shapley and M. Shubik [14], and invoked by [12] and [8].[13] For each asset there operates a trading post: an agent who wants to buy units of the asset declares how much numéraire she wants to spend in that asset; one who wants to sell, declares how many units she offers. All traders commit to these positions simultaneously and, then, the prices are determined as

[11] I am imposing the principle that she takes as given the portfolios of all other Cournot competitors, *à la Nash*.

[12] Note that the dependence of P_A^1 on θ^2 disappears. On the other hand, note that this function takes into account the *strategy* of the follower and not just her portfolio, which guarantees feasibility of the leader's strategy.

[13] See also the work of Koutsougeras and K. Papadapoulos [9], and of J.P. Zigrand [16].

the ratio of the amount of numéraire pledged to the number of units offered. The aggregate amount of numéraire pledged in each post is distributed proportionally between the traders who offered the asset, while the units offered are distributed proportionally between those who pledged numéraire to buy.

In order to write the inverse demand functions of this model, I need some extra notation for the amount of numéraire pledged by buyers. Let m_k^i denote the pledge of numéraire of agent i in the trading post for asset k, and let q_k^i be the number of units offered by her in that post. Both numbers are required to be non-negative, and only one of them can be positive in each post. Then, the price arising for each asset is given by the ratio

$$p_k = \frac{\sum_j m_k^j}{\sum_j q_k^j},$$

assuming that the ratio is defined. If trader i pledged (a positive) m_k^i, she pays that amount and receives m_k^i/p_k units of the asset; if she offered q_k^i, she delivers this quantity and receives an amount $p_k q_k^i$ of numéraire.

Note that, after prices have been determined in this manner, agent i's holdings of asset k are

$$\theta_k^i = \begin{cases} m_k^i/p_k, & \text{if } m_k^i > 0; \\ -q_t^i & \text{if } q_k^i > 0. \end{cases}$$

Instead of adding this new notation to the general writing above, I abuse the concepts a little by including prices, p, as an argument in the definition of the agents' inverse demand functions. Considering that all agents are now non-Walrasian, we can simply state that, for all i,[14]

$$P_A^i(\theta, p) = \left(\frac{p_1 \sum_j (\theta_1^j)_+}{\sum_j (\theta_1^j)_-}, \ldots, \frac{p_K \sum_j (\theta_K^j)_+}{\sum_j (\theta_K^j)_-} \right), \tag{7}$$

where $\theta = (\theta^1, \ldots, \theta^I)$.

Notice that a property of this model is that, by construction, the price of an asset depends *only* on the trades made in that asset.[15] □

2.4 Price impacts

From now on, I assume that each residual inverse demand in the model is continuously differentiable and has Jacobean with respect to the own portfolio of the agent, $D_{\theta^i} P_A^i(\theta)$, which is symmetric and positive semidefinite. The assumption of positive semidefiniteness is made to guarantee that the budget sets defined in Eq. (2) are convex; symmetry is imposed for

[14] In the next expression, I use the following notation, which is standard: for any number r, let $(r)_+ = \max\{0, r\}$ and $(r)_- = \max\{0, -r\}$.

[15] Recall footnote 4. The counterfactual exercise here is that a trader increases her demand for asset k and accommodates this increase through her first-period consumption. This is possible since the latter is not subject to non-negativity constraints. If it were, the analysis would apply only locally, at interior solutions.

technical reasons, but is tenable under the quasilinearity assumption imposed on preferences.

EXAMPLE **5** (Symmetric, positive semidefinite price impacts)**.** It is straightforward that the budget sets of the competitive agents are convex. In the models introduced in Examples 2 to 4, convex budget sets are defined for non-Walrasian players too:

2. *Cournot oligopoly:* Suppose first that A has full column rank. For each $j > n$, function $\theta^j_A(p)$ is characterized by the first-order condition

$$p = Du^j(e^j + A\theta^j_A(p))A. \tag{8}$$

Under our assumptions, by the Implicit Function Theorem, $\theta^j_A(p)$ is of class $\mathbf{C}^1$, and

$$D\theta^j_A(p) = [A^\top D^2u^j(e^j + A\theta^j_A(p))A]^{-1}$$

is symmetric and negative definite. By the Inverse Function Theorem, Eq. (5) has

$$D_{\theta^i}P^i_A(\theta) = -\left[\sum\nolimits_{j>n} D\theta^j_A(P^i_A(\theta))\right]^{-1}, \tag{9}$$

a symmetric, positive definite matrix.

When $\text{rank}(A) < K$, defining A_F, A_R and H as in Example 2, Eq. (6) implies that

$$D_{\theta^i}P^i_A(\theta) = \begin{pmatrix} D_{\theta^i_F}P^i_{A_F}(\theta_F + H\theta_R) & D_{\theta^i_F}P^i_{A_F}(\theta_F + H\theta_R)H \\ H^\top D_{\theta^i_F}P^i_{A_F}(\theta_F + H\theta_R) & H^\top D_{\theta^i_F}P^i_{A_F}(\theta_F + H\theta_R)H \end{pmatrix}. \tag{10}$$

This matrix is symmetric and positive semidefinite.

3. *Stackelberg duopoly:* $D_{\theta^2}P^2_A(\theta)$ is constructed as in the case of a Cournot duopolist. For agent $i = 1$, the Stackelberg leader, proceed as follows: function θ^2_A is characterized by the first-order condition

$$P^2_A(\theta^2_A(\theta^1), \theta^{\neg 2}) + \theta^2_A(\theta^1)^\top D_{\theta^2}P^2_A(\theta^2_A(\theta^1), \theta^{\neg 2})^\top = Du^2(e^1 + A\theta^2_A(\theta^1))A. \tag{11}$$

By direct computation,

$$D_{\theta^1}P^1_A(\theta) = -D\Theta_A(P^1_A(\theta))^{-1}[\mathbb{I} + D\theta^2_A(\theta^1)],$$

whose symmetry and positive semidefiniteness cannot be guaranteed under the assumptions made so far. If we further assume that the Hessian matrix D^2u^i is constant for all $i \geq 3$,[16] it follows from Example 2 that so are the matrices $D_p\Theta_A$ and $D_{\theta_1}P^2_A = D_{\theta_2}P^2_A$. If $\text{rank}(A) = K$, from Eq. (11) we have that

$$D\theta^2_A(\theta^1) = \{A^\top D^2u^2[e^2 + A\theta^2(\theta^1)]A\}^{-1}D_{\theta^1}P^2_A.$$

[16] Namely that their cardinal utility indices are quadratic.

Substituting directly,

$$D_{\theta^1} P_A^1(\theta) = -(D_p \Theta_A)^{-1} + (D_p \Theta_A)^{-1} [A^\top D^2 u^2(e^2 + A\theta^2(\theta^1))A]^{-1} (D_p \Theta_A)^{-1},$$

which is symmetric and positive definite. When $\text{rank}(A) < K$, matrix $D_{\theta^1} P_A^1$ can be constructed as in Example 2, and is symmetric and positive semidefinite.

4. *Market games:* Note from the observation of Example 4 that in this case all price impact matrices are diagonal and non-negative. □

3 Decoupling real and financial variables

3.1 Financial equivalence

I say that two financial structures are *equivalent* if they have the same column span, so that they offer the same opportunities to transfer revenues across states of the world. If A and $\tilde{A}$ are equivalent, I write $A \sim \tilde{A}$.

Given two equivalent financial structures, I say that two allocations are equivalent if for each individual the corresponding portfolios yield the same second-period returns. Formally, if $A \sim \tilde{A}$ I say that allocations θ and $\tilde{\theta}$ are *equivalent* if $A\theta^i = \tilde{A}\tilde{\theta}^i$ for all i; I denote this relation by $(A, \theta) \sim (\tilde{A}, \tilde{\theta})$.

In the same vein, given $A \sim \tilde{A}$, I say that price vectors p and $\tilde{p}$ are *equivalent* if $p \cdot \vartheta = \tilde{p} \cdot \tilde{\vartheta}$ for every pair of portfolios ϑ and $\tilde{\vartheta}$ such that $A\vartheta = \tilde{A}\tilde{\vartheta}$. I denote this equivalence by $(A, p) \sim (\tilde{A}, \tilde{p})$. Intuitively, two price vectors are equivalent if they imply the same cost for portfolios that have equivalent revenues.[17]

For brevity, I write $(A, \theta, p) \sim (\tilde{A}, \tilde{\theta}, \tilde{p})$ when $(A, \theta) \sim (\tilde{A}, \tilde{\theta})$ and $(A, p) \sim (\tilde{A}, \tilde{p})$. In words, equivalence between two triples means that: (i) the two financial structures allow investors the same transfer oppotunities; (ii) all the investors are performing the same transfers in the two situations; and (iii) the cost of any feasible transfer is the same in both situations. Briefly, equivalence means that doing the same is possible, that everybody is doing the same, and that doing the same costs the same.

3.2 Models that decouple financial and real variables

The competitive model of Example 1 displays the following dichotomy between real outcomes and financial details: if $(A, p) \sim (\tilde{A}, \tilde{p})$, then $B_A^i(p) = B_{\tilde{A}}^i(\tilde{p})$ for all i. This is to say that when the conditions are equivalent from the point of view of financial returns, both in terms of what returns are possible and of how much these returns cost, then the spaces of feasible consumption plans of an individual are the same.

[17] We will later see that equivalent prices are those that can be explained by the same vector of state prices, given their corresponding (equivalent) financial structures.

Of course, in any model this same equality holds true for all Walrasian traders, but I want to extend this idea to traders that do not take prices as given. I shall say that a model decouples real outcomes and financial details, if equivalent financial situations do not affect *any* individual's (real) feasible set. Furthermore, I will require that all non-Walrasian individuals face inverse demand functions that are insensitive to changes in financial positions that are irrelevant from the point of view of their real returns.

From now on, given structure A, I say that portfolio ϑ is *return-irrelevant* if it generates null payoffs: $A\vartheta = 0$. Also, I will refer to the Jacobean matrices $D_{\theta^i} P^i_A$ as *price impact* matrices, since $D_{\theta^i} P^i_A(\theta)\vartheta$ represents i's perception of how she would affect asset prices, were she to change her portfolio θ^i in the direction of ϑ.

I next define the key property *of the model* under which the usual principles of arbitrage pricing can be applied, in spite of the presence of non-competitive agents.

Definition **1** (Decoupling). *Model $\mathcal{M}$* decouples real and financial variables *if for all non-Walrasian individuals, all financial structures, and all allocations:*

1. *the individual cannot manipulate prices freely: if ϑ is return-irrelevant with respect to A, then $D_{\theta^i} P^i_A(\theta^i, \theta^{\neg i})\vartheta = 0$; and*
2. *her feasible set is independent of spurious details of the financial structure: if $(\tilde{A}, \tilde{\theta}) \sim (A, \theta)$, then $B^i_{\tilde{A}}(\tilde{\theta}^{\neg i}) = B^i_A(\theta^{\neg i})$.*

The following result is useful to clarify that that the second property in this definition does not imply the first one:

Lemma **1.** *Suppose that model $\mathcal{M}$ decouples real and financial variables, and that $(c^i_0, c^i) \in B^i_A(\theta^{\neg i})$ for some non-Walrasian trader. For all θ^i such that $e^i + A\theta^i = c^i$, one has that $P^i_A(\theta) \cdot \theta^i = -c^i_0$.*

It follows that, in the case of models that decouple, if there are multiple ways to fund a consumption plan for the second period, they are all equivalent in terms of the cost that they imply in first-period consumption.[18] This invariance of first-period consumption to return-irrelevant changes in the portfolios has two meaningful implications.

Theorem **1.** *Suppose that model $\mathcal{M}$ decouples real and financial variables, and that program*

$$\max_{c^i_0, c^i} \left\{ c^i_0 + u^i(c^i) : (c^i_0, c^i) \in B^i_A(\theta^{\neg i}) \right\} \tag{12}$$

has a solution, (c^i_0, c^i), for some non-Walrasian trader.

[18] In particular, what the second property implies is that if ϑ is return-irrelevant, then the product

$$P^i_A(\theta^i + \delta\vartheta, \theta^{\neg i}) \cdot (\theta^i + \delta\vartheta),$$

is constant for all values of $\delta \in \mathbb{R}$; this is weaker than the first property in the definition.

1. Any portfolio θ^i such that $A\theta^i = c^i - e^i$ is a solution to program

$$\max_{\tilde{\theta}^i} \left\{ -P^i_A(\tilde{\theta}^i, \theta^{\neg i}) \cdot \tilde{\theta}^i + u^i(e^i + A\tilde{\theta}^i) \right\}. \tag{13}$$

2. If θ^i solves Program (13) *and ϑ is return-irrelevant, then $P^i_A(\theta) \cdot \vartheta = 0$.*

Proof: For the first statement, fix an optimal (c^i_0, c^i) and let $\tilde{\theta}^i$ be such that such that $-P^i_A(\tilde{\theta}^i, \theta^{\neg i}) \cdot \tilde{\theta}^i = c^i_0$ and $A\tilde{\theta}^i = c^i - e^i$. It is immediate that $\tilde{\theta}^i$ solves Program (13). Now, let θ^i be such that $A\theta^i = c^i - e^i$. By Lemma 1, we have that

$$P^i_{\tilde{A}}(\theta) \cdot \theta^i = -c^i_0 = P^i_A(\tilde{\theta}^i, \theta^{\neg i}) \cdot \tilde{\theta}^i,$$

so it follows that θ^i solves Program (13) too.

For the second statement, note that the first-order conditions of (13) are that

$$P^i_A(\theta) + (\theta^i)^\top D_{\theta^i} P^i_A(\theta) = Du^i(e^i + A\theta^i)A.$$

Then,

$$P^i_A(\theta) \cdot \vartheta = Du^i(e^i + A\theta^i)A\vartheta - (\theta^i)^\top D_{\theta^i} P^i_A(\theta)\vartheta.$$

The first term on the right-hand side of this expression is null, since ϑ is return-irrelevant. The second one is null too, since the model decouples. *Q.E.D.*

Importantly, if θ^i solves Program (13) and ϑ is return-irrelevant, then $\theta^i + \vartheta$ solves that program too. This is a property that is immediate for the competitive model, and which extends to non-Walrasian models that decouple real and financial variables.

To illustrate the property, I next show that the Cournot model of Example 2 decouples variables. Before that, the following lemma is useful, as it will simplify the subsequent analysis.

Lemma **2.** *Fix a model, $\mathcal{M}$, and suppose that for a non-Walrasian individual there exist functions $\pi^i_A : \mathbb{R}^{KI} \to \mathbb{R}^S$ such that, for all A and all θ,*

1. $P^i_A(\theta) = \pi^i_A(\theta)A$; and

2. if $(\tilde{A}, \tilde{\theta}) \sim (A, \theta)$, then $\pi^i_A(\theta) = \pi^i_{\tilde{A}}(\tilde{\theta})$.

Then, $B^i_{\tilde{A}}(\tilde{\theta}^{\neg i}) = B^i_A(\theta^{\neg i})$ whenever $(\tilde{A}, \tilde{\theta}) \sim (A, \theta)$.

This lemma gives us a way to verify the second condition in Definition 1 for models that define state prices in a manner that depends on the financial structure only up to its revenue span. The next example shows that this is indeed the case in the Cournot model, and proves the first property in that definition too.

Example 6 (Cournot oligopoly). Recall the model of Example 2, and let $i \leq n$. For the first property in Definition 1, define A_F, A_R and H as in Example 2. If portfolio $\vartheta = (\vartheta_F, \vartheta_R)$ is return-irrelevant, then $A_F\vartheta_F = -A_F H\vartheta_R$. Since A_F has full column rank, this implies that $\vartheta_F = -H\vartheta_R$. Using Eq. (10), by direct multiplication, $D_{\theta^i} P^i_A(\theta)\vartheta = 0$.

For the second property in the definition, suppose first that $\text{rank}(A) = K$, and define

$$\pi^i_A(\theta) = Du^I(e^I + A\theta^I_A(P^I_A(\theta))), \tag{$*$}$$

where function θ^I_A is defined, as in Example 2, by the unique solution to Program (3). It follows immediately from the first-order conditions of individual I, namely Eq. (8), that $P^i_A(\theta) = \pi^i_A(\theta)A$.

Suppose now that $(A, \theta) \sim (\tilde{A}, \tilde{\theta})$, and that $\tilde{A}$ has full column rank. For simplicity, write $p = P^i_A(\theta)$. By definition, $\Theta_A(p) = \sum_{j>n} \theta^j_A(p)$, while $p = \pi^i_A(\theta)A$. Since $\tilde{A} \sim A$, there exist unique portfolios $\tilde{\theta}^j$, for each j, such that $\tilde{A}\tilde{\theta}^j = A\theta^j_A(p)$. By definition of $\theta^j_A(p)$, if $\tilde{p} = \pi^i_A(\theta)\tilde{A}$,

$$\theta^j_{\tilde{A}}(\tilde{p}) = \text{argmax}_{\hat{\theta}^j} \left\{ -\tilde{p}\hat{\theta}^j + u^j(e^j + \tilde{A}\hat{\theta}^j) \right\} = \tilde{\theta}^j.$$

Now,

$$\tilde{A}\Theta_{\tilde{A}}(\tilde{p}) = \sum_{j>n} \tilde{A}\theta^j_{\tilde{A}}(\tilde{p}) = \sum_{j>n} \tilde{A}\tilde{\theta}^j = \sum_{j>n} A\theta^j_A(p) = A\left(-\sum_{j\leq n} \theta^j\right) = \tilde{A}\left(-\sum_{j\leq n} \tilde{\theta}^j\right).$$

Since $\tilde{A}$ has full column rank, it then follows that $\Theta_{\tilde{A}}(\tilde{p}) = -\sum_{j\leq n} \tilde{\theta}^j$, from which, by construction, $P^i_{\tilde{A}}(\tilde{\theta}) = \tilde{p}$. This implies that

$$e^I + A\theta^I_A[P^I_A(\theta)] = e^I + \tilde{A}\tilde{\theta}^I = e^I + \tilde{A}\theta^I_{\tilde{A}}(P^I_{\tilde{A}}(\tilde{\theta})),$$

so, by Eq. ($*$) above, $\pi^i_A(\theta) = \pi^i_{\tilde{A}}(\tilde{\theta})$. By Lemma 2, we conclude that $B^i_A(\theta^{\neg i}) = B^i_{\tilde{A}}(\tilde{\theta}^{\neg i})$ when $(\tilde{A}, \tilde{\theta}) \sim (A, \theta)$.

Finally, if A has less than full column rank, partition $A = [A_F, A_R]$, as in Example 2. By the construction of that example,

$$P^i_A(\theta) = [P^i_{A_F}(\theta_F + H\theta_R), P^i_{A_F}(\theta_F + H\theta_R)H].$$

Now, let $(c^i_0, c^i) \in B^i_A(\theta^{\neg i})$. By construction, there is some θ^i such that $c^i_0 \leq -P^i_A(\theta) \cdot \theta^i$ and $c^i \leq e^i + A\theta^i$. Construct $\hat{\theta}^j = \theta^j_F + H\theta^j_R$, for all j. Then,

$$P^i_{A_F}(\hat{\theta}) \cdot \hat{\theta}^i = P^i_{A_F}(\hat{\theta}) \cdot (\theta^i_F + H\theta^i_R) = [P^i_{A_F}(\hat{\theta}), P^i_{A_F}(\hat{\theta})H] \cdot \theta^i = P^i_A(\theta) \cdot \theta^i,$$

and

$$A_F\hat{\theta}^i = A_F(\theta^i_F + H\theta^i_R) = (A_F, A_F H)\theta^i = A\theta^i.$$

This suffices to imply that $B^i_A(\theta^{\neg i}) = B^i_{A_F}(\theta^{\neg i}_F)$. Together with the results for the case of full column rank, this implies that if $(\tilde{A}, \tilde{\theta}) \sim (A, \theta)$, then $B^i_A(\theta^{\neg i}) = B^i_{\tilde{A}}(\tilde{\theta}^{\neg i})$. □

4 State prices and state price impacts

4.1 No arbitrage and no price manipulation

An arbitrage opportunity is a portfolio that yields positive returns in a costless and riskless manner. Formally, given financial structure A, prices p allow arbitrage opportunities if there exists a portfolio ϑ for which $p \cdot \vartheta \leq 0$ and $A\vartheta > 0$. It is well known that arbitrage opportunities do not exist if, and only if, there are strictly positive *state prices*, $\pi = (\pi_1, \ldots, \pi_S) \gg 0$, such that $\pi A = p$. It is also a standard argument in mathematical finance that for a price-taking agent, whose budget set is given by Eq. (1), an optimal portfolio exists if, and only if, prices allow no arbitrage opportunities. This immediately implies that if her problem has a solution, then there exists $\pi \gg 0$ such that $\pi A = p$.

For non-competitive agents, a weaker concept is necessary. Note that a strong form of profitable arbitrage is allowed by prices p if there exists a return-irrelevant portfolio ϑ for which $p \cdot \vartheta \neq 0$.[19] In this sense, prices allow (strong) arbitrage opportunities: some risk and its hedge are being priced differently. The (weaker) concept of no arbitrage that we will use for non-Walrasian agents is *the law of one price*: that whenever $A\vartheta = 0$, it must also be true that $p \cdot \vartheta = 0$.

The following result of [4] shows that weakening the definition of no-arbitrage to the requirement that the law of one price be observed amounts to weakening the requirement that state prices be strictly positive.

Lemma **3** (Carvajal and Weretka [4], Lemma 2)**.** *Given financial structure A, prices p obey the law of one price if, and only if, there exists $\pi \in \mathbb{R}^S$ such that $\pi A = p$.*

The lemma says that return-irrelevant portfolios ought to be costless. Here, I want to extend this idea to the effect that such portfolios ought to have in prices: portfolios that are meaningless in terms of their futures returns should be costless and their trade should leave prices invariant. Formally:

Definition **2.** *Given financial structure A, price impact matrix Δ is said to* allow no price manipulation *if $\Delta\vartheta = 0$ whenever portfolio ϑ is return-irrelevant.*[20]

4.2 Existence of state prices and state price impacts

Importantly, the property of price impact matrices that do not allow price manipulation can be characterized by the existence of an impact matrix over state prices, following Lemma 3

[19] It is easy to see that in this case there exists no optimal portfolio for a price-taking investor.

[20] Peck [11] studies price manipulation in a Shapley-Shubik setting. While similar in spirit, his concept of price manipulation addresses informational asymmetries; here, information is symmetric and the interest is on whether market power permits price manipulation.

in [4].

Lemma 4 (Carvajal and Weretka [4]). *Given financial structure A, a symmetric price impact matrix Δ allows no price manipulation if, and only if, there exists a symmetric matrix δ, of dimensions $S \times S$, such that $A^\top \delta A = \Delta$.*

Corollary 1. *Suppose that model $\mathcal{M}$ decouples real and financial variables, and that θ^i solves Program* (13) *for a non-Walrasian agent $i \leq n$. Then, there exist a vector π and a symmetric matrix δ such that $\pi A = P^i_A(\theta)$ and $A^\top \delta A = D_{\theta^i} P^i_A(\theta)$.*

Proof: Existence of π follows from the fact that, by Theorem 1, $P^i_A(\theta)$ obeys the law of one price, using Lemma 3. Existence of δ follows from the first property in Definition 1, using Lemma 4, since $D_{\theta^i} P^i_A$ is symmetric by assumption. *Q.E.D.*

Example 7 (Cournot oligopoly). Consider again the model of Example 2. We know from Example 6 that in this model state prices depend only on the span of the financial structure. If A has full column rank, it follows from Eq. (9) that we can rewrite, for each Cournot competitor,

$$D_{\theta^i} P^i_A(\theta) = -\frac{1}{I-n} \mathcal{H} \{[A^\top D^2 u^j (e^j + A\theta^j) A]_{j=n+1,\ldots,I}\},$$

where the operator $\mathcal{H}$ denotes the harmonic average of an array of invertible matrices. This equation is intuitive: in this model, all individuals $j = n+1, \ldots, I$ act as price-takers, whereas each individual $i = 1, \ldots, n$ acts as a residual monopolist: they recognize as their price impacts the Jacobean of the inverse demand defined by the rest of the consumers.

Assume for simplicity that A is, furthermore, invertible. To see that this model defines impact matrices over state prices, just let

$$\delta^i_A(\theta) = -\frac{1}{I-n} \mathcal{H} \{[D^2 u^j (e^j + A\theta^j_A(P^i_A(\theta))]_{j=n+1,\ldots,I}\}.$$

By direct computation, $A^\top \delta^i_A(\theta) A = D_{\theta^i} P^i_A(\theta)$, and, since $A\theta^j_A(P^i_A(\theta))$ depends on the financial structure A only up to is revenue span, it follows that so does $\delta^i_A(\theta)$. □

5 Equilibrium

By defining the individual budget sets, a model naturally defines an equilibrium concept.

Definition 3. *Given financial structure A, a pair (p, θ) is an* equilibrium for model $\mathcal{M}$ *if:*

1. *for all Walrasian agents, θ^i solves Program* (3)*;*
2. *for all non-Walrasian agents, $P^i_A(\theta) = p$ and θ^i solves Program* (13)*; and*
3. *θ is an allocation.*

I denote by $\Gamma(A, \mathcal{M})$ the set of equilibria under financial structure A for model $\mathcal{M}$.

5.1 Decoupling of variables and equilibrium

The next result proves that models that decouple real and financial variables allow for the application of the principles of no-arbitrage pricing.

THEOREM **2.** *Suppose that model $\mathcal{M}$ decouples real and financial variables. If $(p, \theta) \in \Gamma(A, \mathcal{M})$ and $(\tilde{A}, \tilde{\theta}, \tilde{p}) \sim (A, \theta, p)$, then $(\tilde{p}, \tilde{\theta}) \in \Gamma(\tilde{A}, \mathcal{M})$.*

Proof: Consider first a Walrasian agent. The first condition in Definition 3 is equivalent to the requirement that $(-p \cdot \theta^i, e^i + A\theta^i)$ solve program

$$\max_{c_0^i, c^i} \left\{ c_0^i + u^i(c^i) : (c_0^i, c^i) \in B_A^i(p) \right\}. \tag{14}$$

Since the model decouples and $(\tilde{A}, \tilde{p}) \sim (A, p)$, we have that $B_{\tilde{A}}^i(\tilde{p}) = B_A^i(p)$. This implies that the latter program is equivalent to

$$\max_{c_0^i, c^i} \left\{ c_0^i + u^i(c^i) : (c_0^i, c^i) \in B_{\tilde{A}}^i(\tilde{p}) \right\}. \tag{15}$$

Since $(\tilde{A}, \tilde{p}, \tilde{\theta}) \sim (A, p, \theta)$, we have that $e^i + \tilde{A}\tilde{\theta}^i = e^i + A\theta^i$ and,

$$\tilde{p} \cdot \tilde{\theta}^i = \pi \tilde{A} \tilde{\theta}^i = \pi A \theta^i = p \cdot \theta^i.$$

These two conclusions imply that consumption plan $(-\tilde{p} \cdot \tilde{\theta}^i, e^i + \tilde{A}\tilde{\theta}^i)$ solves Program (15), which suffices to imply the first property in Definition 3, for structure $\tilde{A}$.

Now, consider a non-Walrasian agent. Since $(\tilde{A}, \tilde{p}) \sim (A, p)$, it follows from Lemma 3 that there exists $\bar{\pi}$ such that $p = \bar{\pi} A$ and $\tilde{p} = \bar{\pi} \tilde{A}$. By the second property in Definition 1, $B_{\tilde{A}}^i(\tilde{\theta}^{\neg i}) = B_A^i(\theta^{\neg i})$ and, hence, $[A, P_{\tilde{A}}^i(\tilde{\theta})] \sim [A, P_A^i(\theta)]$ by the first property in that same definition. Again, this implies that $P_A^i(\theta) = \hat{\pi} A$ and $P_{\tilde{A}}^i(\tilde{\theta}) = \hat{\pi} \tilde{A}$, for some $\hat{\pi}$. Since $P_A^i(\theta) = p$, it follows that $\bar{\pi} - \hat{\pi} \in \ker(A^{\mathsf{T}})$. Since $A \sim \tilde{A}$, we further have that $\ker(\tilde{A}^{\mathsf{T}}) = \ker(A^{\mathsf{T}})$, which implies that $P_{\tilde{A}}^i(\tilde{\theta}) = \tilde{p}$.

The latter implies the second part of the second property in Definition 3. The first part of that same property is similar to the previous argument. The third property is immediate since $\tilde{\theta}$ is, by assumption, an allocation. *Q.E.D.*

The argument in the previous proof shows that $(A, p) \sim (\tilde{A}, \tilde{p})$ if, and only if, $\pi \tilde{A} = \tilde{p}$ for any π such that $\pi A = p$. Similarly, given $A \sim \tilde{A}$, I will say that two impact matrices are *equivalent*, denoted $(\tilde{A}, \tilde{\Delta}) \sim (A, \Delta)$, if $\tilde{A}^{\mathsf{T}} \delta \tilde{A} = \tilde{\Delta}$ whenever $A^{\mathsf{T}} \delta A = \Delta$.

DEFINITION **4.** *Model $\mathcal{M}$ is said to* define state price impacts consistently *if, for all non-Walrasian agents, $[A, D_{\theta^i} P_A^i(\theta)] \sim [\tilde{A}, D_{\tilde{\theta}^i} P_{\tilde{A}}^i(\tilde{\theta})]$ whenever $(A, \theta) \sim (\tilde{A}, \tilde{\theta})$.*

An important implication of Theorem 2 is given next. It tells us that pricing derivatives by means of the law of one price is possible when the model decouples real and financial assets. It also shows how to compute price impacts over the derivatives.

Corollary **2** (The fundamental theorem of asset pricing)**.** *Suppose that model $\mathcal{M}$ decouples real and financial assets, and fix a financial structure A. Fix a derivative asset $\alpha \in \langle A \rangle$, and let portfolio ϑ be such that $A\vartheta = \alpha$. Define the structure $\tilde{A} = (A, \alpha)$. If $(p, \theta) \in \Gamma(A, \mathcal{M})$, then:*

1. *There exists $(\tilde{p}, \tilde{\theta}) \in \Gamma(\tilde{A}, \mathcal{M})$ for which $\tilde{p} = (p, p\vartheta)$ and*

$$\tilde{\theta}^i = \begin{pmatrix} \theta^i \\ 0 \end{pmatrix}$$

for all traders.

2. *Moreover, if the model defines state price impacts consistently, then*

$$D_{\tilde{\theta}^i} P^i_{\tilde{A}}(\tilde{\theta}) = \begin{pmatrix} D_{\theta^i} P^i_A(\theta) & D_{\theta^i} P^i_A(\theta)\vartheta \\ \vartheta^\top D_{\theta^i} P^i_A(\theta) & \vartheta^\top D_{\theta^i} P^i_A(\theta)\vartheta \end{pmatrix}$$

for all non-Walrasian traders.

Proof: The first result is immediate from Theorem 2, since $(\tilde{A}, \tilde{\theta}, \tilde{p}) \sim (A, \theta, p)$.

For the second part, note that, since the model decouples, it follows from Corollary 1 that there are symmetric matrices δ^i, for all $i \leq n$, such that

$$D_{\tilde{\theta}^i} P^i_{\tilde{A}}(\tilde{\theta}) = \tilde{A}^\top \delta^i \tilde{A} = \begin{pmatrix} A^\top \delta^i A & A^\top \delta^i A\vartheta \\ \vartheta^\top A^\top \delta^i A & \vartheta^\top A^\top \delta^i A\vartheta \end{pmatrix}.$$

This equality completes the argument, because $A^\top \delta^i A = D_{\theta^i} P^i_A(\theta)$, as the model also defines state price impacts consistently. *Q.E.D.*

5.2 Alternative conditions and necessity

The next result gives us an alternative set of conditions under which the fundamental theorem of asset pricing holds true.

Theorem **3.** *Suppose that model $\mathcal{M}$ satisfies the following properties:*

1. *it defines state price impacts consistently;*

2. *no price impact matrix of a non-Walrasian trader allows price manipulation; and*

3. *if $(A, \theta) \sim (\tilde{A}, \tilde{\theta})$, then $[A, P^i_A(\theta)] \sim [\tilde{A}, P^i_{\tilde{A}}(\tilde{\theta})]$ for all non-Walrasian agents.*

If $(p, \theta) \in \Gamma(A, \mathcal{M})$ and $(\tilde{A}, \tilde{\theta}, \tilde{p}) \sim (A, \theta, p)$, then $(\tilde{p}, \tilde{\theta}) \in \Gamma(\tilde{A}, \mathcal{M})$.

Proof: By definition, the following first-order condition must be satisfied for all $i \leq n$:

$$P^i_A(\theta) + (\theta^i)^\top D_{\theta^i} P^i_A(\theta) = Du^i(e^i + A\theta^i)A.$$

Since $D_{\theta^i} P^i_A(\theta)$ allows no price manipulation, it follows, as in the proof of Theorem 1, that $P^i_A(\theta)$ obeys the law of one price. Using this, Lemmas 3 and 4 imply that we can rewrite this first-order condition as

$$\pi A + (\theta^i)^\intercal A^\intercal \delta_i A = Du^i(e^i + A\theta^i)A.$$

Since $A \sim \tilde{A}$, there exists ϑ such that $A\vartheta = \tilde{A}$. Also, $(A, \theta) \sim (\tilde{A}, \tilde{\theta})$ implies that

$$\pi A\vartheta + (\tilde{\theta}^i)^\intercal \tilde{A}^\intercal \delta_i A\vartheta = Du^i(e^i + \tilde{A}\tilde{\theta}^i)A\vartheta,$$

or, equivalently,

$$\pi \tilde{A} + (\tilde{\theta}^i)^\intercal \tilde{A}^\intercal \delta_i \tilde{A} = Du^i(e^i + \tilde{A}\tilde{\theta}^i)\tilde{A}.$$

By the first and third assumptions, then,

$$P^i_{\tilde{A}}(\tilde{\theta}) + (\tilde{\theta}^i)^\intercal D_{\tilde{\theta}^i} P^i_{\tilde{A}}(\tilde{\theta}) = Du^i(e^i + \tilde{A}\tilde{\theta}^i)\tilde{A}.$$

This implies the second property in Definition 3, for structure $\tilde{A}$. The first and third properties in that definition are straightforward. *Q.E.D.*

Finally, I provide conditions under which the separation of equilibrium real and financial outcomes implies that no price impact matrix gives impact to return-irrelevant trades.

THEOREM **4.** *Suppose that model $\mathcal{M}$ satisfies the following properties:*

1. *if $(p, \theta) \in \Gamma(A)$ and $(\tilde{A}, \tilde{\theta}, \tilde{p}) \sim (A, \theta, p)$, then $(\tilde{p}, \tilde{\theta}) \in \Gamma(\tilde{A})$;*

2. *if $(p, \theta) \in \Gamma(A)$ and $(A, \tilde{\theta}) \sim (A, \theta)$, then $D_{\theta^i} P^i_A(\tilde{\theta}) = D_{\theta^i} P^i_A(\theta)$ for all non-Walrasian agents.*

If $(p, \theta) \in \Gamma(A, \mathcal{M})$, no price impact matrix $D_{\theta^i} P^i_A(\theta)$ allows price manipulation.

Proof: Suppose not, and fix a trader and a portfolio ϑ such that $D_{\theta^i} P^i_A(\theta)\vartheta \neq 0$ even though $A\vartheta = 0$, for some $(p, \theta) \in \Gamma(A, \mathcal{M})$. Define $\tilde{\theta}^i = \theta^i + \vartheta$ and $\tilde{\theta}^j = \theta^j + (I-1)^{-1}\vartheta$ for all $j \neq i$. It is immediate that $(A, \tilde{\theta}, p) \sim (A, \theta, p)$.

By definition of equilibrium, θ^i solves Program (13), so it must be true that

$$p + (\theta^i)^\intercal D_{\theta^i} P^i_A(\theta) = Du^i(e^i + A\theta^i)A.$$

Since $A\vartheta = 0$, we have that $Du^i(e^i + A\theta^i) = Du^i(e^i + A\tilde{\theta}^i)$. Since, by the second assumption, $D_{\theta^i} P^i_A(\tilde{\theta}) = D_{\theta^i} P^i_A(\theta)$,

$$p + (\tilde{\theta}^i)^\intercal D_{\theta^i} P^i_A(\tilde{\theta}) = p + (\theta^i)^\intercal D_{\theta^i} P^i_A(\theta) + \vartheta^\intercal D_{\theta^i} P^i_A(\theta).$$

All this implies that

$$p + (\tilde{\theta}^i)^\top D_{\theta^i} P^i_A(\tilde{\theta}) \neq Du^i(e^i + A\tilde{\theta}^i)A,$$

since $D_{\theta^i} P^i_A(\theta)\vartheta \neq 0$.

It follows that $\tilde{\theta}^i$ does *not* solve Program (13), which implies that $(p, \tilde{\theta}) \notin \Gamma(A, \mathcal{M})$. This contradicts the first assumption. *Q.E.D.*

6 Models where arbitrage pricing is not applicable

If a model does not impose the law of one price at equilibrium, the formulæ for pricing derivatives by constructing an equivalent portfolio of securities and applying the no-arbitrage principle fails to apply.

Example 8 (Market games). The application of Nash equilibrium to the strategic markets game of Shapley and Shubik, introduced in Example 4, is *not* amenable to the application of no-arbitrage pricing of derivatives, as argued by [12], [7] and [8]. From Eq. (7), it follows that the price impact matrices implied by the model, $D_{\theta^i} P^i_A$, are diagonal and positive definite. It follows that $D_{\theta^i} P^i_A \vartheta = 0$ if, and only if, $\vartheta = 0$. But if $\text{rank}(A) < K$, the latter implies that these price impact matrices do give price impact to return-redundant trades, for there are non-zero portfolios that are return-irrelevant. □

This failure of the law of one price, however, is not the only reason why the application of the usual formulæ of mathematical finance may be inadequate. Even if the law applies at equilibrium given a financial structure, one also needs that the equilibrium consumption levels be invariant to changes in the financial structure that leave the space of revenue transfers unchanged. This is important, since the presence of competitive traders in *any* model would suffice for the law of one price to hold at equilibrium: regardless of the behaviour of non-Walrasian agents, the optimization problem of any Walrasian agent, Eqs. (3) and (14), possesses a solution only if the law holds.[21] The following example show that this does not suffice to make a model decouple real and financial variables.

Example 9 (Market games with a competitive fringe). Consider a Shapley-Shubik economy where there are five agents and two future states of the world. The agents' endowments are

$$e^1 = e^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; e^2 = e^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \;\text{ and }\; e^5 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For individuals $i = 1, 2, 3, 4$, ex-ante preferences are

$$u^i(c_0, c_1, c_2) = c_0 + \ln c_1 + \ln c_2,$$

[21] From the point of view of a theory of asset pricing, it would be up to the modeller to decide whether she would like to have a critical equilibrium property induced by the presence of agents who are assumed to be of little significance at the market level.

while for individual 5 they are

$$u^5(c_0, c_1, c_2) = c_0 + \tfrac{2}{7}\ln c_1 + \tfrac{2}{7}\ln c_2.$$

Individual 5 behaves competitive, even in the framework of a strategic market game.

Suppose first that three assets can be traded in the economy: the two elementary securities and a risk-less bond. In that order, let their prices be p_1, p_2 and p_3.

A symmetric Nash equilibrium for this game has

$$m_2^1 = m_2^3 = m_1^2 = m_1^4 = \bar{m},$$

$$q_1^1 = q_1^3 = q_2^2 = q_2^4 = \bar{q},$$

$$q_3^1 = q_3^2 = q_3^3 = q_3^4 = \bar{Q},$$

and $m_3^5 = \bar{M}$, for a tuple $(\bar{m}, \bar{M}, \bar{q}, \bar{Q})$ such that

a. $2\bar{m}/\bar{q} = \bar{M}/4\bar{Q}$;

b. $\bar{M}$ solves program

$$\max_M \left\{ -M + \frac{4}{7}\ln\left(\frac{M}{\bar{p}_3}\right) \right\},$$

where $\bar{p}_3 = \bar{M}/4\bar{Q}$; and

c. $(\bar{q}, \bar{m}, \bar{Q})$ solves program

$$\max_{q,m,Q} \left\{ \frac{2\bar{m}}{q+\bar{q}}q - m + \frac{\bar{M}}{Q+3\bar{Q}}Q + \ln(1-q-Q) + \ln\left(\frac{2\bar{q}}{m+\bar{m}}m - Q\right) \right\}.$$

Here, condition (a) is necessary for the problem of agent 5 to have a solution. Once that condition holds, that agent finds that a unit of the risk-less bond is perfect substitute to a portfolio with one unit of each of the elementary securities. In condition (b), I have used such indifference to simplify the agent's problem so that she demand only the bond. Finally, condition (c) is, using symmetry, the problem of each of the non-Walrasian agents, $i = 1, 2, 3, 4$.

By direct computation, an interior solution for the system is with

$$\bar{m} = \bar{M} = \tfrac{4}{7}, \; \bar{q} = \tfrac{8}{37}, \; \text{ and } \; \bar{Q} = \tfrac{1}{37}.$$

At that equilibrium, consumption of each of the non-Walrasian individuals in their respective good states (where they receive positive endowment) is $28/37 \approx 0.76$.

Suppose, alternatively, that only the two elementary securities can be traded. The only symmetric, interior Nash equilibrium for this game has

$$m_2^1 = m_2^3 = m_1^2 = m_1^4 = \tilde{m},$$

$$q_1^1 = q_1^3 = q_2^2 = q_2^4 = \tilde{q},$$

and

$$m_1^5 = m_2^5 = \tilde{M},$$

for a tuple $(\tilde{m}, \tilde{M}, \tilde{q})$ such that

d. $\tilde{M}$ solves program

$$\max_M \left\{ -M + \frac{2}{7} \ln\left(\frac{M}{\tilde{p}}\right) \right\},$$

where $\tilde{p} = (2\tilde{m} + \tilde{M})/2\tilde{q}$; and

e. $(\bar{q}, \bar{m})$ solves program

$$\max_{q,m} \left\{ \frac{2\tilde{m} + \tilde{M}}{q + \tilde{q}} q - m + \ln(1 - q) + \ln\left(\frac{2\tilde{q}}{m + \tilde{m} + \tilde{M}} m\right) \right\}.$$

Again, we can solve this system to find

$$\tilde{m} = \tfrac{5+\sqrt{137}}{28}, \ \tilde{M} = \tfrac{2}{7}, \ \text{ and } \ \tilde{q} = \tfrac{9+\sqrt{137}}{65+\sqrt{137}}.$$

Upon substitution, the consumption of non-Walrasian agents in the state where they have positive endowments is $56/(65 + \sqrt{137}) \approx 0.73$.

In this case, the presence of a "redundant" derivative has real effects. □

Of course, in the previous example there exists another equilibrium of the economy with the bond, where nobody trades this asset and the real outcomes are the same as in the case where such asset does not exist. What is important, though, is that no equilibrium of the market without the bond can replicate the real outcome of the equilibrium of the market with it. Intuitively, note that the indifference of the Walrasian agent between the bond and the portfolio of the two elementary securities means that we can find equilibria where her level of trade in each of these two submarkets varies. Here, when the bond is available they are the only buyers of the bond, and the two elementary securities are traded solely by the non-Walrasian agents. The level of competition in this case is different from a situation where each elementary security is demanded by the Walrasian and two of the non-Walrasian agents, which drives the result.

These observations are in line with the results of Busetto et Al [2] and of Codognato et al [5], although the research question there is different from mine. In [5], the goal is to determine conditions under which the allocations resulting from Nash equilibrium in a market game are also competitive equilibrium allocations of the same economy. My goal is to understand whether a property of competitive equilibrium (that its allocations decouple) also holds for other concepts of equilibrium, even when their allocations are different. Still, my observations in the previous example can be understood better in the context of Proposition 1 in [5]. From that proposition one concludes that in the context of Example 9, for equilibria where all agents have interior consumption plans, the allocation will not coincide with a competitive equilibrium allocation. The example states that, moreover, unlike competitive equilibrium allocations, the set of Nash equilibrium allocations of a strategic market game is not completely determined by the space of revenue transfers permitted by the available assets. The following example presents another instance of this situation.

EXAMPLE **10** (Misspecified price impacts). Suppose that $n = 1$, so that all agents $i = 2, \dots, I$ take prices as given. Suppose furthermore that, for all of them, $e^i = (0, 0, \dots, 0)$, and $u^i : \mathbb{R}^S \to \mathbb{R}$ is given by the quadratic form

$$u^i(c) = \frac{1}{2}(\bar{c} - c)^\top \mathbf{H}\,(\bar{c} - c),$$

where $\bar{c} \gg 0$ is a vector of parameters, and $\mathbf{H}$ is an $S \times S$, symmetric, negative definite matrix.[22]

Using the notation of Example 2, the aggregate demand of the Walrasian agents is

$$\Theta_A(p) = (I - 1)(A^\top \mathbf{H}\, A)^{-1}(p^\top + A^\top \mathbf{H}\, \bar{c}).$$

The market clearing condition, that $\theta^1 + \Theta_A(p) = 0$, thus requires that

$$p^\top = -A^\top \mathbf{H}\, \bar{c} - \frac{1}{I - 1} A^\top \mathbf{H}\, A \theta^1. \tag{16}$$

Now, suppose that agent $i = 1$, who has the same preferences as the other agents, realizes that her trades have an impact on market prices. Suppose first that she correctly estimates this impact to be the derivative of the right-hand side of Eq. (16). Under this assumption, if the price at which she purchases portfolio θ^1 is p, she estimates that purchasing an alternative portfolio ϑ induces prices

$$p + \frac{1}{I - 1}(\theta^1 - \vartheta)^\top A^\top \mathbf{H}\, A.$$

Let the model require that portfolio θ^1 solve program

$$\max_{\vartheta} \left\{ -\left[p + \frac{1}{I - 1}(\theta^1 - \vartheta)^\top A^\top \mathbf{H}\, A \right] \cdot \vartheta + u^1(e^1 + A\vartheta) \right\},$$

so that individual $i = 1$ correctly estimates the effect of her trade on prices. It follows that the optimal demand, given prices p, is

$$\theta^1(p) = \frac{I - 1}{I}(A^\top \mathbf{H}\, A)^{-1}[p^\top + A^\top \mathbf{H}\, (\bar{c} - e^1)].$$

[22] I assume that $\bar{c}$ is large enough that u^i is increasing in the relevant part of the domain. Also, note that I define u^i over the whole of $\mathbb{R}^S$ so that any point is interior and must satisfy the first-order conditions below, if it is optimal.

At equilibrium, $\theta^1(p) + \Theta_A(p) = 0$, which occurs when

$$p = \left(\frac{1}{I+1}e^1 - \bar{c}\right)^\top \mathbf{H}\, A.$$

By direct substitution, equilibrium consumption plans are

$$c^1 = \left[\mathbb{I} - \frac{I-1}{I+1} A(A^\top \mathbf{H}\, A)^{-1} A^\top \mathbf{H}\right] e^1, \tag{17}$$

and

$$c^i = \frac{1}{1+I} A(A^\top \mathbf{H}\, A)^{-1} A^\top \mathbf{H}\, e^1 \tag{18}$$

for $i = 2, \ldots, I$. Note that these two expressions are invariant to changes in the asset structure that maintain the structure's span unchanged.[23]

Now, suppose that individual $i = 1$ misspecifies her model of price impact, and obtains a biased estimate of the derivative of the right-hand side of Eq. (16),

$$\Delta_A = \gamma \mathbb{I} - \frac{1}{I-1} A^\top \mathbf{H}\, A,$$

for a scalar $\gamma \geq 0$. Then, if the price at which she purchases θ^1 is p, she estimates that ϑ induces prices $p + (\vartheta - \theta^1)\Delta_A$, in which case θ^1 must solve program

$$\max_{\vartheta} \left\{-[p + (\vartheta - \theta^1)\Delta_A] \cdot \vartheta + u^1(e^1 + A\vartheta)\right\}.$$

The optimal portfolio of $i = 1$ is now

$$\theta^1(p) = (A^\top \mathbf{H}\, A)^{-1}[p^\top + A^\top \mathbf{H}\, (\bar{c} - e^1)],$$

and market clearing requires that

$$p = (e^1)^\top \mathbf{H}\, A(A^\top \mathbf{H}\, A - \Delta_A)^{-1}[(A^\top \mathbf{H}\, A - \Delta_A)^{-1} + (I-1)(A^\top \mathbf{H}\, A)^{-1}]^{-1} - \bar{c}^\top \mathbf{H}\, A.$$

By direct substitution, the equilibrium consumption of all Walrasian individuals, $i \geq 2$, is

$$c^i = A(A^\top \mathbf{H}\, A)^{-1}[(A^\top \mathbf{H}\, A - \Delta_A)^{-1} + (I-1)(A^\top \mathbf{H}\, A)^{-1}]^{-1}(A^\top \mathbf{H}\, A)^{-1} A^\top \mathbf{H}\, e^1.$$

By construction,

$$A^\top \mathbf{H}\, A - \Delta_A = \frac{I}{I-1} A^\top \mathbf{H}\, A - \gamma \mathbb{I}.$$

If $\gamma = 0$, individual $i = 1$ has an unbiased estimate of her price impact, the previous

[23] If $\langle \tilde{A} \rangle = \langle A \rangle$, we can construct an invertible matrix H, of dimensions $\text{rank}(A) \times \text{rank}(A)$, such that $\tilde{A} = HA$. If the previous expressions are written for structure $\tilde{A}$, matrix H cancels out.

expression is equivalent to Eq. (18), which depends on the financial structure only up to its span.[24] If $\gamma \neq 0$, this trader's estimate is biased and the agents' equilibrium consumption plans are not completely determined by the space of revenue transfers. □

Finally, the next example illustrates how fundamentals with no microeconomic foundation may make an otherwise decoupling model fail this property.

Example **11** (Cournot competition with noise trading)**.** Suppose that instead of a fringe of competitive agents, the demand faced by the non-Walrasian traders comes from a set of *noise* traders, and is defined for all prices in some set $p \in \mathcal{P}_A \subset \mathbb{R}^K$. Specifically, suppose that $\Theta_A : \mathcal{P}_A \to \mathbb{R}^K$ is the aggregate demand of these noise traders, and that $n = I$ so that there are no Walrasian agents in the economy. Then, in the same vein as Eq. (5), we define, for all agents,

$$P_A^i(\theta) = P_A(\theta) = \Theta_A^{-1}\left(\sum_j \theta^j\right), \tag{19}$$

where $\theta = (\theta^1, \ldots, \theta^I)$, assuming that the right-hand side of the expression exists and is uniquely defined. Here, Eq. (4) has been dismissed, and the inverse demand that determines the (strategic) interaction in the markets has been exogenously imposed.

By construction, one then needs to assume that Θ_A is differentiably invertible, and it follows that

$$D_{\theta^i} P_A^i(\theta) = D_{\theta^i} P_A(\theta) = D\Theta_A\left(\sum_j \theta^j\right)^{-1}. \tag{20}$$

Whether the model decouples real and financial variables depends, thus, on the structure of the family of functions $\{\Theta_A\}_A$. It follows from Lemmas 2 and 4 that the model decouples if one assumes that, for every financial structure A, $D\Theta_A$ is symmetric and negative definite, and there exist functions $\pi : \mathbb{R}^S \to \mathbb{R}^S$ and $\delta : \mathbb{R}^S \times \mathbb{R}^{S\times S}$ such that:

a. for all A and all p, $\pi(A\Theta_A(p))A = p$ and $\left[A^\top \delta(A\Theta_A(p))A\right]^{-1} = D\Theta_A(p)$; and

b. for all θ, matrix $\delta(\theta)$ is symmetric and negative definite.

Without structure like this, there is no guarantee that a perturbation to a financial detail of the setting will not have real effects on the equilibrium of the economy. □

7 Concluding remarks

In this paper I have studied the extent to which the standard formulæ of mathematical finance are applicable in markets where traders realize that their volume of trade affects market prices. While there is only one model of competitive behavior, there are multiple ways to model the interaction between non-competitive agents, and the existing literature

[24] Of course, the same is true of c^1 and Eq. (17)

has obtained results that point in different directions. That is the motivation for this paper. I offer general conditions under which the pricing of derivative assets is possible from the knowledge of the fundamental assets on which the derivatives are based.

The conditions are fairly intuitive. The property of the competitive model that delivers the theory of arbitrage pricing is the fact that in this model the real variables depend on the financial structure only up to its span of revenue transfers. The real variables of most relevance are the state prices that underlie the asset prices. If a change in the financial structure keeps the span fixed, the state prices remain unchanged, and one can use them to determine the asset prices arising in the new structure. This is, indeed, the case when a derivative is introduced to the market.

I refer to this property as the *decoupling* of real and financial variables, and cast it in term of a general model that allows for non-Walrasian behavior. In this general model, I say that decoupling occurs when (i) the real budget sets of the traders depend on the financial structure only up to its transfer span; and (ii) the perception that the traders have of how they can affect asset prices imply that they can only change these prices if they change a real variable.

When a model decouples real and financial variables, asset prices embed state prices that allow for arbitrage pricing of derivatives,[25] and the model also permits the determination of the traders' perceived effects on state prices, from their perceived effects on asset prices. Moreover, the fundamental theorem of asset pricing can be applied within the context of the model: when a derivative is introduced to the market, the transfer span remains unchanged;[26] if both the state prices and the traders' impact on state prices remain unchanged too, one can price the derivative by the observed market cost of the portfolio that replicates it, as in the competitive model, and there is no need to observe trading in the derivative. Importantly, the extension of the theorem also allows for the determination of the impacts that agents have on the price of the derivative: it is the effect that they have on the market cost of that same replicating portfolio.[27]

Some examples of canonical models of non-Walrasian behavior are used to illustrate the concepts and results introduced here. The model of Cournot competition, for instance, is completely amenable to the application of no-arbitrage mathematical finance. The widely used model of strategic market games is not, as has been shown in the literature. My explanation for this result is that under the institutional framework assumed in the model, the (Nash) behavior of agents immediately implies that the model fails to decouple real and financial variables. Consider a derivative that consists of the sum of two fundamental assets. In the Cournot model, all agents think that selling one unit of each of the fundametal assets and buying one of the derivative has no effects on market prices. But note that in a strategic

[25] The theory becomes somewhat weaker, as it does not guarantee that states prices will necessarily be positive.

[26] Recall that there are no constraints on short sales of the assets.

[27] Indeed, the same logic gives us the effects of trading in the derivative on the prices of the fundamental assets, and of trading in the derivative on its own price.

market game, by assumption, these trades would lower the prices of the fundamental assets and increase the price of the derivative; the pricing equation assumed in the model allows for the costless manipulation of prices.

A second important effect, which is perhaps more subtle, is at play too. Suppose that, for some reason, there are agents in the trading posts of the strategic market game that take prices as given. For these agents, the existence of an optimal portfolio immediately implies that any derivative has to be priced equally to any portfolio that replicates it, for otherwise these agents would always think that they can profit from arbitraging on the difference. Beyond the methodological objection that the reader may have to a result where one *negligible* agent dictates an equilibrium property, it is important to note that this analysis neglects a second effect. Suppose that the derivative mentioned above, consisting of the sum of two fundamental assets, is introduced to the market. The presence of a price-taking trader implies that, after this innovation, the price of the derivative must equal the sum of the *new* prices of the fundamental securitities. It does *not* imply, however, that these new prices must equal the ones observed before the derivative was introduced. An example where non-Walrasian agents have biased perceptions of their price impacts is given to illustrate this point.

In a previous paper, [4], M. Weretka and I considered a refinement of the "Nash in supply functions" theory adapted to financial markets with asymmetric information by [10]. There, we showed that if markets are complete and information is symmetric, there always exists at least one equilibrium where the premises of models that decouple financial and real variables apply. We do not know, however, whether other equilibria where this is not the case are possible. That question remains open, as does the one of whether the results obtained here apply when information is not symmetric.

In general, more analysis is needed for a complete theory for non-competitive asset pricing. In this paper I have presented conditions under which the standard counter-factual analysis expends to models that can be cast as deviations from the Walrasian framework, but I did not develop a complete, game-theoretical model. I also focused on global results and not on local analysis: I did not allow for the possibility that a model has equilibria that decouple real and financial variables and equilibria where that is not the case.[28] Again, I leave these questions for further research.

Appendices

A.1. Proofs of the lemmata

Proof of Lemma 1: As in Example 2, partition $A = (A_F, A_R)$ so that A_F has full column rank, equal to rank(A). Partition $\theta^j = (\theta^j_F, \theta^j_R)$ consistently. Let H be such that $A_R = A_F H$, and

[28] The combination of Examples 8 and 9 is an instance of that possibility, but more analysis is needed.

construct $\tilde{A} = A_F$ and $\tilde{\theta}^j = \theta^j_F + H\theta^j_R$. By construction, since the model decouples, $B^i_{\tilde{A}}(\tilde{\theta}^{\neg i}) = B^i_A(\theta^{\neg i})$, so $(c^i_0, c^i) \in B^i_{\tilde{A}}(\tilde{\theta}^{\neg i})$.

Now, since $\tilde{A}$ has full column rank, there exists only one $\tilde{\theta}^i$ such that $\tilde{A}\tilde{\theta}^i = c^i - e^i$. Fixing such unique $\tilde{\theta}^i$, one has that, necessarily, $c^i_0 = P^i_{\tilde{A}}(\tilde{\theta})$. *Q.E.D.*

Proof of Lemma 2: Let $(c^i_0, c^i) \in B^i_A(\theta^{\neg i})$, and fix θ^i such that $c^i_0 = P^i_A(\theta) \cdot \theta^i$ and $c^i - e^i = A\theta^i$. Since $\tilde{A} \sim A$, there is a portfolio $\tilde{\theta}^i$ such that $\tilde{A}\tilde{\theta}^i = A\theta^i$. Also, by assumption, $P^i_A(\theta) = \pi^i_A(\theta)A$ and $P^i_{\tilde{A}}(\tilde{\theta}) = \pi^i_{\tilde{A}}(\tilde{\theta})\tilde{A}$. As $(\tilde{A}, \tilde{\theta}) \sim (A, \theta)$, we have that $\pi^i_A(\theta) = \pi^i_{\tilde{A}}(\tilde{\theta})$, so

$$P^i_{\tilde{A}}(\tilde{\theta}) \cdot \tilde{\theta}^i = \pi^i_{\tilde{A}}(\tilde{\theta})\tilde{A}\tilde{\theta}^i = \pi^i_A(\theta)A\theta^i = P^i_A(\theta) \cdot \theta^i = c^i_0.$$

This shows that $(c^i_0, c^i) \in B^i_{\tilde{A}}(\tilde{\theta}^{\neg i})$, and hence that $B^i_A(\theta^{\neg i}) \subseteq B^i_{\tilde{A}}(\tilde{\theta}^{\neg i})$. The argument that $B^i_{\tilde{A}}(\tilde{\theta}^{\neg i}) \subseteq B^i_A(\theta^{\neg i})$ is identical. *Q.E.D.*

Proof of Lemma 3: This argument is given in [4]; it is included here for completeness. For necessity, without any loss of generality take any p for which there exists no $\bar{\theta}$ for which $p \cdot \bar{\theta} < 0$ and $A\bar{\theta} = 0$; it follows from Farkas's Lemma that for some $(\pi_0, \pi_1) \in \mathbb{R}_{++} \times \mathbb{R}^S$, it is true that $\pi_0 p = A^\top \pi_1$. Letting $\pi = (\pi_0)^{-1}\pi_1^\top$ completes the proof of necessity, while sufficiency is straightforward. *Q.E.D.*

Proof of Lemma 4: The analysis in [4] assumes that structure A is complete, and guarantees uniqueness of the resulting δ. I here show that completeness is not necessary for the existence of δ; in Appendix A.2., I study the possible multiplicity of such matrices.

Define A_F and H as in Example 2, let Ω be the leading principal minor of Δ of order rank(A), and write

$$\Delta = \begin{pmatrix} \Omega & \Psi \\ \Psi^\top & \Gamma \end{pmatrix}.$$

Since A_F has full column rank, we can construct

$$\delta = A_F(A_F^\top A_F)^{-1}\Omega(A_F^\top A_F)^{-1}A_F^\top,$$

which is a symmetric, $S \times S$ matrix.

Now, define $T = (-H^\top, \mathbb{I})^\top$, where $\mathbb{I}$ is the identity matrix of dimension $K - \text{rank}(A)$, and note that $AT = 0$, which means that each column in matrix T is a return-irrelevant portfolio. Since Δ allows no price manipulation, it follows that $\Delta T = 0$, which implies that $\Omega H = \Psi$, that $\Psi^\top H = \Gamma$, and, therefore, that $H^\top \Omega H = \Gamma$. By direct computation,

$$A^\top \delta A = \begin{pmatrix} \Omega & \Omega H \\ H^\top \Omega & H^\top \Omega H \end{pmatrix} = \begin{pmatrix} \Omega & \Psi \\ \Psi^\top & \Gamma \end{pmatrix} = \Delta,$$

as desired. *Q.E.D.*

A.2. Multiplicity of state prices and state price impact matrices

In what follows, the notation B^+ is used for the pseudo-inverse of a matrix B.[29] It is well known that the general solution to equation $A^\top \pi = P$ is given by

$$\pi = (A^\top)^+ P + (\mathbb{I} - (A^\top)^+ A^\top)\bar{\pi},$$

for any $\bar{\pi} \in \mathbb{R}^S$. Moreover, $\dim(\{\pi \mid A^\top \pi = P\}) = S - \operatorname{rank}(A)$.

Now, suppose that some price impact matrix Δ gives no irrelevant impact. A similar decomposition to the one obtained for asset prices is possible for asset price impact matrices.

Theorem **5.** *If price impact matrix Δ is symmetric and allows no price manipulation, then the general solution to equation $A^\top \delta A = \Delta$ is given by*

$$\delta = (A^\top)^+ \Delta A^+ + \bar{\delta} - (A^\top)^+ A^\top \bar{\delta} A A^+,$$

for any $S \times S$ matrix $\bar{\delta}$. Moreover,

$$\dim(\{\delta \mid A^\top \delta A = M\}) = S^2 - \mathit{rank}(A)^2.$$

Proof: It follows from Lemma 4 that the equation $A^\top \delta A = \Delta$ is consistent, so the general solution follows by direct computation, using the properties of the pseudo-inverse. Now, the equation can be rewritten as $(A^\top \otimes A) \cdot \mathbf{vec}(\delta) = \mathbf{vec}(\Delta)$, where the operator **vec** converts a matrix into a vector by stacking its columns vertically.[30] It then follows that the dimension of set $\{\delta | A^\top \delta A = \Delta\}$ is $S^2 - \operatorname{rank}(A^\top \otimes A)$. *Q.E.D.*

This general result, however, does not give a dimension for the set of *symmetric* commodity-price impact matrices on which a symmetric bundle-price impact matrix can be decomposed. It is immediate, however, that the dimension of this set is at least the same as the dimension of $\ker(A)$, namely $S - \operatorname{rank}(A)$, for one can construct symmetric matrices $\delta = (A^\top)^+ \Delta A^+ + \mu\mu^\top$, for any vector $\mu \in \ker(A)$.

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[29] That is, if B is of order $m \times n$, its pseudo-inverse is the matrix of order $n \times m$ for which $BB^+B = B$, $B^+BB^+ = B^+$, $(BB^+)^\top = BB^+$ and $(B^+B)^\top = B^+B$. For the theory of pseudo-inverses, see C. Rao and S. Mitra [13].

[30] The symbol $\otimes$ denotes the standard Kronecker product of matrices.

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