Agreeing to Disagree with Multiple Priors *

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We present an extension of Aumann's Agreement Theorem to the case of multiple priors. If agents update all their priors, then, for the Agreement Theorem to hold, it is sufficient to assume that they have closed, connected and intersecting sets of priors. On the other hand, if agents select the priors to be updated according to the maximum likelihood criterion, then, under these same assumptions, agents may still *agree to disagree*. For the Agreement Theorem to hold, it is also necessary to assume that the maximum likelihood priors are commonly known and not disjoint. To show that these hypotheses are necessary, we give several examples in which agents *agree to disagree*.
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In a celebrated paper, Aumann (1976) established the Agreement Theorem: if two agents have the same prior belief over possible states of the world, and if their posteriors for an event are commonly known by both, then these posteriors must be equal. In this sense, agents cannot *agree to disagree*.[1]

Here, we investigate whether (or to what extent) this result extends to the case in which there is ambiguity, in the sense that the prior beliefs of the agents are described by a set of probability measures. Such a setting has been increasingly considered since the work of Bewley (1986, 1987 and 2002) on Knightian uncertainty, and of Gilboa and Schmeidler (1989) on maxmin expected utility.

We use the same definitions and set-up of Aumann's seminal paper, but for the fact that the two individuals have multiple priors. Since this feature implies that they may also have multiple posteriors for a given event, it may be worthwhile to point out that we say that the agents *disagree* if their sets of posterior distributions are disjoint, and *agree to disagree* if such fact is common knowledge.

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[1] Which implies that bets should not take place, other than for risk-sharing purposes (Milgrom and Stokey, 1982).

Two possibilities for the way in which agents update their priors are considered: (i) full Bayesian updating; and (ii) maximum likelihood updating. In the first case, the agents' sets of posteriors result from the Bayesian updating of *all* the distributions in their sets of priors: agents do not use the information they receive to refine their sets of priors. The second approach assumes that, once they receive their private information, they look at the priors that make the information they have received most likely, and only update these "maximum likelihood" priors.[2]

With agents updating all their priors, we present an Agreement Theorem which is essentially a reformulation of those by Kajii and Ui (2005 and 2009). With respect to Kajii and Ui (2005, Proposition 3), we replace the hypothesis of convexity of posteriors by connectedness of priors, and show that it is not necessary to consider a common set of priors to prevent disagreement: all that is necessary is that the sets of priors intersect. In fact, the more recent result by Kajii and Ui (2009, Corollary 12) only requires the sets of priors to be not disjoint, but it also requires the sets of posterior probability measures to be non-empty, closed and convex. Here, we only assume that the sets of priors are closed, connected and not disjoint.[3]

However, under the same hypotheses, if agents are maximum likelihood maximizers, then it is possible that they agree to disagree. We show this by way of several examples. In particular, we show that even with a common set of priors and intersecting sets of likelihood maximizers, the sets of posteriors may be commonly known but disjoint.[4]

In the context of a maxmin expected utility decision-maker, it is natural to assume that the set of probability distributions that constitutes the individual's beliefs is convex: preferences of this type remain unchanged when a set of probabilities is replaced by its convex hull. And since the convex hull of the set of posteriors that results from fully updating of a set of priors is equal to the set of posteriors that results from fully updating the convex hull of the same set of priors, in this context the convexity assumption used by Kajii and Ui (2005 and 2009) is most natural. But the assumption need not always be plausible to impose: suppose that an agent is to draw a ball from an urn that contains either one third of blue balls and two thirds of red balls, or vice-versa. And, more importantly, in contexts other than maxmin expected utility, the assumption of convexity is not innocuous. In any set-up in which preferences are linear on probabilities, replacing a set of probabilities by its convex hull may be done without loss of generality. But the latter is not the case, for instance, when an agent has variational preferences as introduced by Maccheroni et al (2006), where a convex function on probabilities renders a non-linearity.

With some qualification, if two individuals have maxmin expected utility preferences, then the disjointness of posterior beliefs implies the existence of agreeable bets, as shown by Billot et al. (2000) and Kajii and Ui (2006). In this note we do not consider the problem of betting, nor do we claim that, under our definition, agents who agree to disagree will find a mutually agreeable bet.

[2] The case of maximum likelihood updating has received little attention in the economic theory literature, in spite of its wide usage in statistics and econometrics.

[3] On the other hand, the Agreement Theorem in Kajii and Ui (2009) allows for an arbitrary updating rule.

[4] The body of literature that followed the seminal work of Aumann (1976) has neglected the search for cases in which agents actually agree to disagree. This may occur, for instance, in a countable space of equiprobable states of nature (Correia-da-Silva, 2010).

1. The model

Denote by Δ be the set of all probability measures defined over a finite measurable space $(\Omega, \mathcal{B})$. There are two individuals, $i = 1, 2$, who are asymmetrically informed about such realization. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions of Ω; each partition represents the private information of the corresponding agent.[5]

The two agents have ambiguous beliefs about the realization of the state and may differ in these prior beliefs: individual i has a non-empty set $\Delta_i \subseteq \Delta$ of prior probability distributions. Suppose that these two sets of measures have the property that if $E \in \mathcal{P}_1 \vee \mathcal{P}_2$, then $p(E) > 0$ for all $p \in \Delta_1 \cup \Delta_2$.

For any state $\omega \in \Omega$, denote by $E_i(\omega)$ the event in $\mathcal{P}_i$ that contains ω; this is the event of which agent i is privately informed when ω realizes. Similarly, let $E(\omega)$ be the event in $\mathcal{P}_1 \wedge \mathcal{P}_2$ to which ω belongs. An event E is said to be *common knowledge at* ω if (and only if) $E(\omega) \subseteq E$. The argument of the following lemma is essentially given by Kajii and Ui (2005).

Lemma 1. *Fix an individual i, an event A and a state $\omega \in \Omega$. Let Q be a nonempty subset of the interval $[0, 1]$, and let $\tilde{\Delta}$ be a nonempty subset of Δ. If the event*

$$\{\tilde{\omega} \in \Omega \mid \{q \mid \exists p \in \tilde{\Delta} : p(A \mid E_i(\tilde{\omega})) = q\} = Q\}$$

is common knowledge at ω, then

$$\{q \mid \exists p \in \tilde{\Delta} : p(A \mid E(\omega)) = q\} \subseteq [\inf Q, \sup Q].$$

Proof: By assumption,

$$E(\omega) \subseteq \left\{ \tilde{\omega} \in \Omega \mid \left\{ q \ \mid \ \exists p \in \tilde{\Delta} : \frac{p(A \cap E_i(\tilde{\omega}))}{p(E_i(\tilde{\omega}))} = q \right\} = Q \right\}.$$

Since $E(\omega) \in \mathcal{P}_1 \wedge \mathcal{P}_2$, we can write $E(\omega) = \cup_j E^j$, for some collection $\{E^j\}_j \subseteq \mathcal{P}_i$, and it follows that for all j,

$$\left\{ q \mid \exists p \in \tilde{\Delta} : \frac{p(A \cap E^j)}{p(E^j)} = q \right\} = Q.$$

Now, take any $p \in \tilde{\Delta}$, let $\inf Q = q^l$ and $\sup Q = q^u$, and note that, for each j,

$$q^l \leq \frac{p(A \cap E^j)}{p(E^j)} \leq q^u,$$

so it is immediate that

$$q^l \sum_j p(E^j) \leq \sum_j p(A \cap E^j) \leq q^u \sum_j p(E^j),$$

or, equivalently, that

$$q^l p(E(\omega)) \leq p(A \cap E(\omega)) \leq q^u p(E(\omega)).$$

[5] Denote by $\mathcal{P}_1 \vee \mathcal{P}_2$ the *join* of the two partitions, which is their coarsest common refinement. Their finest common coarsening, or *meet*, shall be denoted by $\mathcal{P}_1 \wedge \mathcal{P}_2$.

This implies that $p(A \mid E(\omega)) \in \left[q^l, q^u\right]$. *Q.E.D.*

The lemma will be the key step in the general results given later. It states that if it is common knowledge that agent i's set of posteriors is Q_i, then the set of posteriors obtained using only the common information is contained in $[\inf Q_i, \sup Q_i]$.

2. Full Bayesian Updating

Let A be an event. An individual carries out *full Bayesian updating*[6] if she updates *all* her priors, given her private information. In this case, the set of posterior probabilities that agent i attributes to the event A, in state $\tilde{\omega}$, is

$$\mathbf{Q}_i(\tilde{\omega}) = \{q \mid \exists p \in \Delta_i : p(A \mid E_i(\tilde{\omega})) = q\}.$$

Given a nonempty set $Q \subseteq [0,1]$, we say that *it is common knowledge at state ω that the set of posteriors of agent i is Q*, if the event consisting of all states $\tilde{\omega} \in \Omega$ for which $\mathbf{Q}_i(\tilde{\omega}) = Q$ is common knowledge at ω.

2.1. An Extension of Aumann's Theorem

The following proposition extends Aumann's Theorem (1976) to the case of multiple priors with full Bayesian updating, strengthening the results of Kajii and Ui (2005 and 2009). It states that if the two individuals have closed, connected and intersecting sets of priors, and their sets of posteriors are common knowledge, then they cannot agree to disagree (in the sense that their sets of posteriors intersect).

Proposition 1 (Aumann's Theorem). *Let $\omega \in \Omega$, and let Q_1 and Q_2 be nonempty subsets of $[0,1]$. Suppose that the sets of priors of the two agents, Δ_1 and Δ_2, are closed and connected. If for both individuals it is common knowledge at ω that $\mathbf{Q}_i(\omega) = Q_i$, then*

$$\{q \mid \exists p \in \Delta_1 \cap \Delta_2 : p(A \mid E(\omega)) = q\} \subseteq Q_1 \cap Q_2.$$

Proof: For each individual i, the mapping $p \mapsto p(A \mid E_i(\omega))$ is continuous over Δ_i, by the assumption that $p(E) > 0$ for all $p \in \Delta_i$ and all $E \in \mathcal{P}_1 \vee \mathcal{P}_2$. Since Δ_i is closed and connected, it then follows that i's set of posterior probabilities of A at ω, $\{q \mid \exists p \in \Delta_i : p(A \mid E_i(\omega)) = q\}$, is a closed interval. Moreover, by the assumption that it is common knowledge at ω that $\mathbf{Q}_i(\omega) = Q_i$,

$$E(\omega) \subseteq \{\tilde{\omega} \in \Omega \mid \{q \mid \exists p \in \Delta_i : p(A \mid E_i(\tilde{\omega})) = q\} = Q_i\},$$

so, since $\omega \in E(\omega)$, we have that

$$\{q \mid \exists p \in \Delta_i : p(A \mid E_i(\omega)) = q\} = Q_i,$$

and, hence, that $Q_i = [\inf Q_i, \sup Q_i]$. By Lemma 1, it follows that $p(A \mid E(\omega)) \in Q_i$ for all $p \in \Delta_i$. *Q.E.D.*

Under the hypotheses of the proposition, it further follows that $Q_1 \cap Q_2 \neq \varnothing$, whenever $\Delta_1 \cap \Delta_2 \neq \varnothing$.

[6] This is also referred to as *Fagin-Halpern* updating – see Kajii and Ui (2005).

The assumptions in Kajii and Ui (2005, Proposition 3) are that the individuals have the same set of priors, and that the sets of posteriors for an event are closed intervals, while the related result of Kajii and Ui (2009, Corollary 12), requires the sets of priors to be not disjoint, and the sets of posterior probability measures to be non-empty, closed and convex. We only assumed that the sets of priors are not disjoint, and that the sets of priors are closed and connected.

The following example shows that, without the connectedness condition, Proposition 1 does not hold: agents can *agree to disagree* even when they share a common set of priors.

Example 1. *Let the set of possible states of nature be* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$*; let the common set of priors,* $\Delta_1 = \Delta_2 = \bar{\Delta}$*, consist of two probability measures,* $p_1 = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ *and* $p_2 = (0, \frac{1}{2}, 0, \frac{1}{2})$*; and suppose that the information partitions are* $\mathcal{P}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ *and* $P_2 = \{\Omega\}$*.*

Consider the event $A = \{\omega_2, \omega_3\}$. We want to show that even though the sets of priors intersect (fully), and the sets of Bayesian posteriors are both (closed and) common knowledge at any $\omega \in \Omega$, these latter sets are disjoint. First, note that for any $\tilde{\omega} \in \Omega$, we have that

$$\{q \mid \exists p \in \bar{\Delta} : p(A \mid E_1(\tilde{\omega})) = q\} = \{0, 1\},$$

while

$$\{q \mid \exists p \in \bar{\Delta} : p(A \mid E_2(\tilde{\omega})) = q\} = \left\{\frac{1}{2}\right\}.$$

So, if we let $Q_1 = \{0, 1\}$ and $Q_2 = \left\{\frac{1}{2}\right\}$, we have that both events

$$\{\tilde{\omega} \in \Omega \mid \{q \mid \exists p \in \bar{\Delta} : p(A \mid E_i(\tilde{\omega})) = q\} = Q_i\}$$

are common knowledge at any $\omega \in \Omega$, yet $Q_1 \cap Q_2 = \varnothing$.[7]

3. Maximum Likelihood Updating

For each individual, let $\Delta_i(E) = \operatorname{argmax}_{p \in \Delta_i} p(E)$, for each $E \in \mathcal{P}_i$. An individual *uses maximum likelihood updating* if, at each state of nature, she updates *only* the priors that make the information she has received most likely: at state ω, her set of posteriors is given by the updating of priors that belong to $\Delta_i(E_i(\omega))$ only.[8] Abusing notation slightly, we will also write $\Delta_i(\omega)$ for the set $\Delta_i(E_i(\omega))$.

As before, fix an event A. Unlike in the setting of a Bayesian individual, for one who uses maximum likelihood updating we cannot define a set of posterior probabilities of A given the individual's information partition, for the set of priors that she updates changes with the event she is informed of. At state ω, the set of posterior probabilities of A is

$$\mathbf{Q}_i(\omega) = \{q \mid \exists p \in \Delta_i(\omega) : p(A \mid E_i(\omega)) = q\};$$

[7] It continues to be true that $\{q \mid \exists p \in \bar{\Delta} : p(A \mid E(\omega)) = q\} \subseteq [\inf Q_1, \sup Q_1] \cap [\inf Q_2, \sup Q_2]$, but without connectedness this does not guarantee that $\{q \mid \exists p \in \bar{\Delta} : p(A \mid E(\omega)) = q\} \subseteq Q_i$.

[8] This type of updating is also known as *Dempster-Shafer* updating.

that is, if $q \in \mathbf{Q}_i(\omega) \subseteq [0,1]$, then there is some $p \in \Delta_i$ that maximizes the probability of observing $E_i(\omega)$ over individual i's set of priors, and for which the posterior for event A, given i's information in state ω, is q.

Given a nonempty set $Q \subseteq [0,1]$, we will say that *it is common knowledge at state ω that the set of posteriors of individual i is Q* if the event consisting of all the states $\tilde{\omega} \in \Omega$ for which $\mathbf{Q}_i(\tilde{\omega}) = Q$ is common knowledge at ω.

3.1. Agreeing to Disagree

The following example shows that in the case of individuals who use maximum likelihood updating, the result of Proposition 1 does no longer hold: under the hypotheses of that proposition, individuals who use maximum likelihood updating can agree to disagree.

EXAMPLE 2. *Let the set of possible states of nature be $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$, and let the common set of priors, $\Delta_1 = \Delta_2 = \bar{\Delta}$, consist of the union of the two following sets of probability measures:*

$$\Delta^a = \left\{ p = \left(0, \frac{1}{2} - x, \frac{1}{2} - x, 0, x, x\right) : 0 \leq x \leq \frac{1}{2} \right\}$$

and

$$\Delta^b = \left\{ p = \left(\frac{1}{2} - x, 0, 0, \frac{1}{2} - x, x, x\right) : 0 \leq x \leq \frac{1}{2} \right\}.$$

Suppose that $\mathcal{P}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\{\omega_5, \omega_6\}\}$ and $\mathcal{P}_2 = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6\}\}$ are the information partitions.

Note that when $\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\}$, the set of likelihood maximizers is common to both agents:

$$\Delta_1(\omega) = \Delta_2(\omega) = \left\{ \left(0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right), \left(\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0\right) \right\}.$$

Moreover, the posteriors for event $A = \{\omega_1, \omega_3\}$ are constant across $\{\omega_1, \omega_2, \omega_3, \omega_4\}$, and, therefore, common knowledge at any $\omega \in A$, but, nevertheless, they do not intersect: $Q_1(\omega) = \{0, 1\}$ while $Q_2(\omega) = \{\frac{1}{2}\}$.[9]

3.2. An Extension of Aumann's Theorem

An extension of Aumann's result is obtained for individuals who use maximum likelihood updating, if (i) one strengthens the requirement of connectedness of the sets of priors to convexity, and (ii) further assumes that the sets of likelihood maximizers are commonly known and intersect. The first assumption is very standard in the literature. For the second assumption, formally, given a nonempty set $\tilde{\Delta} \subseteq \Delta$, we will say that *it is common knowledge at state ω that the set of likelihood-maximizers of individual i is $\tilde{\Delta}$* if the event consisting of all the states $\tilde{\omega} \in \Omega$ for which $\Delta_i(\tilde{\omega}) = \tilde{\Delta}$ is common knowledge at ω. Under these extra hypotheses, if the sets of maximum likelihood priors intersect, then so do the sets of posteriors, if they are both commonly known.

[9] To make the example clearer, we have not required strict positivity of the probability distributions. The same result would be obtained with $\Delta^a = \left\{p = \left(\epsilon, \frac{1}{2} - \epsilon - x, \frac{1}{2} - \epsilon - x, \epsilon, x, x\right) : \epsilon \leq x \leq \frac{1}{2} - 2\epsilon\right\}$ and with Δ^b modified in the same way.

Proposition 2. *Let $\omega \in \Omega$, let Q_1 and Q_2 be nonempty subsets of $[0,1]$, and let $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ be nonempty subsets of Δ. Suppose that the sets of priors, Δ_1 and Δ_2, are closed and convex. If, for both i, it is common knowledge at ω that agent i's set of likelihood maximizers is $\tilde{\Delta}_i$ and that agent i's set of posteriors for an event A is Q_i, then*

$$\{q \mid \exists p \in \Delta_1(\omega) \cap \Delta_2(\omega) : p(A \mid E(\omega)) = q\} \subseteq Q_1 \cap Q_2.$$

Proof: The proof resembles the argument given in the case of Bayesian updaters, so some details can be omitted. For each individual i, note first that the mapping $p \mapsto p(A \mid E_i(\omega))$ is concave over Δ_i, so, since Δ_i is convex, it follows that i's set of likelihood maximizers, $\Delta_i(\omega)$, is convex, and then, as in the proof of Proposition 1, that her set of posterior probabilities of event A,

$$\{q \mid \exists p \in \Delta_i(\omega) : p(A \mid E_i(\omega)) = q\},$$

is a closed interval. Since it is common knowledge at ω that the set of likelihood maximizers is $\tilde{\Delta}_i$, we further have that for all $\tilde{\omega} \in E(\omega)$, $\Delta_i(\tilde{\omega}) = \tilde{\Delta}_i$. By Lemma 1, then,

$$\{q \mid \exists p \in \tilde{\Delta}_i : p(A \mid E(\omega)) = q\} \subseteq [\inf Q_i, \sup Q_i],$$

which implies that

$$\{q \mid \exists p \in \Delta_i(\omega) : p(A \mid E(\omega)) = q\} \subseteq Q_i,$$

since it is common knowledge at ω that i's set of likelihood maximizers is $\tilde{\Delta}_i$ and her sets of posteriors is Q_i. *Q.E.D.*

Not surprisingly, the hypothesis that the sets of likelihood maximizers intersect is indispensable for this result,[10] but a general characterization for this condition remains an open question. This characterization is complicated by the fact that these are the sets of maximizers of different functions over different domains. But if one assumes one of these two features away, it is easy to see that the structure of the problems gives, at least, partial answers. Suppose that both individuals have the same set of priors, namely that $\Delta_1 = \Delta_2 = \bar{\Delta}$.[11] In this case, a necessary condition for $p \in \Delta_1(\omega) \cap \Delta_2(\omega)$ is that p must (also) solve the problem

$$\max_{p \in \bar{\Delta}} \{p\,(E_1(\omega) \cup E_2(\omega)) + p\,(E_1(\omega) \cap E_2(\omega))\}.$$

To see that this is the case, note that if $p \in \Delta_1(\omega) \cap \Delta_2(\omega)$, then, by definition, for any $\tilde{p} \in \bar{\Delta}$ it must be true that $p(E_1(\omega)) \geq \tilde{p}(E_1(\omega))$ and $p(E_2(\omega)) \geq \tilde{p}(E_2(\omega))$. But this means that

$$p\,(E_1(\omega) \cap E_2(\omega)) + p\,(E_1(\omega) \setminus E_2(\omega)) \geq \tilde{p}\,(E_1(\omega) \cap E_2(\omega)) + \tilde{p}\,(E_1(\omega) \setminus E_2(\omega)) \qquad (1)$$

and

$$p\,(E_1(\omega) \cap E_2(\omega)) + p\,(E_2(\omega) \setminus E_1(\omega)) \geq \tilde{p}\,(E_1(\omega) \cap E_2(\omega)) + \tilde{p}\,(E_2(\omega) \setminus E_1(\omega))\,. \qquad (2)$$

If we then add these two inequalities, it follows that for any $\tilde{p} \in \bar{\Delta}$, one has that

$$p\,(E_1(\omega) \cup E_2(\omega)) + p\,(E_1(\omega) \cap E_2(\omega)) \geq \tilde{p}\,(E_1(\omega) \cup E_2(\omega)) + \tilde{p}\,(E_1(\omega) \cap E_2(\omega))\,.$$

[10] This can be seen by considering Example 4 below.

[11] And suppose also that $\mathcal{B}$ is fine enough to allow for all the sets below to be measurable.

On the other hand, sufficient conditions for the sets of likelihood maximizers to intersect are also possible when the sets of priors coincide. For instance, if the setting is sufficiently symmetric, in the sense that for all $p \in \bar{\Delta}$ it is true that $p\left(E_1(\omega) \setminus E_2(\omega)\right) = p\left(E_2(\omega) \setminus E_1(\omega)\right)$, then $\Delta_1(\omega) = \Delta_2(\omega)$. This is because, again by construction, for any $p \in \Delta_1(\omega)$ and any $\tilde{p} \in \bar{\Delta}$, one has that Eq. (1) holds, and, hence, by the symmetry property, so does Eq. (2).

3.3. More Agreeing to Disagree

We now show that the additional hypotheses of Proposition 2 are necessary, by means of examples. If the convexity assumption on the sets of priors is replaced by the weaker assumption of connectedness, we already know that agents may agree to disagree: this was shown in Example 2. Example 3 shows that even if the sets of priors are common, the posteriors are commonly known and the sets of likelihood maximizers intersect, the sets of posteriors can be disjoint.

EXAMPLE 3. *Let the set of possible states of nature be* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9\}$, *let the common set of priors be* $\Delta_1 = \Delta_2 = \bar{\Delta}$, *for*

$$\bar{\Delta} = \left\{p = \left(x, x, x, y, \frac{1}{3} - x - y, y, \frac{1}{3} - x - y, y, \frac{1}{3} - x - y\right) : x, y \geq 0, x + y \leq \frac{1}{3}\right\},$$

and suppose that the information partitions are

$$\mathcal{P}_1 = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9\}\}$$

and

$$\mathcal{P}_2 = \{\{\omega_1, \omega_6, \omega_8\}, \{\omega_2, \omega_3, \omega_5\}, \{\omega_4, \omega_7, \omega_9\}\}.$$

Consider the event $A = \{\omega_2, \omega_3, \omega_6, \omega_7, \omega_8, \omega_9\}$. Depending on the state of nature, the sets of maximum-likelihood priors of individual 1 are

$$\Delta_1(\{\omega_1, \omega_2, \omega_3\}) = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, 0, 0\right)\right\}$$

or

$$\Delta_1(\{\omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9\}) = \left\{p = \left(0, 0, 0, y, \frac{1}{3} - y, y, \frac{1}{3} - y, y, \frac{1}{3} - y\right) : 0 \leq y \leq \frac{1}{3}\right\},$$

but, in any case, her posteriors are the singleton set $Q_1 = \{\frac{2}{3}\}$. On the other hand, the sets of maximum likelihood priors of individual 2 are

$$\Delta_2(\{\omega_1, \omega_6, \omega_8\}) = \left\{\left(0, 0, 0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0\right)\right\},$$

$$\Delta_2(\{\omega_2, \omega_3, \omega_5\}) = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, 0, 0\right)\right\}$$

and

$$\Delta_2(\{\omega_4, \omega_7, \omega_9\}) = \left\{\left(0, 0, 0, 0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}\right)\right\}.$$

In any case, her posteriors for event A are all the singleton set $Q_2 = \{1\}$. With both Q_1 and Q_2 constant across the states in A, one has that these sets of posteriors are common knowledge at ω_2, and $\Delta_1(\omega_2) \cap \Delta_2(\omega_2) \neq \varnothing$, but the two individuals, still, agree to disagree: Q_1 and Q_2 are disjoint.

Next, Example 4 will show a case in which agents agree to disagree even though their information partitions are the same, their sets of priors intersect, and their sets of posteriors are commonly known. In this case, the result fails because individuals update disjoint sets of likelihood maximizers; while one should not expect individuals to agree in such situation, what the example highlights is the possibility that such disagreement in priors can occur between people whose information partitions are identical and whose original priors are not disjoint.

Example 4. *Taking Δ^a and Δ^b as defined in Example 2, let $\Delta_1 = \Delta^a$ and $\Delta_2 = \Delta^b$, and let agent 2 have the same information as agent 1, with $\mathcal{P}_1 = \mathcal{P}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_0, \omega_5\}\}$.*

When $\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\}$, each agent has a single maximum likelihood prior:

$$\Delta_1(\omega) = \left\{ \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0 \right) \right\}$$

and

$$\Delta_2(\omega) = \left\{ \left(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0 \right) \right\}.$$

Their posteriors for the event $A = \{\omega_2, \omega_3\}$ are constant across $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ and are, as before, common knowledge, but are completely opposite: $Q_1 = \{1\}$ while $Q_2(\omega) = \{0\}$.

Finally, it is worthwhile to observe that while the hypothesis that there is at least one commonly likelihood maximizer that is commonly known is required for our conclusion that the sets of (commonly known) posteriors intersect, this condition is not necessary from a logical perspective. That is, while it is true that if, in addition to the hypotheses of Proposition 2, one has that $\Delta_1(\omega) \cap \Delta_2(\omega) \neq \varnothing$, then $Q_1 \cap Q_2 \neq \varnothing$, the former condition is not implied by the latter: the following presents, straightforwardly, a case in which $Q_1 \cap Q_2 \neq \varnothing$ and $\Delta_1(\omega) \cap \Delta_2(\omega) \neq \varnothing$, but where both Q_1 and Q_2 are commonly known while $\Delta_1(\omega) \cap \Delta_2(\omega)$ is not.

Example 5. *Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, let the (common) set of priors be $\Delta_1 = \Delta_2 = \bar{\Delta}$, for*

$$\bar{\Delta} = \left\{ p = \left(x, y, \frac{1}{2} - x, \frac{1}{2} - y \right) : x, y \geq 0, x, y \leq \frac{1}{2} \right\},$$

and suppose that $\mathcal{P}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and $\mathcal{P}_2 = \{\Omega\}$.

Consider the event $A = \{\omega_2, \omega_3\}$. Note that the sets of likelihood maximizers for individual 1 are

$$\Delta_1(\{\omega_1, \omega_2\}) = \left\{ \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right) \right\}$$

or

$$\Delta_1(\{\omega_3, \omega_4\}) = \left\{ \left(0, 0, \frac{1}{2}, \frac{1}{2} \right) \right\},$$

but, in any case, the set of posteriors for event A is $Q_1 = \{\frac{1}{2}\}$ at all ω. Individual 2 is uninformed, so $\Delta_2(\Omega) = \bar{\Delta}$, but her set of posteriors is also $Q_2 = \{\frac{1}{2}\}$. The sets of posteriors are common and commonly known (since they are the same for all $\omega \in \Omega$), as are the sets of priors. And while the sets of likelihood maximizers trivially intersect (since individual 2 cannot refine her set of priors), no common likelihood maximizer is commonly known, for $\Delta_1(\{\omega_1, \omega_2\}) \cap \Delta_1(\{\omega_3, \omega_4\}) = \varnothing$ and agent 2 is uninformed.

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